

# Inequality with Ordinal Data

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## **Abstract**

The standard theory of inequality measurement assumes that the equalisand is a cardinal quantity with known cardinalisation. However, one may need to make inequality comparisons where either the cardinalisation is unknown or the underlying data are categorical. We propose a natural way of evaluating individuals' status in such situations, based on their position in the distribution and develop axiomatically a class of inequality indices, conditional on a reference point. We examine the merits of mean, median and maximum status as reference points. We also show how the approach can be applied to perceived health status and reported happiness.

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# 1 Introduction

It is very common to find problems of inequality comparison where conventional inequality-measurement tools just will not work. A principal cause of this difficulty lies in the nature of the thing that is being studied. By contrast to the inequality analysis of income or wealth distributions where the underlying variable is measurable and interpersonally comparable, “ordinal data” may be measurable but with an unknown scale of measurement, or the underlying concept may not be measurable at all. The term “ordinal data” covers such things as access to amenities, educational achievement, happiness, health, each of which has a sub-literature on inequality in its own right.<sup>1</sup> This paper provides a new approach to the generic ordinal-data problem.

Why is there a problem with ordinal data? Although we can handle a small number of standard tools of distributional analysis, several key concepts are not well defined. For example the mean will depend on the particular cardinalisation that is used and so there is no meaning to points in a simplex. Therefore we cannot implement something like the Principle of Transfers. The literatures on inequality in happiness, health and so on contain a number of work-rounds that address this problem but none of these work-rounds is entirely satisfactory. In some cases first-order dominance criteria have been applied and quantiles have been used to characterise inequality comparisons. But difficulties can arise even with these methods.<sup>2</sup>

However, in this paper we show that, as long as the underlying data can be ordered, it is possible to construct a fully-fledged approach to inequality measurement. In section 2 we discuss the basic building blocks of inequality analysis, section 3 develops our formal approach to the problem of ordinal data and section 4 discusses the properties of the class of inequality measures that emerges from our analysis. Section 5 investigates the empirical properties of the class of inequality measures and discusses two important applications; section 6 concludes.

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<sup>1</sup>On health status, see for example the discussion by Van Doorslaer and Jones (2003) of the McMaster Health Utility Index (page 65). On life satisfaction and inequality of happiness see for example, Oswald and Wu (2011), Stevenson and Wolfers (2008b), Yang (2008).

<sup>2</sup>See, for example, Abul Naga and Yalcin (2010), Allison and Foster (2004).

## 2 Income, utility, status

There are three basic ingredients of the income-inequality measurement problem:

- the definition of “income”
- the definition of the “income-receiving unit”
- method of aggregation

The same issues arise in cases where “income” is replaced by some other concept that is essentially ordinal rather than cardinal. We briefly deal first with the standard income-inequality problem before modelling the ordinal-data problem.

### 2.1 Income and inequality

The conventional approach to inequality measurement is quite demanding informationally. It takes income (or wealth, expenditure...) as the first ingredient; income is assumed to be both measurable and inter-personally comparable.<sup>3</sup> so that we can represent it formally as  $x \in X \subseteq \mathbb{R}$ . For a given population consisting of  $n$  persons an income distribution is simply a vector  $\mathbf{x} \in X^n$ . In aggregating information about the income distribution in order to measure inequality we usually work with  $X_\mu^n$ , a subset of  $X^n$  consisting of all the  $n$ -person distributions that have a specified mean  $\mu$ . The economic meaning of  $X_\mu^n$  is the set of all distributions obtainable from a given total income  $n\mu$  using lump-sum transfers.

However, we obviously cannot adopt such an approach in the present case, but we can use it as a basis for developing a new approach that covers the case of ordinal data.

### 2.2 Utility

Now let us consider cases where the equalisand is less rich in information than the case of income as described in section 2.1. suppose inequality is in terms of something which we will call utility. Of course utility may be derived from income  $u = U(x)$ . However, even if we assume that  $U$  has cardinal

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<sup>3</sup>See the discussion in Cowell (2011).

significance in the sense that it is defined up to affine transformations (as in the standard treatment of individual choice under risk) one still has a problem as far as inequality-measurement is concerned. As an example consider the Gini coefficient if utility (income) is subjected to an affine transformation: multiplying every person's utility by  $\lambda \in (0, 1)$  and adding an amount  $\delta = \mu[1 - \lambda]$ : if  $\lambda < 1$  this transformation of the scale and level of utility will automatically reduce measured inequality.

There is an analogy with the Atkinson (1970) revolution in inequality measurement. In his path-breaking paper Dalton (1920) had worked with utilities<sup>4</sup> and Atkinson pointed out that, although the Dalton insight of using (social) utility was a valuable step in the appropriate direction for introducing social values into inequality measurement, looking at the relative shortfall of average utility of income below the utility of average income was flawed, in that it was dependent on the assumed origin of the utility scale. Atkinson's solution was to introduce the concept of equivalent income.

However, the Atkinson and Dalton approaches properly belonged to the third of the basic ingredients mentioned above – the aggregation process. They were concerned with the way in which social values could be introduced into an inequality-evaluation of income distribution, not the inequality-evaluation of a distribution of utilities. In some cases these two problems can be seen as effectively equivalent; what is more it may be possible to infer the utility function from, for example, people's attitude to risk. But there are other important cases where this is not true and where utility has no natural income equivalent:

**Example 1.** Suppose utility depends on some measurable quantity  $x$  that has no agreed valuation. If the form of  $U(\cdot)$  is unknown and cannot be inferred from individual behaviour, then increased dispersion of  $x$  leads to increased inequality, but we cannot say by how much. If  $U$  differs across individuals we cannot say even that.

**Example 2.** Suppose utility depends on a categorical variable. “Amenity inequality” – inequality in access to specific public facilities – provides an illustrative example of the concept. Suppose there is general agreement that, other things being equal, a person is better off with access to both gas and electricity supplies than with access to electricity only, a person is better off with access to electricity only than gas only and

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<sup>4</sup>See also Aigner and Heins (1967).

	Case 1	Case 2
	$n_k$	$n_k$
<u>B</u> oth Gas and Electricity	25	0
<u>E</u> lectricity only	25	50
<u>G</u> as only	25	50
<u>N</u> either	25	0

Table 1: Access to amenities: categorical variable

is better off with access to gas only than access to neither utility; we know nothing about how much energy is consumed or about how much could be afforded; so trying to put a dollar equivalent may be inappropriate or meaningless. In Table 1 suppose  $n_k$  is the number of persons in category  $k \in \{B, E, G, N\}$ . If there are 100 people in the population, intuition suggests that Case 1 in Table 1 represents more amenity inequality than does Case 2.

It is easy to find practical cases that have some of the characteristics of one or other example: quality of life, happiness may be represented by our Example 1; inequality of educational attainment can be thought of in terms of Example 2. Some important aspects of inequality, such as inequality in health status, display elements of both examples. Conventional tools of inequality analysis are not of much immediate help in quantifying inequality comparisons in either example. In the face of these difficulties two main approaches have been adopted in the literature.

The first is to impute some artificial index of individual well-being  $y$  as a function of  $x$  or of the relevant category. In some cases the imputation is achieved through subjective evaluation by individuals (for example on a Likert scale) and in some cases by official institutions (for example the Quality-Adjusted Life Year or the Human Development Index). Clearly the same procedure can be applied explicitly or implicitly to entities that do not have a natural ordering, such as vectors of attributes or endowments; one uses the utility function to force an ordering of the data.<sup>5</sup> This approach runs into some obvious objections such as the arbitrariness of the cardinalisation,

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<sup>5</sup>This is similar to one of the standard theoretical approaches to the measurement of multi-dimensional inequality – one computes the “utility” of factors and then computes inequality of utility where the utility function is an appropriate aggregator (Maasoumi 1986, Tsui 1995) .

the arbitrariness of aggregating apples and oranges, and the arbitrariness of attempting to include measures of dispersion into the index as well. Even if the resulting  $y$  appears reasonable over a wide subset of the possible values of  $x$  we might still be concerned about the way extreme values are represented in the index and their consequences for inequality comparisons.

The second approach, quite common in the health literature,<sup>6</sup> involves a reworking of traditional inequality-ranking approaches focusing on first-order dominance criteria. It is commonly suggested that the median could be used as an equality concept corresponding to the use of the mean in conventional inequality analysis, although it has been noted that comparing distributions with different medians raises special issues (Abul Naga and Yalcin 2010). Clearly such an approach may run into difficulty if quantiles are not well-defined, as may happen in the case of categorical variables.<sup>7</sup>

### 2.3 Status

In the broad literature on the assessment and comparison of income distributions it is common to find that a person’s location in the distribution is related to a concept of an individual’s status in society and then have used this to develop measures of individual and social deprivation. Some have incorporated into the measurement of inequality. Here it is central to the approach.

Denote by  $u_i$  person  $i$ ’s endowment of utility and let the distribution function of  $u$  for a population of size  $n$  be  $F$ . Each person’s status  $s_i$  is uniquely defined for a given distribution of as follows:

$$s_i = \psi(u_i, F(\cdot), n), \tag{1}$$

where the function  $\psi$  is such that status is independent of the cardinalisation of utility. One simple way of specifying status for an individual is the standard definition of position, the proportion of the population that is no better off than oneself; so if the distribution cumulative function for utility in the population is  $F$  then we could use  $s_i = F(u_i)$  as the status measure.<sup>8</sup>

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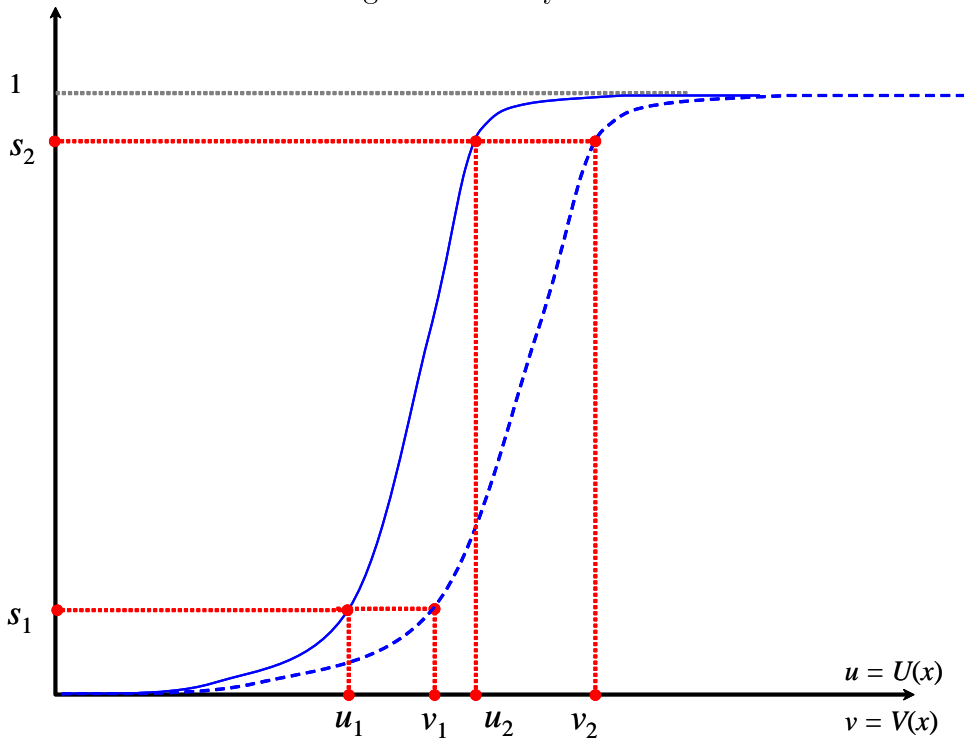
<sup>6</sup>For the application to health, see Abul Naga and Yalcin (2008), Allison and Foster (2004), Zheng (2011).

