

TURNOUT AND POWER SHARING

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ABSTRACT. Differences in electoral rules and/or legislative, executive or legal institutions across countries induce different mappings from election outcomes to distributions of power. We explore how these different mappings affect voters' participation in a democracy. Assuming heterogeneity in the cost of voting, the effect of such institutional differences on turnout depends on the distribution of voters' preferences for the parties: when the two parties have similar support, turnout is higher in a winner-take-all system than in a power sharing system; the result is reversed when one side has a larger base. The results are robust to a wide range of modeling approaches, including the instrumental voting model, ethical voter models, and voter mobilization models. Findings from laboratory experiments provide empirical support for most of the theoretical predictions.

Keywords: Voter turnout; Proportional representation; Electoral systems

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1. INTRODUCTION

Voter participation is an essential component of democracy, and changes in the level of electoral participation may affect the political positioning of the competing parties, electoral outcomes and ultimately public policy. At the same time, the level of electoral participation, electoral outcomes, political parties and other aspects of the political landscape are all endogenous and widely believed to be consequences of the electoral rules. A key property of electoral systems is the degree of proportionality in translating votes to seats, and an important literature has developed in comparative politics that studies the empirical relationship between proportionality on voter turnout.¹ Many conjectures have been offered about whether or not more proportional systems lead to higher turnout and why. Heuristic arguments have been offered on both sides. Many empirical studies have looked at the relationship, with mixed results. While the overall picture is not as blurry as it was twenty five years ago, it is still out of focus.

This article develops a formal approach in order to provide some essential foundations for the study of the complex relationship between proportionality and turnout. Our approach is two-pronged. First, we provide a theoretical analysis of the fundamental causes of the variation in turnout based on differences in institutions for political power sharing, or proportionality. Second, we report results from a laboratory experiment designed to test the key hypotheses that emerge from the theory about the relationship between turnout and proportionality. The results of the experiment starkly show how the theoretical forces in the model play out largely as predicted in the laboratory elections. The experiment, while obviously a stylized version of elections in the field, enjoys the equally obvious advantage of avoiding confounding factors that have challenged empirical studies.²

¹The comparison is usually stated in terms of differences between SMP (single-member plurality) systems and PR (proportional representation). However, there are different incarnations of PR that imply varying degrees of proportionality in the translation of votes to seats, as well as different constitutional arrangements at the legislative and executive levels that lead to variations in how seats translate into political power. As in Lizzeri and Persico (2001), the approach in this paper is to consider both parts of the equation. Hence we use the terms "power sharing" and "proportionality" interchangeably.

²These include the measurement of competitiveness, properly controlling for social/cultural factors, endogeneity of the choice of electoral system, isolating the effects of district magnitude or multimember districts, and taking into account institutional variations in government formation, to name a few. Acemoglu (2005) argues that cultural and idiosyncratic characteristics that are difficult to control for make it difficult to assess the causal effects of institutional differences, since institutions are themselves endogenous. Blais (2006) in his

In order to develop a theory as robust and general as possible, our first decision has been to consider all possible determinants of the degree of proportionality in a reduced form, considering as equally important all the different institutional systems impacting the mapping from vote shares to the relative weight of different parties in policy making (henceforth *power shares*).³ We introduce a power sharing, or proportionality, parameter, γ , that allows us to embed a wide array of electoral systems ranging from a fully proportional power sharing system ($\gamma = 1$) to a pure winner-take-all system ($\gamma = \infty$). Hence we try to assess in a general way the role of these institutions on electoral participation by characterizing how that vote-shares-to-power-shares mapping affects voters' incentives to vote and parties' campaign efforts. Second, and most importantly, in order for our theory to be robust we allow for multiple alternative behavioral assumptions about the turnout mechanics, rather than limiting our analysis to one single approach such as rational instrumental voting or mobilization.

The theoretical results we obtain from all models, from instrumental voting to mobilization models, depend on a key variable, namely the expected winning margin or “closeness” of the election. We are able to identify theoretically how proportionality and turnout interact with the closeness variable, and find that these interaction effects are quite subtle. Before explaining how the vote-share to power-share mapping and the expected closeness of an election jointly determine turnout, we first highlight the basic modeling approach taken in the paper. In all models, turnout is costly, be it individual voting costs or mobilization costs. We take as the baseline model in our analysis the standard rational voter model (see e.g. Ledyard (1984) and Palfrey and Rosenthal (1985)) under population uncertainty⁴, extending the analysis to turnout survey concludes that “many of the findings in the comparative cross-national research are not robust, and when they are, we do not have a compelling microfoundation account of the relationship.”