<sup>7</sup>For examples of this see footnote 12 below.

<sup>8</sup> $F(u_i)$  and  $1 - F(u_i)$  correspond to Yitzhaki (1979)’s concepts of “Satisfaction” and “Deprivation.”

Such a status measure is familiar to anyone who has ever taken a GRE or TOEFL test and is similar to that sometimes used to measure opportunity – see de Barros et al. (2008). Working with the status thus implied by utility rather than with utility itself would obviously dispose of the problem of cardinalisation, as illustrated in Figure 1: if  $U$  and  $V$  are two alternative cardinalisations of the utility of income  $x$  ( $V$  is a monotonic increasing transformation of  $U$ ), then the two utilities for person 1 under the two cardinalisations  $u_1$  and  $v_1$  each map into  $s_1$  and, for person 2,  $u_2$  and  $v_2$  each map into  $s_2$ .

Figure 1: Utility and Status



However, for categorical data – where we know only the ordering of the categories  $k$  and the number of persons  $n_k$  in each category,  $k = 1, 2, \dots, K$  – there is a stronger reason for taking a status measure of this form. Consider the following “merger principle.”:

**Mergers.** If non-empty adjacent categories  $k^*$  and  $k^* + 1$  are merged then



this has no effect on the status of a person who belongs to neither category.

We can easily see that this simple principle induces an additive structure. Suppose there are three categories and that person  $i$  belongs to category 1. Now let categories 2 and 3 be merged; in view of the mergers principle person  $i$ 's status is given by  $s_i = f_i(n_1, n_2, n_3) = f_i(n_1, n_2 + n_3, 0)$ . If the status function is anonymous, so that every person in a given category has the same status, it is clear that status must take the form

$$s_i = f \left( \sum_{\ell=1}^{k(i)-1} n_{\ell}, n_{k(i)}, \sum_{\ell=k(i)+1}^K n_{\ell} \right).$$

where  $k(i)$  is the category to which person  $i$  belongs. Hence we could take person  $i$ 's status to be

$$s_i = \frac{1}{n} \sum_{\ell=1}^{k(i)} n_{\ell}. \quad (2)$$

The form (2) provides a “downward-looking” measure of position (my status is determined by all those below me or equal to me) that is independent of population replications. But other formulations of status may be appropriate. We could also consider the “upward-looking” counterpart of (2):

$$s'_i = \frac{1}{n} \sum_{\ell=k(i)}^K n_{\ell}; \quad (3)$$

here my status is determined as the proportion of those above me or equal to me. It can also be useful to express status in absolute terms, as  $ns_i$  or  $ns'_i$ ; for example if status is interpreted as position in the football league then we would measure it as  $ns'_i$ , an absolute, upward-looking concept. However, we do not need to specify a particular form for our theoretical approach; in section 4 we will provide particular examples based on equations (2) and (3).

## 3 Inequality measurement: theory

### 3.1 Approach

Here we offer a new approach to inequality measurement that draws together many of the themes discussed in section 2. It involves two main steps.

Step 1 is to define status. We assume that an individual's status is given by

$$s \in S \subseteq \mathbb{R},$$

The precise definition will depend on the structure of information and may also depend on the purpose of the inequality analysis. In some cases a person's status is self-defining from the data: for example if we want to focus on income or wealth inequality. In some cases status is defined once one is given additional distribution-free information: for example if there are observations on some variable  $x$  and it is known that utility is  $\log(x)$ . In some cases status requires information dependent on distribution of the underlying data: for example it could be determined as in (2) or (3).

Step 2 is to aggregate information about the distribution of status  $\mathbf{s} \in S^n$ , where  $n$  is the size of the population. To make progress on this step we need (1) a concept of equality and (2) a way of characterising departures from equality.

Define  $e \in \mathbb{R}$  as an equality-reference point. We have to go carefully here since, unlike the conventional income-inequality case where the summation of incomes is well defined and it makes sense to assume that equality is where everyone receives mean income, we cannot similarly aggregate utility and we cannot meaningfully aggregate status without specifying  $\psi$ . The reference point could be specified exogenously or it could also depend on the status vector

$$e = \eta(\mathbf{s}). \tag{4}$$

The specification of  $e$  is discussed further in Section 4.2 below. Denote by  $S_e^n$ ; the subset of  $S^n$  that consists of vectors with the same value of  $e$ .

To capture inequality we could define a specific distance function  $d : S^2 \rightarrow \mathbb{R}_+$ , where  $d(s, e)$  means the distance that a person with status  $s$  is from equality  $e$ : this is analogous to, say, the interpretation of Generalised-Entropy measures of income inequality that can be thought of as average distance from mean income. We could then introduce a number of principles (axioms) to characterise an inequality ordering  $\succeq$  and the associated distance concept. However, we can make progress without an explicit function  $d(\cdot)$  *a priori*: by characterising  $\succeq$  axiomatically the distance-from-equality concept will emerge .

## 3.2 Inequality ordering on ordinal data

Consider inequality as a weak ordering  $\succeq$  on  $S^n$ ; denote by  $\succ$  the strict relation associated with  $\succeq$  and denote by  $\sim$  the equivalence relation associated with  $\succeq$ . We also need one more piece of notation: for any  $\mathbf{s} \in S^n$  denote by  $\mathbf{s}(\varsigma, i)$  the member of  $S^n$  formed by replacing the  $i$ th component of  $\mathbf{s}$  by  $\varsigma \in S$ . The first step is to characterise the general structure of the inequality relation using just four axioms:<sup>9</sup>

**Axiom 1** [*Continuity*]  $\succeq$  is continuous on  $S^n$ .

**Axiom 2** [*Monotonicity in distance*] If  $\mathbf{s}, \mathbf{s}' \in S_e^n$  differ only in their  $i$ th component then (a) if  $s'_i \geq e : s_i > s'_i \iff \mathbf{s} \succ \mathbf{s}'$ ; (b) if  $s'_i \leq e : s'_i > s_i \iff \mathbf{s} \succ \mathbf{s}'$ .

In other words, if two distributions differ only in respect of person  $i$ 's status, then the distribution that registers greater individual distance from equality for  $i$  is the distribution that exhibits greater inequality.

**Axiom 3** [*Independence*] For  $\mathbf{s}, \mathbf{s}' \in S_e^n$ , if  $\mathbf{s} \sim \mathbf{s}'$  and  $s_i = s'_i$  for some  $i$  then  $\mathbf{s}(\varsigma, i) \sim \mathbf{s}'(\varsigma, i)$  for all  $\varsigma \in [s_{i-1}, s_{i+1}] \cap [s'_{i-1}, s'_{i+1}]$ .

Suppose that the distributions  $\mathbf{s}$  and  $\mathbf{s}'$  are equivalent in terms of inequality and that there is some person  $i$  who has the same status in  $\mathbf{s}$  and in  $\mathbf{s}'$ . Then, the same small change in  $i$ 's status in both distributions  $\mathbf{s}$  and  $\mathbf{s}'$  still leaves  $\mathbf{s}$  and  $\mathbf{s}'$  as equivalent in terms of inequality.

**Axiom 4** [*Anonymity*] For all  $\mathbf{s} \in S^n$  and permutation matrix  $\mathbf{\Pi}$ ,  $\mathbf{\Pi}\mathbf{s} \sim \mathbf{s}$ .

Permuting the labels on the individuals leaves inequality unchanged.

**Theorem 1** Given Axioms 1 to 4  $\succeq$  is representable by the continuous function  $I : S_e^n \rightarrow \mathbb{R}$  given by

$$I(\mathbf{s}; e) = \Phi \left( \sum_{i=1}^n d(s_i, e), e \right), \quad (5)$$

where  $d : S \rightarrow \mathbb{R}$  is a continuous function that is strictly increasing (decreasing) in its first argument if  $s_i > e$  ( $s_i < e$ ).

<sup>9</sup>Here we follow the method of Cowell and Ebert (2004) and Ebert and Moyes (2002).

**Proof.** Axiom 1 implies that  $\succeq$  is representable by a continuous function  $I(\mathbf{s}; e)$ . By Axiom 2  $I$  is increasing in  $s_i$  if  $s_i > e$  and *vice versa*. From Theorem 5.3 of Fishburn (1970) Axiom 3 further implies that  $\succeq$  must be expressible as

$$\sum_{i=1}^n d_i(s_i, e), \forall \mathbf{s} \in S_e^n \quad (6)$$

up to an increasing transformation, where, for each  $i$ ,  $d_i : S \rightarrow \mathbb{R}$  is a continuous function. In view of Axiom 4 the functions  $d_i$  must all be identical and the result follows. ■

Theorem 1 establishes inequality as total “distance” from equality and that the function  $d$  is continuous, satisfies  $d(e, e) = 0$  and has the property that, for a given  $e$ ,  $d(s, e)$  is increasing in status  $s$  if  $s$  is above the reference point, decreases in  $s$  if  $s$  is below the reference point.<sup>10</sup> The next steps involve introducing more structure on  $d$  and hence on  $\succeq$ .

### 3.3 Characterisation of inequality

The following is perhaps a natural assumption, although the implication of alternatives will be discussed later.

**Axiom 5** [*Scale irrelevance*] For all  $\lambda \in \mathbb{R}_+$ : if  $\mathbf{s}, \mathbf{s}' \in S_e^n$  and  $\lambda \mathbf{s}, \lambda \mathbf{s}' \in S_{\lambda e}^n$  then  $\mathbf{s} \sim \mathbf{s}' \Rightarrow \lambda \mathbf{s} \sim \lambda \mathbf{s}'$ .

In the case where status is defined proportional to position in the distribution, as in (??) then this Axiom is equivalent to the Dalton principle of population – see also Yaari (1988) for a similar assumption.

**Theorem 2** Given Axioms 1 to 5  $\succeq$  is representable by (5) where the function  $d$  takes the form

$$d(s, e) = e^c \phi\left(\frac{s}{e}\right) \quad (7)$$

where  $\phi$  is a continuous function and  $c$  is an arbitrary constant.

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<sup>10</sup>Note that  $d$  is not a conventional distance function since the symmetry property is not required.

**Proof.** From (5)  $\mathbf{s}, \mathbf{s}' \in S_e^n$  and  $\mathbf{s} \sim \mathbf{s}'$  imply

$$\sum_{i=1}^n d(s_i, e) = \sum_{i=1}^n d(s'_i, e)$$

and so Axiom 5 implies

$$\sum_{i=1}^n d(\lambda s_i, \lambda e) = d(\lambda s'_i, \lambda e).$$

Therefore we have

$$\frac{\sum_{i=1}^n d(\lambda s_i, \lambda e)}{\sum_{i=1}^n d(s_i, e)} = \frac{\sum_{i=1}^n d(\lambda s'_i, \lambda e)}{\sum_{i=1}^n d(s'_i, e)} = f(\lambda),$$

where  $f$  is a continuous function  $\mathbb{R} \rightarrow \mathbb{R}$ , so that

$$\sum_{i=1}^n d(\lambda s_i, \lambda e) = f(\lambda) \sum_{i=1}^n d(s_i, e).$$

This implies , for any  $s \in S$ :

$$d(\lambda s, \lambda e) = f(\lambda) d(s, e).$$

and there must exist  $c \in \mathbb{R}$  and a function  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that<sup>11</sup>

$$d(s, e) = e^c \phi\left(\frac{s}{e}\right) \tag{8}$$

■

This establishes “distance from equality” in terms of proportions rather than, say, absolute differences. However, because  $\phi$  is an arbitrary function, we need to impose more structure on  $d$  in order to obtain a usable inequality measure. As the next step we impose the following

**Axiom 6** [*Ratio scale irrelevance*] Suppose there are  $\mathbf{s} \in S_e^n$  and  $\mathbf{s}^\circ \in S_{e^\circ}^n$  such that  $\mathbf{s} \sim \mathbf{s}^\circ$ . Then for all  $\lambda > 0$ ,  $\mathbf{s}' \in S_e^n$  and  $\mathbf{s}'' \in S_{e^\circ}^n$  such that for each  $i$ ,  $s'_i/e = \lambda s_i/e$  and  $s''_i/e^\circ = \lambda s_i^\circ/e^\circ$ :  $\mathbf{s}' \sim \mathbf{s}''$ .