³The relative power of the majority party for a given election outcome varies with the degree of separation of powers, the organization of chambers, the assignment of committee chairmanships and institutional rules on agenda setting, allocation of veto powers, and obviously electoral rules. Electoral rules determine the mapping from vote shares to seat shares in a legislature, whereas the other institutions determine the subsequent mapping from seat shares to power shares across parties.

⁴Viewing the size of the electorate as a random variable (see Myerson (1998) and (2000)) has the advantage of simplifying the computations without altering the incentives driving the results. Krishna and Morgan (2011) obtain important results in a model similar to ours, but with common values, in which population uncertainty is key, see also McMurray (2012). For multiple candidate elections results obtained with Poisson games see Bouton (2012), Bouton & Castanheira (2012). In our setting population uncertainty serves only the purpose of allowing us to obtain analytical results, but numerical computations we performed with fixed population sizes confirm that all our comparative results do not depend on population uncertainty.

the proportional influence or proportional power sharing system. We also analyze the same comparative questions using other prominent approaches from the turnout theory literature: mobilization and ethical voting,⁵ both of which share with the baseline model not only the fact that each voter has a cost of voting, but also the fact that each voter has a preference for one of the alternatives (candidates or parties or coalitions of parties), so that the only relevant decision by each citizen is whether or not to vote.⁶ The robustness of the theoretical comparative findings is then put to a test in a large study of 1700 laboratory elections, where we experimentally control and manipulate the proportionality parameter, voting costs and the competitiveness of the election.

The key to understanding the interaction between proportionality, turnout, and closeness is what we refer to as the *underdog compensation effect*. Underdog compensation refers to the phenomenon whereby smaller parties turnout in higher proportions than larger parties. This is due in large part to the greater free rider problem that must be overcome in large parties.⁷ The robust theoretical result within the large class of models that we consider is that any model of large elections featuring either a partial or zero underdog compensation effect yields the prediction that a winner take all system induces higher turnout in competitive races. Proportional systems induce higher turnout than winner take all systems in less competitive races. Thus, there is a crossing point. So the answer to the big question of whether PR or SMP produces higher turnout is complicated: it can go either way, and depends on the ex ante competitiveness of the election. The intuition is that in the winner-take-all system when preferences are not evenly split the non full underdog compensation preserves the ex-ante leading party as the ex-post leading party in equilibrium, hence preserving a high expected winning margin, which discourages participation. In a more proportional power sharing system, a less competitive election (i.e. a higher expected winning margin) does not affect the incentives to vote as much.

⁵See Morton (1987, 1991), Uhlaner (1989), and Shachar and Nalebuff (1999) for the mobilization approach). See Coate and Conlin (2004) and Feddersen and Sandroni (2006) for the ethical voting approach.

⁶Even though elite mobilization models and pivotal voting models put different emphasis on the voter (passive versus active, etc.), closeness affects turnout both at the level of masses and at that of elites. What we find is that these effects are differentiated in the same way across electoral rules, so that masses and elites can be thought of as reacting very similarly to electoral rule reforms.

⁷The underdog compensation phenomenon has an important implication for empirical research. It implies that using margin of victory (or any other ex post measure of closeness) as a measure of *ex ante* competitiveness of an election will produce a biased measure of that critical variable.

It is therefore crucial to explain the important role of the *partial underdog compensation effect* mentioned above, because without such a common feature, the instrumental rational voting model could not possibly yield comparative predictions in line with the mobilization and ethical voting models. Several recent theoretical papers⁸ have identified and used a neutrality result for equilibrium turnout models of winner-take-all elections, which we call the *full underdog compensation effect*. This theoretical claim is that in pivotal voting models the expected vote shares of the two parties are *equal* independent of the distribution of partisan preferences in the population. Our paper shows that this result is a fragile finding, based on special and empirically suspect technical assumptions about the distribution of costs in those models. The property of full underdog compensation identified in earlier papers is due to either an assumption that the distribution of voting costs is degenerate (Goeree and Grosser (2007), Taylor and Yildirim (2010)), or is bounded below by a strictly positive minimum voting cost (Krasa and Polborn (2009)).⁹ It is, in fact, not a general property of pivotal voting models: if the two parties have supporters with heterogeneous costs of voting with a distribution of voting costs with lower bound of support less than or equal to zero, then the underdog compensation effect is always partial or zero. The intuition for this is easy to see in the extreme case where some fraction of voters have zero or negative voting costs. In this case, for large elections they are the only ones who vote, and there is no free rider problem among these voters because they get direct utility (or zero cost) from voting. Hence there is zero underdog compensation in large elections. In the intermediate case that occurs in our model, the underdog party supporters turn out in higher percentages than the supporters of the favorite to win. Hence, the party with higher ex-ante support is always expected to win, but by a smaller margin of victory than the ex-ante support advantage (e.g. the opinion polls) would predict, which we refer to as partial underdog compensation.