<sup>11</sup>See Aczél and Dhombres (1989), page 346

This requirement means that a proportionate increase in all status measures relative to the reference point leaves inequality comparisons unchanged. We may then state:

**Theorem 3** *Given Axioms 1 to 6  $\succeq$  is representable as  $\Phi(I(\mathbf{s}; e), e)$  where*

$$I_\alpha(\mathbf{s}; e) = \frac{1}{\alpha[\alpha - 1]} \left[ \frac{1}{n} \sum_{i=1}^n \left[ \frac{s_i}{e} \right]^\alpha - c(e) \right], \quad (9)$$

$\alpha \in \mathbb{R}$   $c$  is a constant, conditional on  $e$  and  $\Phi$  is increasing in its first argument

**Proof.** Take the special case where

$$e^\circ = e,$$

$$s_1^\circ = s_1,$$

$$s_i^\circ = \bar{s}, \quad i = 2, \dots, n.$$

Using (8)  $\mathbf{s} \sim \mathbf{s}^\circ$  implies

$$\begin{aligned} \sum_{i=2}^n e^c \phi\left(\frac{s_i}{e}\right) &= \sum_{i=2}^n e^c \phi\left(\frac{\bar{s}}{e}\right) \\ \sum_{i=2}^n \phi\left(\frac{s_i}{e}\right) &= [n - 1] \phi\left(\frac{\bar{s}}{e}\right) \end{aligned} \quad (10)$$

From Axiom 6 equation (10) implies, for any  $\lambda > 0$ :

$$\sum_{i=2}^n \phi\left(\lambda \frac{s_i}{e}\right) = [n - 1] \phi\left(\lambda \frac{\bar{s}}{e}\right)$$

and so, taking the inverse of  $\phi$ :

$$\bar{s} = e \phi^{-1} \left( \frac{1}{n - 1} \sum_{i=2}^n \phi\left(\frac{s_i}{e}\right) \right) \quad (11)$$

$$\lambda \bar{s} = e \phi^{-1} \left( \frac{1}{n - 1} \sum_{i=2}^n \phi\left(\lambda \frac{s_i}{e}\right) \right) \quad (12)$$

$$\phi \left( \lambda \phi^{-1} \left( \frac{1}{n-1} \sum_{i=2}^n \phi \left( \frac{s_i}{e} \right) \right) \right) = \frac{1}{n-1} \sum_{i=2}^n \phi \left( \lambda \frac{s_i}{e} \right)$$

Introduce the following change of variables

$$u_i := \phi \left( \frac{s_i}{e} \right), i = 2, \dots, n \quad (13)$$

$$\phi \left( \lambda \phi^{-1} \left( \frac{1}{n-1} \sum_{i=2}^n u_i \right) \right) = \frac{1}{n-1} \sum_{i=2}^n \phi \left( \lambda \phi^{-1} (u_i) \right), \text{ for all } \lambda > 0. \quad (14)$$

Also define the following functions

$$\theta_0(u, \lambda) := \sum_{i=2}^n \phi \left( \lambda \phi^{-1} \left( \frac{1}{n-1} u \right) \right) \quad (15)$$

$$\theta_1(u, \lambda) := \frac{1}{n-1} \phi \left( \lambda \phi^{-1} (u) \right). \quad (16)$$

Substituting (15),(16) into (14) we get the Pexider functional equation

$$\theta_0 \left( \sum_{i=2}^n u_i, \lambda \right) = \sum_{i=2}^n \theta_1(u_i, \lambda)$$

which has as a solution

$$\theta_0(u, \lambda) = a(\lambda) u + [n-1]b(\lambda), \quad (17)$$

$$\theta_1(u, \lambda) = a(\lambda) u + b(\lambda), \quad (18)$$

– see Aczél (1966), page 142. Therefore, from (13), (16) and (18) we have

$$\frac{1}{n-1} \phi(\lambda v) = a(\lambda) u + b(\lambda), \quad (19)$$

where  $v$  is an arbitrary value of  $s_i/e$ . From Eichhorn (1978), Theorem 2.7.3 the solution to (19) is of the form

$$\phi(v) = \begin{cases} \beta v^\alpha + \gamma, & \alpha \neq 0 \\ \beta \log v + \gamma, & \alpha = 0 \end{cases} \quad (20)$$

	Case 0			Case 1			Case 2			Case 3		
	$n_k$	$s_i$	$s'_i$	$n_k$	$s_i$	$s'_i$	$n_k$	$s_i$	$s'_i$	$n_k$	$s_i$	$s'_i$
<b>B</b>	0			25	1	1/4	0			25	1	1/4
<b>E</b>	50	1	1/2	25	3/4	1/2	50	1	1/2	25	3/4	1/2
<b>G</b>	25	1/2	3/4	25	1/2	3/4	50	1/2	1	50	1/2	1
<b>N</b>	25	1/4	1	25	1/4	1	0			0		
$\mu(\mathbf{s})$	11/16			5/8			3/4			11/16		

Table 2: Distributional comparisons for a categorical variable

where  $\beta$  is an arbitrary positive number. Substituting for  $\phi(\cdot)$  from (20) into (7) gives the result. ■

Note that the general function  $\Phi$  takes into account the possibility that one might choose non-zero value of  $c$  in (8).

## 4 Inequality measures

As we have seen, the conventional approach to inequality measurement only works within a narrowly defined information structure. In this alternative approach how do we proceed to get a usable inequality index? Equation (9) provides us with the structure for inequality analysis of ordinal data; to get an operational inequality measure from this requires some further steps. We need to check that measures of the form  $\Phi(I(\mathbf{s}; e), e)$  have sensible properties when taken applied as inequality measures, we need to consider how the reference point for inequality comparisons is to be determined, and we need to clarify the way in which different members of the class of indices of the form (9) behave.

### 4.1 The transfer principle?

In standard approaches to inequality measurement the transfer principle plays a central role; but in its pure form it is clearly not relevant here. The problem is that the transfer principle is simply inappropriate in the case of ordinal data because there is no natural “compensation” to consider. However, might the transfer principle apply in modified form? Let us work through a simple example using our access-to-utilities story.



Table 2 represents four distributional scenarios based on the example in Table 1. As before  $n_k$  is the number of persons in category  $k \in \{B, E, G, N\}$  and in each scenario there are 100 persons in the population. We consider the two versions of status  $s_i$  and  $s'_i$  introduced in subsection 2.3 – see equations (2) and (3). The last row in the table contains the value of mean status,

$$\mu(\mathbf{s}) = \mu(\mathbf{s}') = \frac{1}{n} \sum_{i=1}^n s_i = \frac{1}{n} \sum_{i=1}^n s'_i, \quad (21)$$

If the relevant scenario changes from Case 0 to Case 1 then 25 people are promoted from category E to category B and, by the principle of monotonicity, if  $e$  were a constant equal to any of the values taken by  $\mu(\mathbf{s})$  (namely  $5/8, 11/16$  or  $3/4$ ) then inequality would increase; if instead the relevant scenario changes from Case 0 to Case 2 then 25 people are promoted from category N to category G and, by the same principle, inequality would decrease.

Now suppose we try to invoke a principle that appears to be related to the transfer principle. If  $e$  is a constant, independent of  $\mathbf{s}$  or if  $e$  depends only on  $\mu(\mathbf{s})$  then, the following is true for all values of  $\alpha$ : if there is a change in the underlying distribution such that  $i$ 's status increases  $\delta > 0$  and  $j$ 's status decreases by  $\delta$  (where  $s_i < s_j$  and  $s_i + \delta < s_j - \delta$ ), then, from (9) inequality is reduced. However, this “transfer property” is not particularly attractive, as we can see from Table 2. If the relevant scenario changes from Case 0 to Case 3 this exactly fits the balanced change-of-status story: person  $i$ , on getting promoted from N to G, experiences an increase in status of  $1/4$ ; if a person is promoted from E to B this reduces the status of person  $j$  by  $1/4$ . But the change from Case 0 to Case 3 is, as we have seen, a combination of an inequality-increasing and inequality-decreasing change in scenario (it combines the two changes that we discussed in the previous paragraph). What is important is the individual move towards or away from the reference point.

## 4.2 The reference point

The importance of specifying a reference point is clear from this example. We know that, for any given reference point  $e$ , the inequality measure must take the form (9). However we are not restricted to situations where the reference point is an exogenously given number. To include the possibility that the reference point is a function of  $\mathbf{s}$  as in equation (4) we take the

natural extension of (9) given by

$$I_\alpha(\mathbf{s}; \eta(\mathbf{s})) = \frac{1}{\alpha[\alpha - 1]} \left[ \frac{1}{n} \sum_{i=1}^n \left[ \frac{s_i}{\eta(\mathbf{s})} \right]^\alpha - c(\eta(\mathbf{s})) \right], \quad (22)$$

It is clear that inequality as measured by  $I_\alpha(\mathbf{s}; \eta(\mathbf{s}))$  is invariant under replications of the population and that if  $c \equiv 1$  then  $I_\alpha(\mathbf{e}; e) = 0$ . Other properties will depend on the specification of  $\eta(\cdot)$ . Let us consider three alternative specifications that might appear to make sense.

### I mean status

In the conventional approach to inequality measurement the reference point is taken to be the mean value of the quantities to which the inequality measure is being applied – mean income, mean wealth or whatever. Although one does not suppose that this mean value is what individuals would actually receive if a policy were implemented to enforce equality, it is still a useful and informative reference point. Taking the lead from this we might consider  $e = \eta(\mathbf{s}) = \mu(\mathbf{s})$  given in (21) as in the example of Table 1. If we use this specification and set  $c(e) = 1$  then it is easy to show that  $I_\alpha(\mathbf{s}; \mu(\mathbf{s})) \geq 0$  with equality if and only if, for all  $i$ :  $s_i = \mu(\mathbf{s})$ . It is also the case that  $I_\alpha(\mathbf{s}; \mu(\mathbf{s}))$  is continuous in  $\alpha$ . Note that, by contrast to standard inequality analysis where one conventionally assumes a fixed total of income or wealth, in the case of categorical data  $\mu(\mathbf{s})$  cannot be an exogenously given constant (see, for example, Table 2).

### II median status

Consider the median,  $e = \eta(\mathbf{s}) = \text{med}(\mathbf{s})$ , defined as  $e \in S$  such that  $\#(s_i \leq e) \geq \frac{n}{2}$  and  $\#(s_i \geq e) \geq \frac{n}{2}$ , as a possible reference point. We immediately find a fundamental problem for the kind of problem considered here: in the case of categorical data the median is not well-defined.<sup>12</sup> For

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<sup>12</sup>Two examples illustrate the kind of problem that can arise. (1) With three ordered categories and the same proportion of individuals in each category, the median is ambiguous. The status vector is  $\mathbf{s} = (1/3, 2/3, 1)$ . The conventional definition of the median gives  $\text{med}(\mathbf{s}) = m := 2/3$ . But, nevertheless we can see that  $2/3$  of the population has a status less or equal to  $m$  and  $1/3$  of the population has a status greater than  $m$ . (2) With two ordered categories (where B is better than A) and a thousand persons consider the following three distributions: (i)  $n_A = 500$ ,  $n_B = 500$ ; (ii)  $n_A = 499$ ,  $n_B = 501$ ; (iii)  $n_A = 999$ ,  $n_B = 1$ . Distributions (i) and (ii) have very different medians, but distributions (ii) and (iii) have almost the same median!

example in Table 2 the median could be any value in an interval  $M(\mathbf{s})$  where  $M(\mathbf{s}) = [1/2, 1)$  in cases 0 and 2 and  $M(\mathbf{s}) = [1/2, 3/4)$  in cases 1 and 3. Even if we resolve this problem by picking one specific value  $e \in M(\mathbf{s})$  as the reference point, for example the lower bound of the interval  $M(\mathbf{s})$  it is not clear that this provides an appropriate reference point with categorical data. Furthermore, there is nothing in the formula (22) that prevents the index taking a negative value; a redefinition of the inequality index as

$$\frac{1}{\alpha[\alpha - 1]} \left[ \frac{1}{n} \sum_{i=1}^n \left[ \frac{s_i}{e} \right]^\alpha - \left[ \frac{\mu(\mathbf{s})}{e} \right]^\alpha \right], \quad (23)$$

would produce a measure that is continuous in  $\alpha$  and always positive. But this requires  $c = [\mu(\mathbf{s})/e]^\alpha$  which is not expressible as  $c(\text{med}(\mathbf{s}))$  and so (23) cannot be expressed in the basic form (22).