Especially in rational instrumental models of voting without any assumed coordination among voters, comparing turnout across systems boils down to comparing the individual benefits of voting across systems. In a proportional power sharing system the expected marginal benefit of a single vote is proportional to the marginal change in the vote share determined by that vote. Whereas in a winner take all system the marginal benefit of a vote is proportional

⁸See Goeree and Grosser (2007), Taylor and Yildirim (2010), Krasa and Polborn (2009).

⁹In contrast, the original Palfrey-Rosenthal (1985) model rules out the first two of these special cases, and explicitly allows for different distributions of voting costs for the two parties. Taylor and Yildirim (2010) show that the neutrality result generally fails if the lower bound of two parties' voting costs is different.

to the probability of that vote being pivotal. Both marginal benefits obviously decrease as the number of voters increases. In large elections the comparison of turnout across systems hence depends on the asymptotic speed with which a larger population reduces the individual benefit of voting, i.e. the magnitude of the “size effect.”¹⁰ Quantitatively we show that in a proportional system the benefit of voting decreases asymptotically according to an intermediate power law when N , the expected size of the electorate, increases; whereas in a winner-take-all system such asymptotic speed is slower when the election is expected to be a tie and much faster otherwise. This fact determines the main conclusion, namely that turnout is higher in a proportional system when the election has a clear favorite party while a winner-take-all system induces higher turnout otherwise.¹¹ In sum, consider the three magnitudes, turnout in a close majority election, in a non-close majority election and in a proportional election: we show that the first magnitude is larger than the second, which is expected and intuitive;¹² but, more importantly, we also show that the third magnitude lies exactly between the first two. The latter is a novel quantitative result: ex-ante, without an explicit model and computation, it was not at all clear what to expect. This result is robust across all the costly voting models that we consider.

As we show in the extensions section, the qualitative results we obtain are also robust to various power sharing regimes. We use the ‘contest success function’ (see Tullock (1980)) to span all power sharing regimes. This is a methodological innovation. This function is extensively used in several economic contexts, especially in the contest literature (see among many others Skaperdas (2006)) typically as a mapping from efforts or resources to the chance of victory. But to our knowledge this modeling approach has not been used in the voting literature so

¹⁰As in Levine and Palfrey (2007) the “size effect” refers the equilibrium effect of overall turnout decreasing with the size of the electorate.

¹¹We conduct the bulk of the analysis for the case of two parties, but show in the appendix the robustness of all comparisons to changes in the number of parties: in a proportional power sharing system the order of magnitude of the size effect does not depend on the distribution of ex ante support of parties, nor does it depend on the number of parties present in the election. We also show that in a proportional power sharing system turnout increases as the number of parties increases.

¹²For the relationship between closeness and turnout in a majoritarian election one could go back to Downs (1957) and Riker and Ordeshook (1968). Their premise is that citizens will participate in elections if and only if the expected benefits of voting exceed the costs, implying that closeness and turnout will be positively correlated because higher closeness implies higher instrumental value of voting.

far, which is surprising because the contest function captures nicely the mapping between vote shares and power shares in power sharing systems.

If one moves away from the convenient population uncertainty world (convenient in terms of tractability) it is straightforward to derive *exact* equilibrium conditions to be used for numerical computation of the predictions for any known number of voters. Hence it is possible to test the comparative results in the laboratory. Since Levine and Palfrey (2007) already provide a preliminary set of data with winner-take-all rules, we adopted the same treatments even in the new proportional experiments, so that the data could be pooled together. The experimental results confirm the theoretical predictions of the general model, as well as other predictions on the closeness effects that came out specifically from the known-population model computations. Experimental evidence (see Schram and Sonnemans (1996)) suggests turnout is higher in a majoritarian system than in proportional representation, but the experimental design featured only the case of perfect symmetry in the ex-ante supports for the two parties. They find higher turnout in the winner take all elections than in PR, which is consistent with our theoretical results. However, the theory also predicts that the turnout ordering will be reversed if the parties' ex-ante supports are sufficiently asymmetric.¹³

The paper is organized as follows. Section 2 contains the complete analysis of a rational voter model of turnout, comparing the properties of proportional power sharing system and winner-take-all system. Section 3 contains a number of important extensions and robustness results, including the very important study of our comparative question using mobilization and ethical voting models, clustered together as group-based models. Section 4 contains the experimental analysis, where one can see that the rational costly voting model actually performs very well in terms of comparative statics in small elections, and where one can see the general findings of the theory further confirmed. Section 5 offers some concluding remarks. All proofs of the model and of all its extensions are in the Appendix, as well as a sample of the instructions from one of the experimental sessions.