### III maximum status

Finally, consider the situation where the reference point  $e$  is independent of  $\mathbf{s}$ . What values could or should  $e$  take? Consider what happens in the case of perfect equality where the distance from the reference point must be zero for everyone. If status is defined as individual position in the distribution, there is only one case for which we can have  $s_i = e$ ,  $i = 1, \dots, n$ : this is where  $e = 1$ , the maximum possible value of  $s$ .

With  $e = 1$ , a natural normalisation of the index is  $c = 1$ . This ensures that the inequality index equals 0 in the situation of equality,  $I(\mathbf{1}; 1) = 0$ . However, once again there is no guarantee that the inequality measure will be non-negative for every value of  $\alpha$ .

Which of these these alternative concepts of the reference point should be used? Let us compare them using the amenity-inequality discussed earlier. The first three rows of Table 3 shows the values of a number of reference points  $e$ , the mean  $\mu(\mathbf{s})$  and two possible interpretations of the median:  $\text{med}_1(\mathbf{s})$  is the midpoint of the interval  $M(\mathbf{s})$  and  $\text{med}_2(\mathbf{s})$  is the lower bound of  $M(\mathbf{s})$ . The four columns correspond to the four cases in Table 2; notice that  $\mu(\mathbf{s})$  and  $\text{med}_1(\mathbf{s})$  differ from one distribution to another. Rows 4-7 of Table 3 give the values of the index  $I_0(\mathbf{s}; e)$  (see (24) below) with  $c = 1$  for each of these three endogenous specifications of  $e$  and for the case  $e = 1$ .

The specifications  $I_0(\mathbf{s}; \mu(\mathbf{s}))$  and  $I_0(\mathbf{s}; \text{med}_1(\mathbf{s}))$  appear to produce counterintuitive results: inequality *decreases* when one person is promoted from

	Case 0	Case 1	Case 2	Case 3
$\mu(\mathbf{s})$	11/16	5/8	3/4	11/16
$\text{med}_1(\mathbf{s})$	3/4	5/8	3/4	5/8
$\text{med}_2(\mathbf{s})$	1/2	1/2	1/2	1/2
$I_0(\mathbf{s}; \mu(\mathbf{s}))$	0.1451	0.1217	0.0588	0.0438
$I_0(\mathbf{s}; \text{med}_1(\mathbf{s}))$	0.2321	0.1217	0.0588	-0.0515
$I_0(\mathbf{s}; \text{med}_2(\mathbf{s}))$	-0.1732	-0.1013	-0.3465	-0.2746
$I_0(\mathbf{s}; 1)$	0.5198	0.5917	0.3465	0.4184

Table 3: Inequality comparisons for a categorical variable: Downward-looking status

E to B (Case 0 to Case 1, or Case 2 to Case 3) . We can understand why this happens when we see that the movement of this person changes both the  $\mu(\mathbf{s})$  and  $\text{med}_1(\mathbf{s})$  reference points. By contrast, if we use the  $\text{med}_2(\mathbf{s})$  specification, the reference point does not change and the inequality changes are in the direction that accords with intuition; the major problem, of course, is that  $I_0(\mathbf{s}; \text{med}_2(\mathbf{s}))$  is negative for *all* the cases in this example!

Only one specification of the reference point appears to produce an inequality measure that behaves consistently with what one might expect: it is the last row with  $e = 1$ . Here we find that  $I_0(\mathbf{s}; 1)$  increases when one individual moves away from the (fixed) reference point (Case 0 to Case 1, Case 3 to Case 1) and  $I_0(\mathbf{s}; 1)$  decreases when one individual moves toward the reference point (Case 0 to Case 2, Case 3 to Case 2). Moreover measured inequality is always positive if  $\mathbf{s} \neq \mathbf{1}$ .

It seems that the use of either definition of the median produces unsatisfactory results – some or all of the inequality values are negative in our simple example. Use of mean status seems to produce strange results when the mean changes significantly as people’s position changes. This appears to leave only maximum status as a candidate reference point. To get further insight on this let us consider examine the way the class (22) behaves for different values of  $\alpha$ .

### 4.3 The sensitivity parameter $\alpha$

The parameter  $\alpha$  in the generic formula (22) captures the sensitivity of measured inequality to different parts of the distribution, for any reference point  $e$  and any normalisation constant  $c$ . In the case of high values of  $\alpha$  the index

is particularly sensitive to high-status inequality. For low and negative values of the parameter the opposite is true.

### Special Cases

For any specification of the reference point  $e$  there is at least one special form of (22). In the case where  $\alpha = 0$  and  $c = 1$  the expression (22) takes the form used in Table 3:

$$I_0(\mathbf{s}; e) = -\frac{1}{n} \sum_{i=1}^n \log \left( \frac{s_i}{e} \right); \quad (24)$$

this follows from a simple application of L'Hopital's rule. Furthermore, in Specification I – where the reference point is the mean value of status,  $e = \mu(\mathbf{s})$  – it is clear that we have  $I_0(\mathbf{s}; e) \geq 0$ ; this is also true in Specification III (exogenously given reference point) if  $e = 1$ . The second remark follows because  $0 < s_i \leq 1$ .

Under certain circumstances there is a second special case where  $\alpha = 1$ . If we take the Specification I where  $e = \mu(\mathbf{s})$  and assume  $c = 1$  then :

$$I_1(\mathbf{s}; \mu(\mathbf{s})) = \frac{1}{n} \sum_{i=1}^n \frac{s_i}{\mu(\mathbf{s})} \log \left( \frac{s_i}{\mu(\mathbf{s})} \right). \quad (25)$$

Again it is clear that  $I_1(\mathbf{s}; \mu(\mathbf{s})) \geq 0$ .

### Negative values

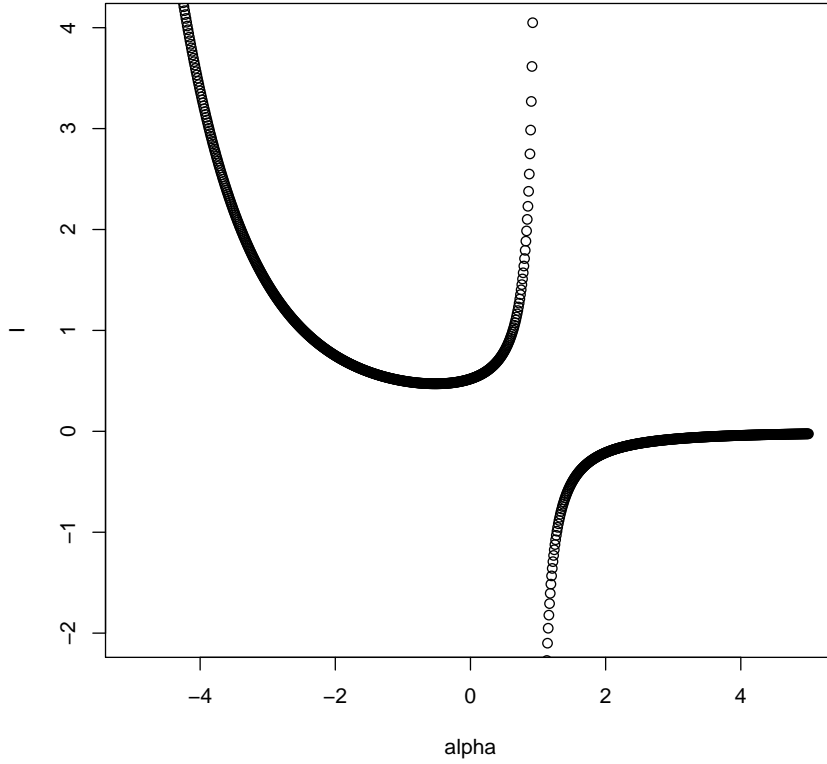
In cases where  $e \neq \mu(\mathbf{s})$ , the index  $I_\alpha$  can be negative and is undefined for  $\alpha = 1$ . These properties are illustrated in Figure 2 which plots values of  $I(\mathbf{s}; 1)$  for different values of  $\alpha$ , using the distribution labeled Case 0 in Table 2. The problem with the index in the neighbourhood of  $\alpha = 1$  is clear:  $I_\alpha(\mathbf{s}; 1) \rightarrow +\infty$  when  $\alpha \uparrow 1$  and  $I_\alpha(\mathbf{s}; 1) \rightarrow -\infty$  when  $\alpha \downarrow 1$ .

In the case where  $c = 1$  and  $e = 1$ , using (24) we may rewrite (22) as

$$I_\alpha(\mathbf{s}, 1) = \begin{cases} \frac{1}{\alpha(\alpha-1)} \left[ \frac{1}{n} \sum_{i=1}^n s_i^\alpha - 1 \right], & \text{if } \alpha \neq 0, 1, \\ -\frac{1}{n} \sum_{i=1}^n \log s_i. & \text{if } \alpha = 0. \end{cases} \quad (26)$$

Notice that  $I_\alpha(\mathbf{s}, 1)$  can also be written

Figure 2: The behaviour of  $I_\alpha(\mathbf{s}; 1)$  as  $\alpha$  varies



$$I_\alpha(\mathbf{s}, 1) = \frac{1}{\alpha - 1} \left[ \frac{1}{n} \sum_{i=1}^n \frac{s_i^\alpha - 1}{\alpha} \right], \text{ if } \alpha \neq 0, 1. \quad (27)$$

For any  $\alpha \in \mathbb{R}$ , it is clear that if  $0 < s < 1$  then  $[s^\alpha - 1]/\alpha < 0$  and if  $s = 1$  then  $[s^\alpha - 1]/\alpha = 0$ . So it is evident from (27) that  $I_\alpha(\mathbf{s}; 1)$  is only well behaved under the parameter restriction  $\alpha < 1$ . But this parameter restriction is exactly the same as that required in order to obtain the Atkinson family of inequality indices from the family of generalised entropy indices. So our admissible class of inequality indices could also be written in the

ordinally equivalent form

$$A_\alpha(\mathbf{s}) := \begin{cases} 1 - \left[\frac{1}{n} \sum_{i=1}^n s_i^\alpha\right]^{1/\alpha} & \text{if } \alpha < 0 \text{ or } 0 < \alpha < 1, \\ 1 - \left[\prod_{i=1}^n s_i\right]^{1/n} & \text{if } \alpha = 0. \end{cases} \quad (28)$$

#### 4.4 Downward- or upward-looking status

Finally, let us examine the inequality outcomes if we use an upward-looking rather than downward-looking concept of status. The bottom two rows of Table 4 report the values of  $I_0$  for the two concepts where the reference status is 1; in the upper part of this table the numbers of persons in each category are reproduced for reference (taken from Table 2). It comes as no surprise that the inequality outcome for each version of status is high in Case 1 and low in Case 2. For downward-looking status inequality is higher when the distribution is skewed towards the higher categories (Case 0); for upward-looking status inequality is higher when the distribution is skewed towards the lower categories (Case 3).

The reason for this is suggested by equation (26) where the number of active categories remains unchanged: the index decreases when the vector of status gets closer to a vector of 1. Label individuals in increasing order of utility. Then from (2), we have  $0 < s_1 \leq s_2 \leq \dots \leq s_n \leq 1$  for downward-looking status: the status vector becomes closer to the vector unity when people move to the lower categories. For upward-looking status we have  $1 \geq s'_1 \geq s'_2 \geq \dots \geq s'_n > 0$ : the status vector becomes closer to the vector unity when people move to the higher categories.

What happens if the number of active categories changes? For downward-looking status measures inequality must increase if a person migrates upwards from a category with multiple occupants to an empty category; for upward-looking status measures inequality must increase if a person migrates downwards from a category with multiple occupants to an empty category.<sup>13</sup>

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<sup>13</sup>We can see this if we consider what happens to inequality when there is a merger. Suppose status is downward-looking, given by (2). Now let the non-empty categories  $k^*$  and  $k^* + 1$  be merged. This is equivalent to increasing the size of category  $k^*$  and emptying category  $k^* + 1$ . For every  $i$  such that  $k(i) = k^*$  we see that  $s_i$  increases by  $n_{k^*+1}/n$ ; for every other  $i$  status  $s_i$  remains unchanged. So the status of some persons moves closer to  $e$  and remains unchanged for others. By Axiom 2 this means that inequality must fall. By contrast, if a person migrates from category  $k^*$  (where  $n_{k^*} > 1$ ) to empty category  $k^* + 1$

	Case 0	Case 1	Case 2	Case 3
	$n_k$	$n_k$	$n_k$	$n_k$
<b>B</b>	0	25	0	25
<b>E</b>	50	25	50	25
<b>G</b>	25	25	50	50
<b>N</b>	25	25	0	0
$I_0(\mathbf{s}; 1)$	0.5198	0.5917	0.3465	0.4184
$I_0(\mathbf{s}'; 1)$	0.4184	0.5917	0.3465	0.5198

Table 4: Inequality outcomes for downward-looking status ( $\mathbf{s}$ ) and upward-looking status ( $\mathbf{s}'$ )

## 5 Inequality measurement: practice

To be used in practice, we need to study the statistical properties of the inequality measure with ordinal data. In this section, we establish asymptotic distribution and finite sample performance of such inequality measures. Two empirical applications, in health and in happiness, illustrate the usefulness of the inequality measures in practice. In the light of the arguments in Section 4 we focus on ordered categorical variables, for which the appropriate reference point is the maximum status  $e = 1$  and for which a natural normalisation of the index is  $c = 1$ : we take the family of inequality indices (26). We consider samples with independent observations on  $K$  ordered categories. Without loss of generality, any sample can be represented as follows:

$$x_i = \begin{cases} 1 & \text{with sample proportion } p_1 \\ 2 & \text{with sample proportion } p_2 \\ \dots & \\ K & \text{with sample proportion } p_K \end{cases}, \quad (29)$$

where  $p_l$  is the number of observations in the  $l^{\text{th}}$  category, divided by the sample size, so that  $\sum_{l=1}^K p_l = 1$ . For  $K > 2$ , the sample proportions  $(p_1, p_2, \dots, p_K)$  are known to follow a multinomial distribution with  $n$  observations and a vector of probabilities  $(\pi_1, \pi_2, \dots, \pi_K)$ . The status of observation  $i$  is its position in the distribution, computed as the proportion of

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this is the exact opposite of the merger just described: so inequality must rise. A similar argument can be constructed for upward-looking status.



observations in the sample with a value less than or equal to  $x_i$ :

$$s_i = \frac{1}{n} \sum_{j=1}^n \iota(x_j \leq x_i) = \sum_{j=1}^{x_i} p_j \quad (30)$$

where  $\iota(\cdot)$  is the indicator function, equals to 1 if its argument is true and 0 otherwise. We provide the derivations for downward-looking status only, with status defined in (30). The extension to upward-looking status is straightforward: all one needs to do is to reverse the order of the categories.

## 5.1 Statistical properties

With a sample of categorical data, as defined in (29), and status given by the individual position, as defined in (30), we can rewrite the inequality measure (26) as follows:

$$I_\alpha = \begin{cases} \frac{1}{\alpha(\alpha-1)} \left[ \sum_{i=1}^K p_i \left[ \sum_{j=1}^i p_j \right]^\alpha - 1 \right] & \text{if } \alpha \neq 0, 1, \\ - \sum_{i=1}^K p_i \log \left[ \sum_{j=1}^i p_j \right] & \text{if } \alpha = 0. \end{cases} \quad (31)$$

This measure is expressed as a non-linear function of  $K$  parameter estimates  $(p_1, p_2, \dots, p_K)$  following a multinomial distribution. From the Central Limit Theorem,  $I_\alpha$  follows asymptotically a Normal distribution with a covariance matrix which can be calculated by the delta method. Specifically, an estimator of the covariance matrix of  $(p_1, p_2, \dots, p_k)$  is given by

$$\Sigma = \frac{1}{n} \begin{bmatrix} p_1(1-p_1) & -p_1p_2 & \dots & -p_1p_K \\ -p_2p_1 & p_2(1-p_2) & \dots & -p_2p_K \\ \vdots & \vdots & \ddots & \vdots \\ -p_Kp_1 & -p_Kp_2 & \dots & p_K(1-p_K) \end{bmatrix}. \quad (32)$$

The variance estimator for  $I_\alpha$  is equal to:

$$\widehat{\text{Var}}(I_\alpha) = D\Sigma D^\top \quad \text{with} \quad D = \left[ \frac{\partial I_\alpha}{\partial p_1} ; \frac{\partial I_\alpha}{\partial p_2} ; \dots ; \frac{\partial I_\alpha}{\partial p_K} \right], \quad (33)$$

where the  $l^{\text{th}}$  element of  $D$  is defined as

$$\frac{\partial I_\alpha}{\partial p_l} = \begin{cases} \frac{1}{\alpha(\alpha-1)} \left( \left[ \sum_{i=1}^l p_i \right]^\alpha + \alpha \sum_{i=l}^{K-1} p_i \left[ \sum_{j=1}^i p_j \right]^{\alpha-1} \right) & \text{if } \alpha \neq 0, 1, \\ -\log \left[ \sum_{j=1}^l p_j \right] - \sum_{i=l}^{K-1} p_i \left[ \sum_{j=1}^i p_j \right]^{-1} & \text{if } \alpha = 0. \end{cases} \quad (34)$$

For instance, in the case of three ordered categories, the inequality measures defined in (26) are equal to

$$I_\alpha = \begin{cases} \frac{p_1^{\alpha+1} + p_2(p_1+p_2)^\alpha + p_3 - 1}{\alpha(\alpha-1)} & \text{if } \alpha \neq 0, 1, \\ -p_1 \log p_1 - p_2 \log(p_1 + p_2) & \text{if } \alpha = 0, \end{cases}$$

where  $p_1, p_2$  and  $p_3$  are the proportions of observations in the respective ordered categories. Their variance estimators are given by (33), with  $\Sigma$  defined in (32) where we use  $K = 3$ , and with  $D$  equal to

$$D = \begin{cases} \frac{1}{\alpha(\alpha-1)} \left[ (\alpha+1)p_1^\alpha + \alpha p_2(p_1+p_2)^{\alpha-1}; (p_1+p_2)^\alpha + \alpha p_2(p_1+p_2)^{\alpha-1}; 1 \right] & \text{if } \alpha \neq 0, 1, \\ \left[ -\log p_1 - 1 - p_2/(p_1+p_2); -\log(p_1+p_2) - p_2/(p_1+p_2); 0 \right] & \text{if } \alpha = 0. \end{cases}$$

We can use the variance estimators of  $I_\alpha$  to compute test statistics and confidence intervals.

We now turn to the finite sample properties of the index. The coverage error rate of a confidence interval is the probability that the random interval does not include, or cover, the true value of the parameter. A method of constructing confidence intervals with good finite sample properties should provide a coverage error rate close to the nominal rate. For a confidence interval at 95%, the nominal coverage error rate is equal to 5%. We use Monte-Carlo simulation to approximate the coverage error rate of asymptotic confidence intervals in several experimental designs.

In our experiments, samples are drawn from a multinomial distribution with probabilities  $\pi = (\pi_1, \pi_2, \dots, \pi_K)$ . The status of observation  $i$  is its position in the distribution, computed as the proportion of observations in the sample with a value less than or equal to  $x_i$ . For fixed values of  $\alpha$ ,  $n$ ,  $K$  and  $\pi$ , we draw 10 000 samples. For each sample we compute  $I_\alpha(\mathbf{s}, 1)$  and its confidence interval at 95%. The coverage error rate is computed as the

$\alpha$	-1	0	0.5	0.99	1.01	1.5	2
$n = 20$	0.0606	0.0417	0.0598	0.0491	0.0491	0.0472	0.0431
$n = 50$	0.0553	0.0518	0.0704	0.0601	0.0603	0.0483	0.0394
$n = 100$	0.0499	0.0513	0.0684	0.0619	0.0619	0.0489	0.0416
$n = 200$	0.0544	0.0476	0.0617	0.0613	0.0607	0.0543	0.0451
$n = 500$	0.0523	0.0492	0.0521	0.0523	0.0526	0.0498	0.0466
$n = 1000$	0.0485	0.0540	0.0552	0.0549	0.0551	0.0546	0.0528

Table 5: Coverage error rate of asymptotic confidence intervals at 95% of  $I_\alpha$ , 10 000 replications,  $K = 3$  and  $x \sim Multinomial(0.3, 0.5, 0.2)$ .

proportion of times the true value of the inequality measure is not included in the confidence intervals.<sup>14</sup> Confidence intervals perform well in finite sample if the coverage error rate is close to the nominal value.

Table 5 shows coverage error rates of confidence intervals at 95% of  $I_\alpha$  for different values of  $\alpha = -1, 0, 0.5, 0.99, 1.01, 1.5, 2$ , as the sample size increases,  $n = 20, 50, 100, 200, 500, 1000$ . We consider 3 ordered categories: samples are drawn from a multinomial distribution with probabilities  $\pi = (0.3, 0.5, 0.2)$ . If the asymptotic distribution is a good approximation of the exact distribution of the statistic, the coverage error rate should be close to the nominal error rate, 0.05. The results show that asymptotic confidence intervals perform well in finite sample, they are still reliable for  $\alpha = 0.99, 1.01$  when the index is undefined for  $\alpha = 1$ .

## 5.2 Application

In the application, we use the data from the 5th wave of the World Values Survey 1981-2008, conducted in 2005-2008 over 56 countries.<sup>15</sup> We focus our empirical study on two questions:

Life satisfaction question:

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<sup>14</sup>The true values are  $I_\alpha^{(0)} = \frac{1}{\alpha(\alpha-1)} \left[ \sum_{i=1}^K \pi_i \left[ \sum_{j=1}^i \pi_j \right]^\alpha - 1 \right]$   $\alpha \neq 0$  and  $I_0^{(0)} = -\sum_{i=1}^K \log \left[ \sum_{j=1}^i \pi_j \right]$ .

<sup>15</sup>The bibliographic citation of the datafile is: WORLD VALUES SURVEY 1981-2008 OFFICIAL AGGREGATE v.20090901, 2009. World Values Survey Association (<http://www.worldvaluessurvey.org>). Aggregate File Producer: ASEP/JDS, Madrid. It can be downloaded at <http://www.worldvaluessurvey.org>.