2. RATIONAL VOTER TURNOUT

Consider two parties, A and B, competing for power. Citizens have strict political preferences for one or the other, chosen exogenously by Nature. We denote by $q \in (0, 1)$ the preference split, i.e. the chance that any citizen is assigned (by Nature) a preference for party A (thus $1 - q$ is the expected fraction of citizens that prefer party B). Without loss of generality, we

¹³Related experimental findings can be found in Kartal (2012), a laboratory study developed independently.

assume that $q \leq 1/2$, so that the A party is the underdog party (with smaller ex-ante support) and the B party is the leader party (with larger ex-ante support). The indirect utility for a citizen of preference type i , $i = A, B$, is increasing in the share of power that party i has. For normalization purposes, we let the utility from “full power to party i ” equal 1 for type i citizens and 0 for the remaining citizens.¹⁴

Beside partisan preferences, the second dimension along which citizens differ from one another is their cost of voting: each citizen’s cost of voting c is drawn from a distribution with infinitely differentiable pdf $f(c)$ over the support $c \in [0, \bar{c}]$, with $\bar{c} > 0$ (we denote the cdf as $F(c)$).¹⁵ The cost of voting and the partisan preferences are two independent dimensions that determine the type of a voter.

For any vote share V obtained by party A, an institutional system γ determines power shares $P_\gamma^A(V) \in [0, 1]$ and $P_\gamma^B(V) = 1 - P_\gamma^A(V)$. Given the above normalization, these are the reduced form “benefit” components of parties’ (respectively, voters’) utility functions that will determine the incentives to campaign (respectively, vote) in an institutional system. In this section we study the base model in which parties do not campaign nor attempt to coordinate or mobilize voters, hence turnout depends exclusively on voters’ comparison between the policy benefits of voting for the preferred candidate and the opportunity costs of voting.

In terms of the size of the electorate, we find it convenient to assume that the population is finite but uncertain. There are n citizens who are able to vote at any given time, but such a number is uncertain and distributed as a Poisson distribution with mean N :

$$n \sim \frac{e^{-N} (N)^n}{n!}$$

Most analytical statements in the first part of the paper are made for a large enough population, namely they are true for every N above a given \bar{N} . However, we will easily establish very similar results for small elections via numerical computations.

Citizens have to choose to vote for party A, party B, or abstain. If a share α of A types vote for A and a share β of B types vote for B, the expected turnout T is

$$T = q\alpha + (1 - q)\beta$$

¹⁴This normalization will allow us to match party utility and voters’s utilities in a simple way under all the institutional systems that will be considered.

¹⁵One could allow for the support to include negative voting costs. This trivially implies a zero compensation effect, as explained in the introduction.

We look for a Bayesian equilibrium in which all voters of type A with a cost below a threshold c_α vote for type A and voters of type B with a cost below c_β vote for B. So on aggregate, type A citizens vote for A with chance $\alpha = F(c_\alpha)$ and type B citizens vote for B with chance $\beta = F(c_\beta)$.

In any equilibrium strategy profile (α, β) , the expected marginal benefit of voting, B_γ , must be equal to the cutoff cost of voting (indifference condition for the citizen with the highest cost among the equilibrium voters). Hence the equilibrium conditions can be written as

$$B_\gamma^A(\alpha, \beta) = F^{-1}(\alpha), \quad B_\gamma^B(\alpha, \beta) = F^{-1}(\beta)$$

We compare the above equilibrium conditions in two systems which differ on the benefit side: a winner-take-all system ($\gamma = M$) and a proportional power sharing system ($\gamma = P$).¹⁶

2.1. Winner take all system ($\gamma = M$). In the M system the expected marginal benefit of voting B_M^A is the chance of being pivotal for a type A citizen, namely

$$B_M^A = \sum_{k=0}^{\infty} \left(\frac{e^{-qN\alpha} (Nq\alpha)^k}{k!} \right) \left(\frac{e^{-(1-q)N\beta} ((1-q)N\beta)^k}{k!} \right) \frac{1}{2} \left(1 + \frac{(1-q)N\beta}{k+1} \right)$$

namely the chance that an A citizen by voting either makes a tie and wins the coin toss or breaks a tie where it would have lost the coin toss. Likewise, for the type B citizens we have

$$B_M^B = \sum_{k=0}^{\infty} \left(\frac{e^{-qN\alpha} (Nq\alpha)^k}{k!} \right) \left(\frac{e^{-(1-q)N\beta} ((1-q)N\beta)^k}{k!} \right) \frac{1}{2} \left(1 + \frac{qN\alpha}{k+1} \right)$$

Equating the benefit side to the cost side we obtain a system of two equations in (α, β) (the M system henceforth). We now show that asymptotically turnout for each party is zero as a percentage of the population, but is infinite in absolute numbers. Moreover, the ratio of turnouts for each party remains finite.