*All things considered, how satisfied are you with your life as a whole these days? Using this card on which 1 means you are “completely dissatisfied” and 10 means you are “completely satisfied” where would you put your satisfaction with your life as a whole? (code one number):*

*Completely dissatisfied – 1 2 3 4 5 6 7 8 9 10 – Completely satisfied*

Health question:

*All in all, how would you describe your state of health these days? Would you say it is (read out): 1 Very good, 2 Good, 3 Fair, 4 Poor.*

The relationship between life satisfaction and income has been studied extensively in the literature and is a subject of some disagreement.<sup>16</sup> In a seminal paper, Easterlin (1974) shows that, for a given country, people with higher incomes are likely to report higher life satisfaction, whereas for cross-country comparisons and for higher income countries, the average level of life satisfaction does not vary much with higher income, it is known as the Easterlin or happiness-income paradox. Several studies confirm the lack of impact of income on life satisfaction for higher income countries,<sup>17</sup> while other studies find a significant positive impact.<sup>18</sup> To illustrate it, Figure 3 presents a cross-country comparison of the average of answers to the life satisfaction question ( $y$ -axis) and the GDP per capita<sup>19</sup> ( $x$ -axis). It is clear that the mean of life satisfaction is higher in countries with higher GDP per capita. However, the relationship is not linear and it is not obvious how to choose between a lognormal function and a piecewise linear regression model with the slope of the second line not significantly different from zero beyond \$15 000 per head.<sup>20</sup> Then, the presence or absence of income effect on life satisfaction among the higher income countries is not clear, and the controversy continues.

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<sup>16</sup>See Clark and Senik (2011) for a recent survey.

<sup>17</sup>See Easterlin (1995), Easterlin et al. (2010).

<sup>18</sup>See Hagerty and Veenhoven (2003), Deaton (2008), Stevenson and Wolfers (2008a), Inglehart et al. (2008).

<sup>19</sup>GDP per capita in 2005 is measured in purchasing parity power chained dollars at 2005 constant prices, and comes from the Penn World Tables 7.0.

<sup>20</sup>Layard (2003, p.23) noticed that “once a country has over\$15 000 per head, its level of happiness appears to be independent of its income per head”. Deaton (2008) argues that different results are obtained from different datasets.

Many empirical studies make comparisons of the average of answers to questions, or they use specific transformations to calculate a composite index of well-being.<sup>21</sup> In fact, they interpret the answers as cardinal with a linear scale, i.e., the values given to each successive answers are supposed to be equidistant. Such an interpretation is a matter of disagreement in the literature on Likert-type scale questionnaires. Some people consider that ordinal data provide information on ranks only and nothing else, so they should be treated as purely ordinal data with nonparametric statistics.<sup>22</sup> For example, a horse race result provides information on the rank order 1st, 2nd, 3rd, etc., without any information on the arrival time and differences in time between horses.<sup>23</sup> Others consider that ordinal data can be interpreted as cardinal, with a linear scale, in some contexts – the scale presents a suitable symmetry of items around a clear middle category, the intervals between points are approximately equal – so they can be analyzed parametrically.<sup>24</sup>

In our application, the main issue is whether we can assume that the values assigned in each questions can be treated as equidistant. In the health question – *Very Good, Good, Fair, Poor* – the items are unlikely to be equidistant: they are not symmetric (only one item can receive a below-average rating), and a bias would be introduced in favour of better outcomes. In the life satisfaction question, equidistant items implies that the distance between ‘2’ and ‘3’ is similar to the distance between ‘5’ and ‘6’ or any other two successive items. This position is quite common.<sup>25</sup> For our particular purpose, the relationship between the average of life satisfaction and the GDP per capita in a cross-country comparison, we can illustrate how sensitive the results are to this hypothesis. Indeed, if we assume that life satisfaction is highly correlated to personal incomes and that the distribution of incomes is lognormal, it makes sense to consider the answers on an exponential scale rather than on a linear scale.<sup>26</sup> Figure 4 presents a cross-country comparison

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<sup>21</sup>For instance, Deaton (2008) uses the average of life satisfaction, and Inglehart et al. (2008) use the index of subjective well-being: SWB=life satisfaction - 2.5 × happiness.

<sup>22</sup>See Kuzon, Urbanchek, and McCabe (1996), Jamieson (2004).

<sup>23</sup>Another example is the IQ score, which can be used for rank comparisons, but not as interval data for many authors: the difference between an IQ of 130 and one of 100 is not equivalent to the difference between an IQ of 100 and one of 70 (Mackintosh 1998, Bartholomew 2004).

<sup>24</sup>See Knapp (1990), Norman (2010) .

<sup>25</sup>See Ng (1997), Ferrer-i-Carbonell and Frijters (2004) and Kristoffersen (2011) for a discussion of this point.

<sup>26</sup>Here, we assume that everybody use the same scale, which is also questionable.

of the average of the exponential transformation of the code numbers from the life satisfaction question ( $y$ -axis) and the GDP per capita ( $x$ -axis). There is no clear relationship between the average of life satisfaction and national income. Column ( $i$ ) in Table 6 shows OLS regression results of the average of life satisfaction (on the exponential scale) on GDP per capita. The slope coefficient is not significantly different from zero and, thus, we conclude that there is no significant relationship between life satisfaction and GDP per capita. Compared to the conclusions drawn from Figure 3 and based on a linear scale, we obtain very different results. Clearly, the results are sensitive to the cardinal interpretation of the answers.

A simple solution would be to replace the mean by the median in the empirical analysis. However, the median is not well-defined with ordinal data, in particular when there are a few categories (see the discussion in section 4.2 and footnote 12). Another solution would be to use the fraction of people with a score higher than  $c$ . However, in our example, different choices of  $c$  produce different results: the relationship between the fraction of people with an answer higher than  $c = 5$  and the GDP per capita is significantly positive from an OLS regression, while it is not significant with  $c = 8$  (results not reported). Hereafter, we use the index developed in this paper, which avoids all such problems.

The inequality index developed in this paper is defined on individual ranks alone and is thus insensitive to any cardinal interpretation of the answers. Whether an upward-looking or downward-looking status concept is appropriate will depend on the context. If we require inequality to fall if answers are skewed toward more desirable categories, then it is clear that, for the life satisfaction question, where the categories are (*Completely dissatisfied* - 1, 2, ..., 10 - *Completely satisfied*) an upward-looking version is required. For the health question the lowest category is assigned to the “very good” answer; a downward-looking version ensure that inequality decreases when people tend to report better health states.

Figure 5 presents a cross-country comparison of the inequality index of life satisfaction and the GDP per capita. The inequality index,  $I_\alpha$ , is computed with  $\alpha = 0$ , as defined in (31). A negative relationship seems to appear, but it is not so clear. An OLS regression of the inequality index of life satisfaction,  $I_0(\text{life satisfaction})$ , on the GDP per capita shows that the slope coefficient is significantly different from zero at 1% (see Table 6, column ( $ii$ )). However, the  $p$ -value is very close to 0.01 and the  $R^2 = 0.12$  is small, it suggests that GDP per capita is not strongly related to the inequality of

	(i)	(ii)	(iii)	(iv)
constant	4079 ( <i>&lt;0.001</i> )	0.7358 ( <i>&lt;0.001</i> )	0.5245 ( <i>&lt;0.001</i> )	0.3450 ( <i>&lt;0.001</i> )
GDP per capita	0.0225 ( <i>0.292</i> )	-1.53e-06 ( <i>0.008</i> )	-1.03e-06 ( <i>0.014</i> )	-
$I_0$ (life satisfaction)	-	-	-	0.2274 ( <i>0.019</i> )
$R^2$	0.02	0.12	0.11	0.10

Table 6: Cross-country regressions, with  $p$ -values in parenthesis. Column (i) shows the OLS estimation of the average of life satisfaction (on the exponential scale) on the GDP per capita ; column (ii) the estimation of the inequality of life satisfaction,  $I_0$ (life satisfaction), on the GDP per capita ; column (iii) the estimation of the inequality of health,  $I_0$ (health), on the GDP per capita ; and column (iv) the estimation of  $I_0$ (health) on  $I_0$ (life satisfaction).

life satisfaction. In other words, people in high income countries are not necessarily those who tend to report higher life satisfaction. Indeed, we can see from Figure 5 that the two countries with the lowest inequality index, Colombia and Mexico, have very small values of GDP per capita. When we look at the detailed data, presented in Table 7, we can see that more than 73.9% of the individuals in Colombia and more than 77.1% of the individuals in Mexico, report values higher or equal to ‘8’ in the original life satisfaction question. The Figure suggests that, depending on the countries included in the dataset, we could have more or less significant results on the happiness-income relationship. Nevertheless, it is clear from those results that the happiness-income relationship is weak in cross-country comparisons.<sup>27</sup>

We now turn to the health question, detailed in Table 8.<sup>28</sup> When we look at the relationship between the fraction of people *satisfied* with their health

<sup>27</sup>We obtain similar results with different values of  $\alpha$ , the  $R^2$  is slightly higher with  $\alpha = 0.5$  and slightly lower with  $\alpha = -0.5, -1$ .

<sup>28</sup>Over all the countries, Guatemala, Turkey, Uruguay and South Africa report a few values equal to 5, associated to a “Very poor” answer. From the *Documentation of the Values Surveys*, for the other countries, the “Very poor” is not a possible answer in the questionnaire. We thus remove those four countries from the database.

– answering “Very good” or “Good” – and the GDP per capita, we find a significant positive relationship, as obtained in Deaton (2008). But, when we look at the fraction of people *very satisfied* with their health – answering “Very good” only – the relationship is not significant from an OLS estimation. Once again, the results appear to be sensitive to the fraction of people used (with a code number less than ‘c’=2 or ‘c’=1). It may be explained by some countries, which behave very differently when we consider *satisfied* and *very satisfied* people. For instance, Hong-Kong and Rwanda have, respectively, 63.5% and 33.7% of *satisfied* people, but only 5.6% and 2.5% of *very satisfied* people (see Table 8). The index developed in this paper takes into account the cumulative mass of individuals to each answers; for downward-looking status is defined to decrease when the distribution of answers gets closer to the case where everybody give the answer associated with the lowest value (“Very good” for the health question). In the following, we use this index to study health-income and health-life satisfaction relationships.

Figure 6 presents a cross-country comparison of the inequality index of health and the GDP per capita. The inequality index,  $I_\alpha$ , is computed with  $\alpha = 0$ , as defined in (31). There is no clear relationship, and the OLS estimation of  $I_0(\text{health})$  on GDP per capita produces a slope coefficient not significantly different from zero at 1%, and a  $R^2 = 0.11$  (see Table 6, column (iii)). These results suggest that the health-income relationship is not significant.<sup>29</sup> In other words, people in higher income countries do not tend to report higher health satisfaction.

Figure 7 presents a cross-country comparison of the inequality index of health and the inequality of life satisfaction. Once again, there is no clear relationship, and the OLS estimation of  $I_0(\text{health})$  on  $I_0(\text{life satisfaction})$  produces a slope coefficient not significantly different from zero at 1%, and a  $R^2 = 0.10$  (see Table 6, column (iv)). These results suggest that the health-life satisfaction relationship is not significant, that is, countries where people tend to report higher life satisfaction are not necessarily those where people tend to report higher health satisfaction.

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<sup>29</sup>We obtain similar results with  $\alpha = -1, -0.5, 0.5$ .



## 6 Conclusion

We can provide a precise answer to the problem of measuring inequality of the distribution of ordinal data interpreted as categorical variables.

There are three basic ingredients: the concept of status within a distribution, a reference point and a set of axioms. Status can be downward- or upward-looking depending on the context of the analysis. The axiomatisation is general and provides an approach for a variety of data types including the case of categorical data – it characterises a family of indices that is conditional on a sensitivity parameter and a reference point. The specific class of inequality measures that emerges from the axiomatisation is related to the Generalised Entropy and Atkinson classes; but, by contrast to conventional inequality analysis, the reference point for categorical data is not the mean of the distribution but the maximum possible value of status.

The approach is straightforward to implement empirically. As we have shown in section 5.2, how you treat categorical data in empirical studies really matters: if we were to treat ordinal variables as though they were cardinal the cardinalisation that is imposed affects conclusions about whether there is a relationship between inequality and income.

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	Life satisfaction question										Inequality	
	1	2	3	4	5	6	7	8	9	10	$I_0$	s.e.
Argentina	14	9	19	15	75	53	197	279	131	203	0.658	0.00077
Australia	17	18	29	40	109	117	303	472	195	110	0.689	0.00065
Burkina Faso	55	71	103	186	399	233	171	126	48	107	0.775	0.00049
Bulgaria	57	69	115	116	184	109	147	81	49	30	0.804	0.00032
Brazil	29	15	21	34	162	130	185	352	201	366	0.653	0.00094
Canada	12	17	25	52	123	158	361	726	348	335	0.663	0.00064
Switzerland	8	7	12	16	64	78	170	445	241	191	0.647	0.00061
Chile	11	3	24	56	120	125	161	197	130	165	0.710	0.00069
China	72	72	86	85	222	278	244	427	198	275	0.731	0.00068
Colombia	44	20	20	40	172	157	332	601	485	1147	0.553	0.00123
Cyprus	28	8	16	41	79	108	187	284	157	142	0.701	0.00059
Egypt	305	122	234	251	510	311	439	370	162	346	0.773	0.00061
Spain	4	2	16	25	84	166	324	335	170	69	0.696	0.00052
Ethiopia	50	108	249	200	255	281	197	86	41	23	0.793	0.00044
Finland	7	18	19	20	40	34	137	347	290	102	0.639	0.00067
France	15	9	41	43	135	106	219	263	102	67	0.726	0.00057
United Kingdom	9	3	11	20	73	100	208	346	157	111	0.681	0.00058
Georgia	128	92	154	204	377	160	161	111	25	64	0.790	0.00047
Germany	24	27	86	112	262	208	370	530	290	141	0.729	0.00053
Ghana	97	84	117	157	114	181	221	273	120	164	0.767	0.00059
Guatemala	9	12	26	21	69	82	122	176	171	311	0.610	0.00112
Hong Kong	23	22	49	77	191	272	236	222	85	67	0.755	0.00042
Indonesia	58	22	47	64	296	256	358	372	159	274	0.723	0.00064
India	71	1	208	0	821	0	437	0	0	316	0.605	0.00049
Iran	101	103	117	180	453	364	385	396	229	339	0.754	0.00061
Iraq	465	221	232	318	616	310	207	149	63	89	0.804	0.00042
Italy	12	10	16	47	93	182	280	216	84	66	0.724	0.00050
Jordan	99	22	22	37	148	73	131	180	121	364	0.634	0.00121
Japan	10	12	41	38	92	171	219	301	146	50	0.715	0.00061
Korea	30	22	47	88	172	191	273	246	83	45	0.748	0.00057
Morocco	27	31	115	195	373	206	130	58	22	40	0.761	0.00057
Moldova	46	69	99	150	172	151	131	127	66	30	0.800	0.00030
Mexico	39	10	15	18	86	60	117	380	256	531	0.565	0.00113
Mali	113	31	79	95	328	154	184	148	96	202	0.749	0.00072
Malaysia	17	9	28	41	145	230	287	265	92	86	0.729	0.00049
Netherlands	2	5	5	16	40	82	235	406	156	102	0.653	0.00065
Norway	7	3	7	12	47	51	142	387	245	122	0.638	0.00058
New Zealand	7	8	16	20	50	67	149	224	182	204	0.651	0.00078
Peru	29	24	59	72	194	165	278	280	111	278	0.703	0.00082
Poland	14	15	34	34	152	117	149	238	112	124	0.723	0.00058
Romania	93	82	163	143	283	179	247	285	107	76	0.785	0.00045
Russia	117	74	104	151	362	217	326	342	146	177	0.768	0.00049
Rwanda	91	146	136	166	365	299	122	96	53	29	0.790	0.00049
Serbia	30	53	84	97	177	170	262	210	54	38	0.760	0.00062
Slovenia	10	6	19	25	167	122	173	240	104	167	0.702	0.00068
Sweden	5	5	16	22	47	61	188	367	177	114	0.661	0.00060
Thailand	12	11	36	53	137	217	307	429	172	158	0.710	0.00056
Trinidad&Tobago	35	7	14	29	120	124	176	195	80	219	0.679	0.00091
Turkey	41	18	46	36	77	134	220	291	211	272	0.681	0.00080
Taiwan	38	16	34	54	190	207	217	280	74	117	0.734	0.00059
Ukraine	45	51	97	81	147	147	170	149	66	43	0.787	0.00038
Uruguay	19	2	12	19	76	125	218	245	115	162	0.689	0.00065
United States	9	11	31	54	91	117	275	358	224	71	0.698	0.00061
Vietnam	6	10	41	61	163	287	278	292	138	206	0.719	0.00059
South Africa	96	75	97	141	257	298	419	588	435	571	0.699	0.00081
Zambia	107	52	89	114	204	189	249	229	99	131	0.773	0.00050

Table 7: Number of people per code number in the life satisfaction question (source: World Values Surveys, 5th wave) and the inequality index with its standard error (upward looking status).

	Health question				Inequality	
	1	2	3	4	$I_0$	s.e.
Argentina	326	463	185	27	0.480	0.00069
Australia	402	675	274	61	0.496	0.00068
Burkina Faso	427	670	350	72	0.511	0.00065
Bulgaria	149	389	310	150	0.575	0.00062
Brazil	408	700	355	36	0.501	0.00061
Canada	857	880	341	81	0.461	0.00086
Switzerland	400	635	173	33	0.462	0.00068
Chile	196	475	277	52	0.524	0.00067
China	619	617	486	290	0.550	0.00080
Colombia	674	1517	760	72	0.502	0.00065
Cyprus	413	387	184	65	0.478	0.00092
Egypt	471	1578	766	236	0.515	0.00090
Spain	237	716	195	48	0.463	0.00091
Ethiopia	386	582	390	139	0.544	0.00063
Finland	232	437	284	61	0.534	0.00059
France	300	418	225	58	0.513	0.00069
United Kingdom	345	412	196	87	0.508	0.00080
Georgia	189	439	574	296	0.600	0.00042
Germany	458	908	538	148	0.534	0.00063
Ghana	571	661	208	93	0.471	0.00083
Hong Kong	68	723	407	48	0.435	0.00139
Indonesia	386	1144	408	63	0.477	0.00086
India	438	934	489	136	0.526	0.00068
Iran	499	1294	680	156	0.520	0.00074
Iraq	476	1285	592	317	0.536	0.00081
Italy	188	562	228	34	0.487	0.00084
Jordan	524	518	130	27	0.425	0.00086
Japan	148	440	405	95	0.554	0.00063
Korea (Republic of)	175	809	208	8	0.416	0.00115
Morocco	461	434	262	43	0.482	0.00086
Moldova	107	443	354	117	0.547	0.00086
Mexico	344	654	503	54	0.532	0.00049
Mali	437	510	467	93	0.536	0.00064
Malaysia	332	739	125	5	0.426	0.00066
Netherlands	247	497	262	42	0.513	0.00063
Norway	426	388	165	46	0.460	0.00093
New Zealand	348	427	154	20	0.462	0.00074
Peru	112	606	723	59	0.511	0.00085
Poland	175	365	331	127	0.575	0.00051
Romania	175	776	569	254	0.551	0.00092
Russia	134	721	911	256	0.548	0.00086
Rwanda	37	469	740	255	0.523	0.00103
Serbia	226	431	406	147	0.574	0.00049
Slovenia	220	369	302	144	0.574	0.00054
Sweden	353	430	187	33	0.480	0.00075
Thailand	331	834	295	66	0.488	0.00079
Trinidad&Tobago	321	375	248	56	0.515	0.00073
Taiwan	280	772	117	58	0.439	0.00086
Ukraine	76	412	378	132	0.545	0.00094
United States	350	645	206	47	0.480	0.00069
Vietnam	155	736	473	131	0.519	0.00101
Zambia	415	618	282	128	0.520	0.00072

Table 8: Number of people per code number in the health question (*source*: World Values Surveys, 5th wave) and the inequality index (downward-looking status) with its standard error.

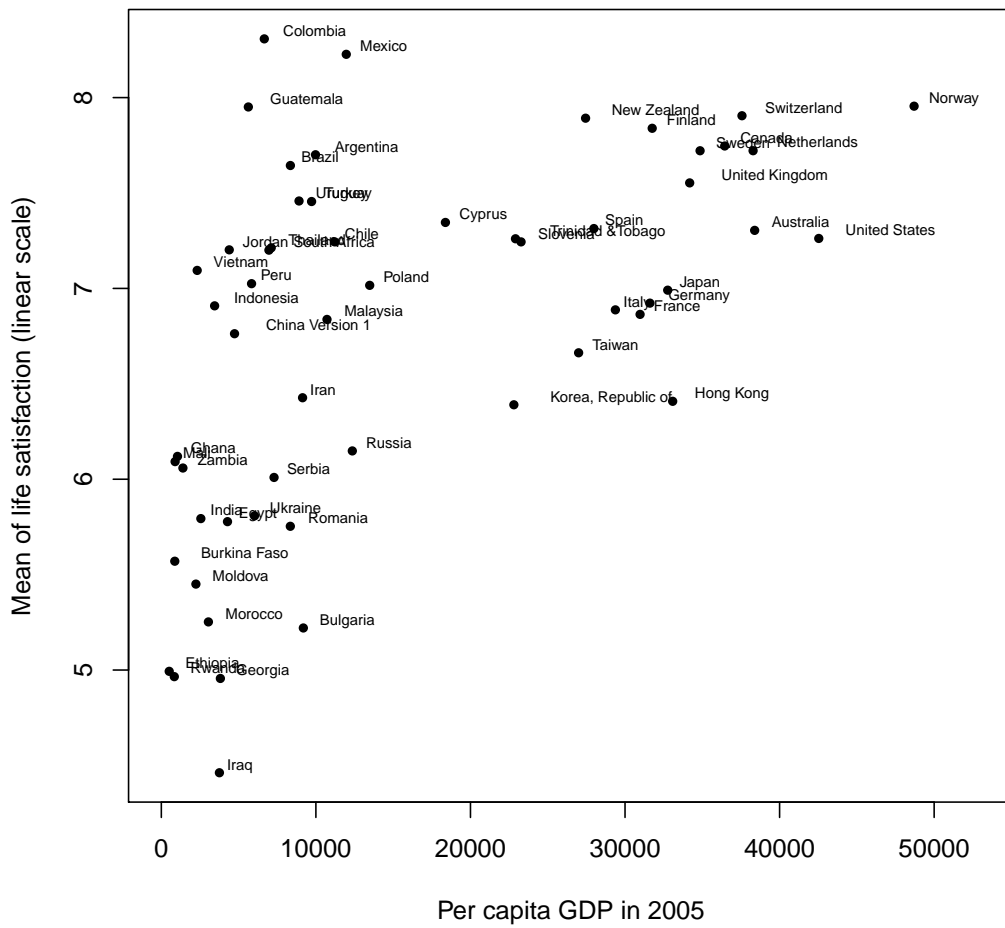


Figure 3: Cross-country comparison of average life satisfaction and GDP per capita.