Lemma 1. *Any equilibrium solution (α_N, β_N) to the M system (if it exists) has the following three properties*

$$\lim_{N \rightarrow \infty} \alpha_N = \lim_{N \rightarrow \infty} \beta_N = 0, \quad \lim_{N \rightarrow \infty} N\alpha_N = \lim_{N \rightarrow \infty} N\beta_N = \infty, \quad \lim_{N \rightarrow \infty} \frac{\alpha_N}{\beta_N} \in (0, \infty)$$

¹⁶Recall that the interpretation is not restricted to electoral rules, as explained in the introduction. Two countries with the same electoral rule can have very different mappings from electoral outcomes to power shares, and this is the summary or reduced form variable that we are interested in and that affects turnout.

The above lemma allows us to use some approximations to show existence and uniqueness of an equilibrium for N large and also the following characterization results.¹⁷

Lemma 2. *There exists an equilibrium (α, β) in the M system. For uniqueness it suffices that F is weakly concave.¹⁸ The equilibrium has the following properties:*

- Size effect:

$$\frac{dT_M}{dN} < 0$$

- Partial underdog compensation effect:

$$q < 1/2 \implies \alpha > \beta, \quad q\alpha < (1 - q)\beta$$

The size effect shows how the benefit of voting declines for larger electorates, although we will show that the rate of decline depends crucially on whether the parties do or do not have the same support ex-ante. The partial underdog compensation shows that the party with less supporters has higher relative expected turnout but lower expected turnout overall. We discuss all these effects in the following section.

2.2. Discussion of the M System. The partial underdog compensation arises from the equilibrium relationship between the turnout rates for the two parties which for large electorates can be expressed simply as (see Appendix)

$$(1) \quad q\alpha (F^{-1}(\alpha))^2 = (1 - q)\beta (F^{-1}(\beta))^2$$

Why the above expression takes the specific quadratic form is a quantitative result, but in what follows we highlight important qualitative findings that derive from it. Since for heterogeneous costs $F^{-1}(\alpha)$ is increasing, then $q < 1/2$ implies an underdog compensation (i.e. $\alpha > \beta$) that must be partial (i.e. $q\alpha < (1 - q)\beta$). As a consequence, we have a balanced election with a 50% expectation of victory from each side *only* when $q = 1/2$. With homogeneous costs the result would be different: homogeneous costs mean that $F^{-1}(\alpha) = c = F^{-1}(\beta)$, which implies $q\alpha = (1 - q)\beta$, i.e. full underdog compensation and a 50% chance of victory regardless of the ex-ante preference split q .

To understand why the heterogeneity of the cost distribution is so important, assume for instance that $q = 1/3$ so that the leader party has double the ex-ante support than the underdog

¹⁷We thank John Morgan for pointing out the importance of proving this non-trivial lemma for the approximation results and proofs that will follow.

¹⁸Or alternatively $\alpha F^{-1}(\alpha)$ weakly convex, which is a less straightforward but weaker condition.

party. To have an election with a 50-50 chance of victory (i.e. $q\alpha = (1 - q)\beta$), the underdog party would have to turn out twice as much as the leader party. We claim that the latter cannot happen unless citizens have homogenous costs. Suppose not; then on the benefit side, in a strategy profile with an ex-ante even outcome, the gross benefit of voting is the same across all voters (as they all individually face the same even environment). On the cost side, since the underdog party has to turn out more, then we must have $\alpha = F(c_\alpha) > \beta = F(c_\beta)$. With heterogenous cost this means that the equilibrium cost thresholds would have to be different $c_\alpha > c_\beta$ which, in turn, implies that the cost thresholds cannot both be equal to the benefit. In other words, the underdog supporters cannot fully rebalance the election because turning out in a higher proportion means that types with a higher cost would have to turn out as well. To have an equilibrium with full underdog compensation (same benefit) we must have $c_\alpha = c_\beta$ (same cost), which happens when F is constant so costs are homogeneous.

Conversely, as the costs become equal, the equilibrium must exhibit full underdog compensation. The intuition is as follows. Suppose, to the contrary, that with homogenous costs a pure strategy equilibrium with partial underdog compensation existed so the ex-ante underdog is expected to lose the election. With such a strategy profile, a supporter of the underdog party who is abstaining, by deviating and going to vote would bring the election closer to a tie, hence he would have a higher benefit than the benefit of his fellow supporters of the underdog party that were voting according to that strategy profile, a contradiction.

Assuming heterogenous costs determines more appealing features. First of all the underdog compensation being just partial is what guarantees that the party with more ex-ante support is the more likely winner of the election. This natural outcome is corroborated by observed winning margins. Second, on the normative side, having the election result be determined by a coin toss as in the homogenous cost full underdog compensation case is clearly unappealing from a welfare perspective.¹⁹

¹⁹The fact that the 50-50 benchmark result is pervasive in the literature prompted the question of whether it is of any use to have people vote at all as the preferences of the electorate are not reflected in the outcome. See e.g. Borgers (2004) and Krasa and Polborn (2010). A different line of work that tries to avoid the full compensation undesirable outcome assumes that the preference split q remains unknown to voters: if so, then the compensation effect which rebalances the election and lowers welfare cannot be triggered properly. Hence opinion polls, which reduce uncertainty about q , may be welfare reducing. See Goeree and Grosser (2007) and Taylor and Yildirim (2010).

The distinction between homogenous and heterogeneous costs and hence between full and partial underdog compensation is also key for turnout predictions. The different equilibria with different cost assumptions, namely a 50-50 outcome versus a non 50-50 outcome, imply very different overall turnout numbers in large elections. In fact, the benefit of voting is proportional to

$$B_M \sim \frac{e^{-\left(\sqrt{q\alpha} - \sqrt{(1-q)\beta}\right)^2 N}}{\sqrt{N}}$$

In the homogenous cost case, in which $q\alpha = (1-q)\beta$, this implies that it declines at the rate $N^{-1/2}$. In the heterogeneous cost case, where $q\alpha \neq (1-q)\beta$ unless $q = 1/2$, the benefit of voting declines at an exponential rate for $q \neq 1/2$ ²⁰ and declines at the algebraic rate $N^{-1/2}$ when $q = 1/2$.²¹

Even though the nature of our work is positive, we want to conclude this discussion of the M system with a simple welfare corollary:

Corollary 3. *Asymptotically, for the population N going to infinity, neither subsidies nor penalties for voters can improve total expected utility in the M system.*

This could be easily shown by adapting the proof of proposition 5 in Krasa and Polborn (2009), since their model is similar to our model of the M system but with a positive voting cost lower bound $\underline{c} > 0$. They show that in the limit the optimal subsidy to voters converges to \underline{c} . Thus, when one considers the same model but with zero as lower bound $\underline{c} = 0$, the optimal subsidy in the limit must be zero.²² Intuitively, on the one hand introducing a subsidy is unnecessary since asymptotically the party with larger ex-ante support always wins the

²⁰For $q \neq 1/2$ the argument of the exponential function diverges to $-\infty$ (see Lemma 1 and Proposition 7).

²¹Chamberlain and Rothschild (1981) obtain a similar result on rates of convergence in a model in which two candidates receive votes as binomial random variables. They assume no abstention ($\alpha = \beta = 1$), so their analysis is statistical and not an equilibrium analysis of turnout. In a non-strategic framework, the number of votes for each side can be seen as flips of identical coins with a certain bias q . They show that if you toss an even number n of coins, the chance of obtaining the same number of heads and tails (the chance of a tie) drops asymptotically like $N^{-1/2}$ when the coins are unbiased ($q = 1/2$) and exponentially if the coins are biased ($q < 1/2$).

²²Krasa and Polborn (2010) obtain the inefficient full compensation result with a non degenerate cost distribution because its support $[\underline{c}, \bar{c}]$ is bounded away from zero. Hence, unlike what we obtain in Lemma 1, only a finite number of voters will go to vote even when the population N grows unboundedly large. Asymptotically their model is isomorphic to a homogenous cost model with cost $\underline{c} > 0$.

election in any case. On the other hand, introducing a penalty for voting would bring us back the inefficient lower bound $\underline{c} > 0$ in the voting cost distribution.