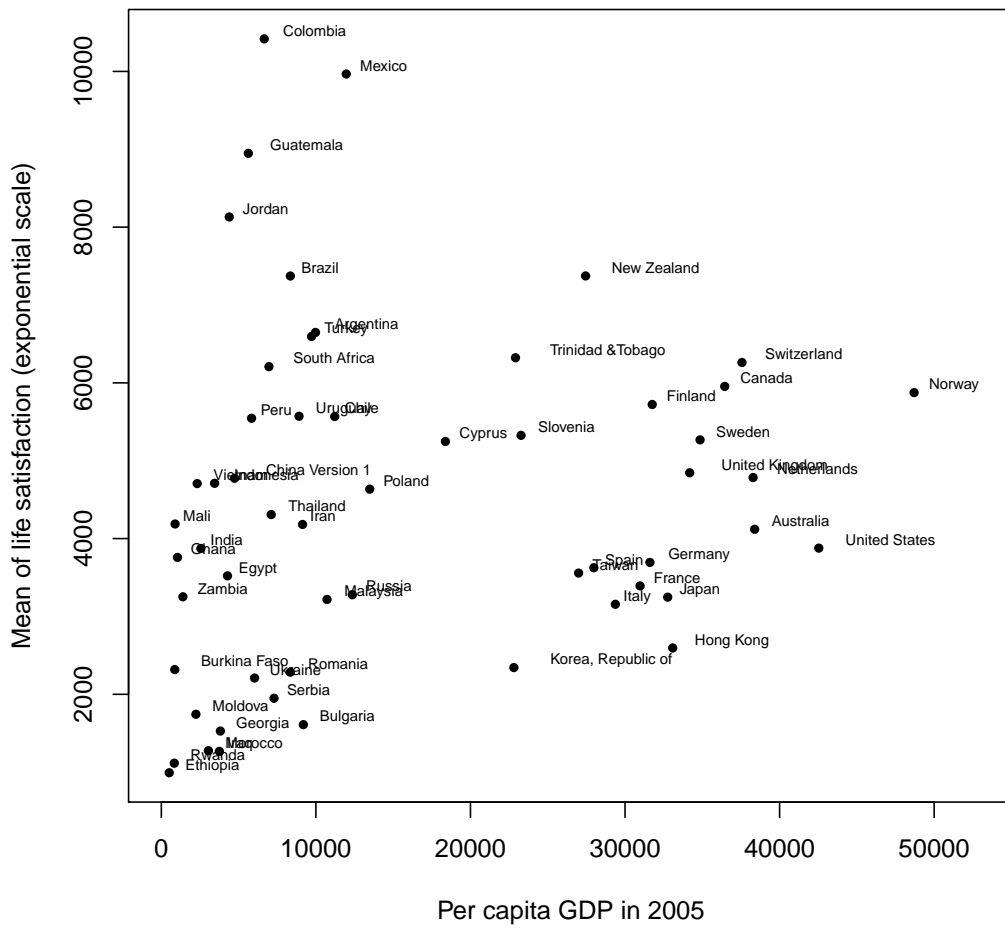


Figure 4: Cross-country comparison of average life satisfaction and GDP per capita, using an exponential scale for values assigned to the answers

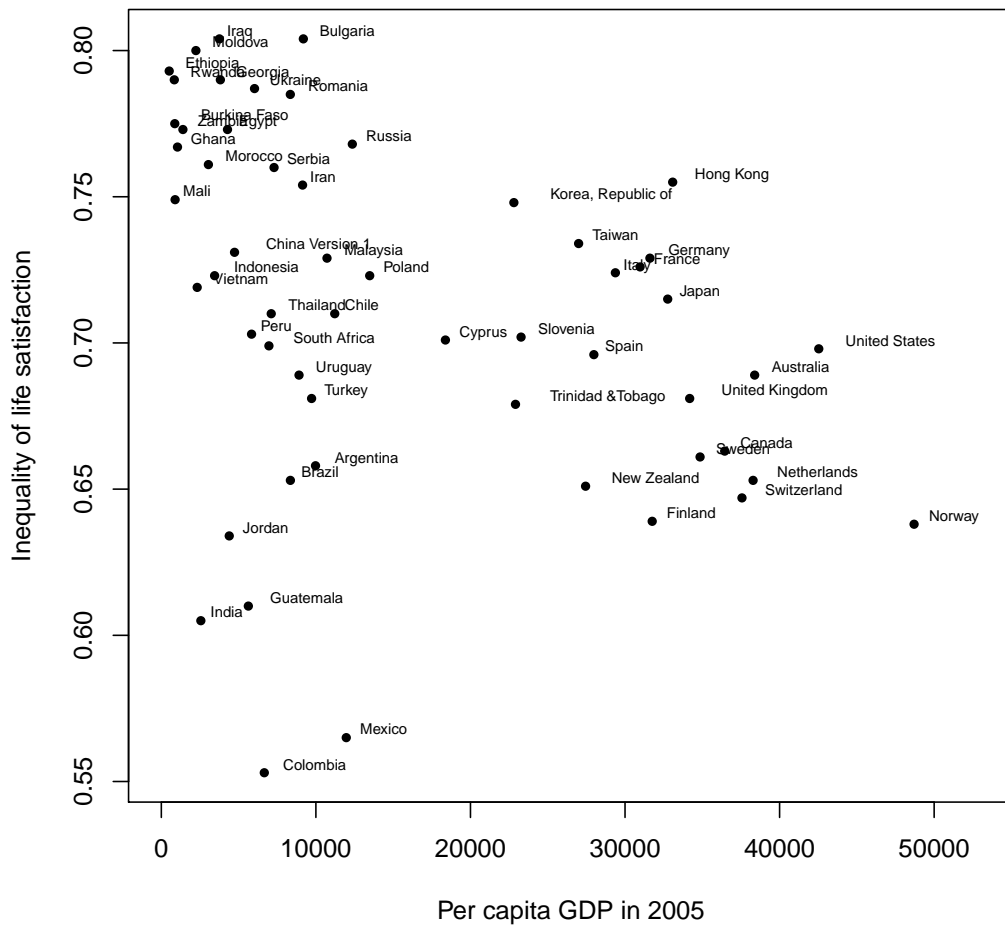


Figure 5: Cross-country comparison of inequality of life satisfaction and GDP per capita

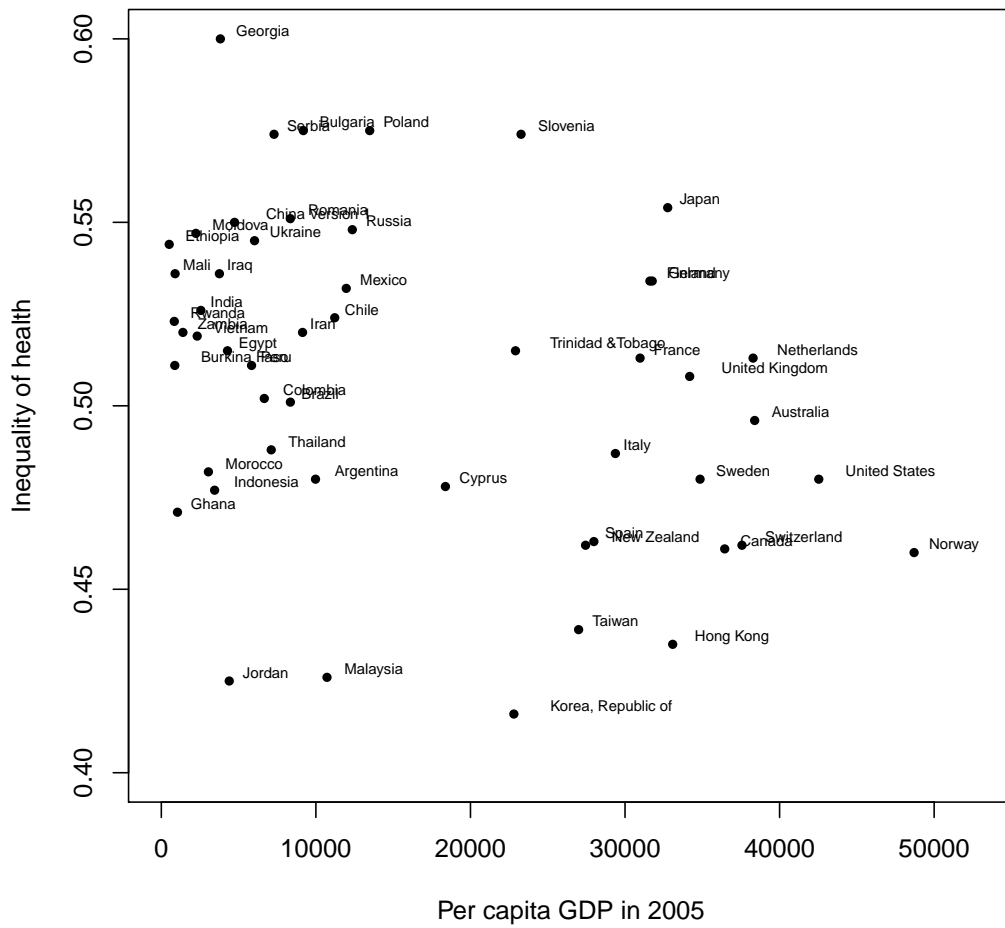


Figure 6: Cross-country comparison of inequality of health and GDP per capita

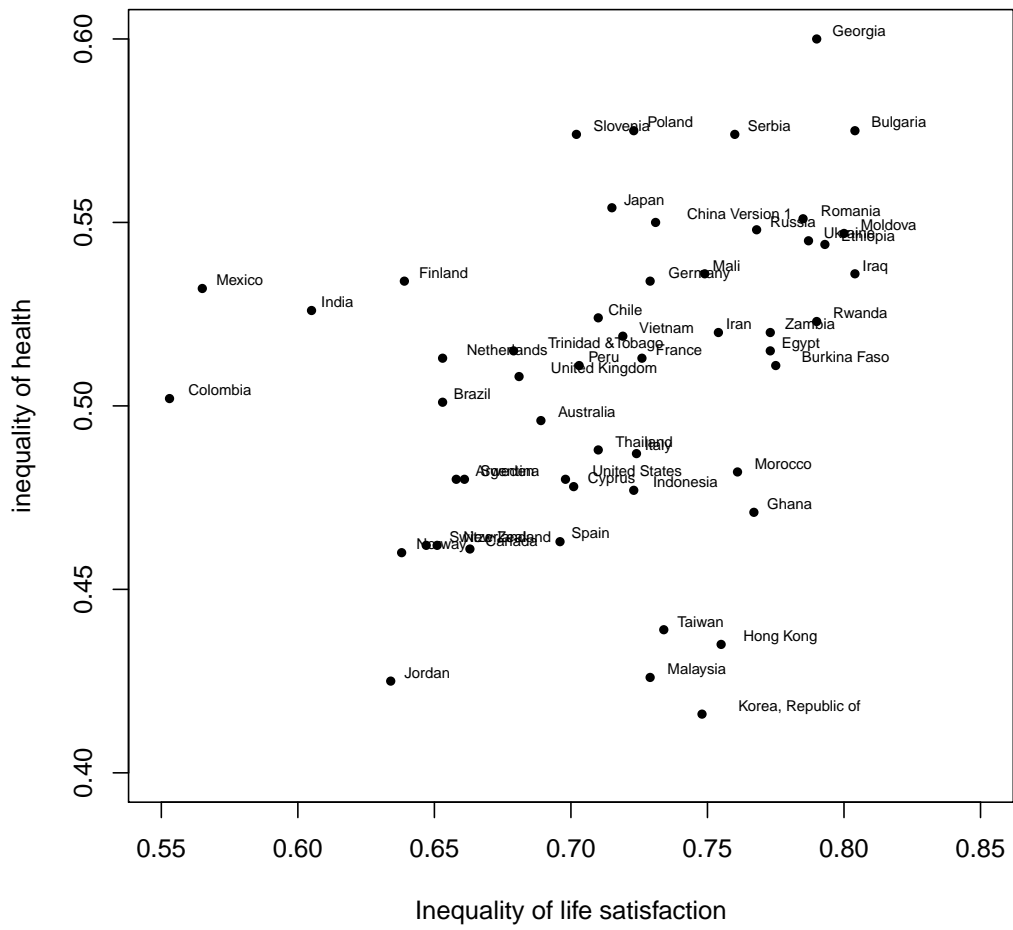


Figure 7: Cross-country comparison of inequality of health and inequality of life satisfaction