2.3. Proportional Power Sharing System ($\gamma = P$). With proportional power sharing (P system) the share of power is proportional to the vote share obtained in the election. So if (a, b) are the absolute numbers of votes for each party, the power of parties A and B would be respectively $\left(\frac{a}{a+b}, \frac{b}{a+b}\right)$.²³

The expected marginal benefit of voting B_P^i for party i is the expected increase in the vote share for the preferred party induced by a single vote, namely

$$B_P^A = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left(\left(\frac{e^{-qN\alpha} (qN\alpha)^a}{a!} \right) \left(\frac{e^{-(1-q)N\beta} ((1-q)N\beta)^b}{b!} \right) \left(\frac{a+1}{a+b+1} - \frac{a}{a+b} \right) \right)$$

$$B_P^B = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left(\left(\frac{e^{-qN\alpha} (qN\alpha)^a}{a!} \right) \left(\frac{e^{-(1-q)N\beta} ((1-q)N\beta)^b}{b!} \right) \left(\frac{b+1}{a+b+1} - \frac{b}{a+b} \right) \right)$$

In this case, unlike in the M system, we have double summations because an A supporter, for instance, has an impact on the electoral outcome not only in the event of a tied election ($a = b$ and $a = b - 1$), but also in all the other cases $a \neq b$. In the P system voters always have some impact on the electoral outcome albeit very small, whereas in the M system voters have a large impact in the very small chance event that $a = b$ and zero impact otherwise. A non obvious quantitative question is to compare how the expected impacts of a voter in the M and in the P systems decline with the electorate size N . After some manipulation the double summations above can be expressed in a simple form.

Lemma 4. *The marginal benefit of voting in the P system has the closed form*

$$(2) \quad B_P^A = \frac{(1-q)\beta}{NT^2} - \left(\frac{((1-q)\beta)^2 - (q\alpha)^2 + (1-q)\beta\frac{1}{N}}{2T^2} \right) e^{-NT}$$

$$B_P^B = \frac{q\alpha}{NT^2} + \left(\frac{((1-q)\beta)^2 - (q\alpha)^2 - q\alpha\frac{1}{N}}{2T^2} \right) e^{-NT}$$

The above closed form is a very lucky outcome and it allows us to show that asymptotically turnout for each party is zero as a percentage of the population, but is infinite in absolute

²³We assume that if nobody votes, power is shared equally, namely

$$\frac{a}{a+b} = \frac{b}{a+b} = \frac{1}{2} \text{ for } a = b = 0$$

numbers, moreover the party turnout ratio stays finite, similarly to what we obtained for the M model.

Lemma 5. *Any solution (α_N, β_N) (if it exists) to the P system has the following three properties*

$$\lim_{N \rightarrow \infty} \alpha_N = \lim_{N \rightarrow \infty} \beta_N = 0, \quad \lim_{N \rightarrow \infty} N\alpha_N = \lim_{N \rightarrow \infty} N\beta_N = \infty, \quad \lim_{N \rightarrow \infty} \frac{\alpha_N}{\beta_N} \in (0, \infty)$$

As in the M system, the above lemma allows us to use some approximations to show existence and uniqueness of an equilibrium for N large and also the following characterization results.

Lemma 6. *In the P system there is always a unique equilibrium (α, β) . The equilibrium has the following properties:*

- Size effect:

$$\frac{dT_P}{dN} < 0$$

- Partial underdog compensation effect:

$$q < 1/2 \implies \alpha > \beta, \quad q\alpha < (1 - q)\beta$$

The relation describing quantitatively the underdog compensation under the P system can be expressed simply for large electorates as (see Appendix)

$$(3) \quad q\alpha F^{-1}(\alpha) = (1 - q)\beta F^{-1}(\beta)$$

the above linear expression is slightly different from the quadratic expression (1) describing the underdog compensation under the M system.

2.4. Main Comparison. The size effect and the underdog compensation effect, though qualitatively similar, are quantitatively different across the two institutional systems. We now turn to the implications of these differences and to the comparison of turnout incentives across systems. Turnout is higher in a proportional power sharing system when there is a favorite party, while it is higher in a winner take all system if the election is even.

Proposition 7. .

- *Comparative turnout: for any $q \in (0, 1)$, $\exists \bar{N}_q$ such that for $N > \bar{N}_q$*

$$T_M > T_P \quad \text{for } q = 1/2$$

$$T_P > T_M \quad \text{for } q \neq 1/2$$

- *Comparative underdog compensation.*²⁴

$$\frac{1-q}{q} = \left(\frac{\alpha_P}{\beta_P}\right)^{n+1} = \left(\frac{\alpha_M}{\beta_M}\right)^{2n+1}$$

where $n \geq 1$ is the lowest integer for which $\frac{d^n F^{-1}}{dx^n}|_{x=0} \in (0, \infty)$.

Regarding the comparative underdog compensation, we have already explained in section 2.2 that with heterogeneous costs full compensation is impossible in equilibrium, and a similar explanation holds for the proportional power sharing system. In both systems the underdog compensation is partial: the ex-ante favorite party obtains the majority of the votes in a large election, but the underdog party has a higher turnout of its supporters. The above proposition shows that the underdog compensation is larger in the P system, namely

$$(4) \quad q < 1/2 \quad \Rightarrow \quad \alpha_P/\beta_P > \alpha_M/\beta_M > 1$$

Compared to the P system, in the M system minority voters are always more discouraged to vote relative to majority voters. This result could also be stated as a higher *relative winning margin* in the M system than in the P for any given preference split q , where the relative winning margin W is defined as²⁵

$$W := \frac{|q\alpha - (1-q)\beta|}{T}$$

Regarding turnout, the intuition behind the turnout result relies on how fast the marginal benefit of voting decreases in the two models as the electorate gets larger. The M system has two asymptotic regimes: it decreases exponentially for $q \neq 1/2$ and for $q = \frac{1}{2}$ it decreases at the algebraic rate of $N^{-1/2}$. Since we have only partial underdog compensation, then for any $q \neq 1/2$ the majority party is always the more likely side to win. Hence the chance of a tied election, which is what drives rational voters to turn out, is much smaller than in the case $q = 1/2$ for any population size N .²⁶

²⁴We thank Faruk Gul for providing the main insight for proving this second result.

²⁵This result is not obvious ex-ante as there are two competing effects: in the M system, while minority voters α are discouraged to vote, also (and for the same reason) majority voters β are. So, it is not clear whether α/β should be smaller or greater in the M system than in the P system (where neither effect is present).

²⁶The two rates of convergence derived above do not depend on the (Poisson) population uncertainty in this model. For instance, Herrera and Martinelli (2006) analyze a majority rule election without population uncertainty. They introduce aggregate uncertainty in a different way, which allows to obtain a closed form for the chance of being pivotal, namely $\frac{(a+b)!}{2^{a+b+1}a!b!}$. As it can be seen using Stirling's approximation, that marginal

The benefit from voting in the P system drops asymptotically at the intermediate rate of N^{-1} . This rate is independent of q as in the power sharing system the event that a voter is pivotal or the chance of a tied election have no special relevance.

It is perhaps now intuitive that a winner take all system, unlike a proportional power sharing one, should have two quite different rates of convergence regimes (although as we explained this is not the case with a degenerate cost distribution). Be that as it may, only an explicit computation could determine that the rate of convergence in the P system is quantitatively *in between* the two rates of convergence in the M system.

In order to illustrate the comparison in terms of turnout as well as underdog compensation effects, we now turn to a numerical example.

2.5. Example. Consider the cost distribution family ($z > 0$): $F(c) = c^{1/z}$ with $c \in [0, 1]$.

This example yields an explicit solution for the P system, i.e.

$$\alpha_P = \left(\frac{1}{N} \frac{(1-q) q^{\frac{1}{z+1}} (1-q)^{\frac{1}{z+1}}}{\left(q(1-q)^{\frac{1}{z+1}} + (1-q) q^{\frac{1}{z+1}} \right)^2} \right)^{\frac{1}{z+1}} \quad \beta_P = \left(\frac{1}{N} \frac{q}{1-q} \frac{(1-q) q^{\frac{1}{z+1}} (1-q)^{\frac{1}{z+1}}}{\left(q(1-q)^{\frac{1}{z+1}} + (1-q) q^{\frac{1}{z+1}} \right)^2} \right)^{\frac{1}{z+1}}$$

The M system equilibrium has no closed form solution, namely (α_M, β_M) jointly solve

$$\beta_M = \left(\frac{q}{1-q} \right)^{\frac{1}{2z+1}} \alpha_M, \quad \alpha_M^z = \frac{e^{-N(\sqrt{(1-q)\beta_M} - \sqrt{q\alpha_M})^2}}{\sqrt{N}} \left(\frac{\sqrt{q\alpha_M} + \sqrt{(1-q)\beta_M}}{4\sqrt{\pi}(q(1-q)\alpha_M\beta_M)^{1/4}} \right)$$

Setting $N = 3000$ and $z = 5$, the numerical solutions to the M system yield a clear illustration of the comparative result of proposition 7. In the picture below we compare, as the preference split q varies, the turnout T in the M system (continuous line) and in the P system (dashed line).

benefit for large a and b has exactly the square root decline on the diagonal ($a = b$) and the exponential decline off the diagonal ($a = \omega b$, $\omega \neq 1$).

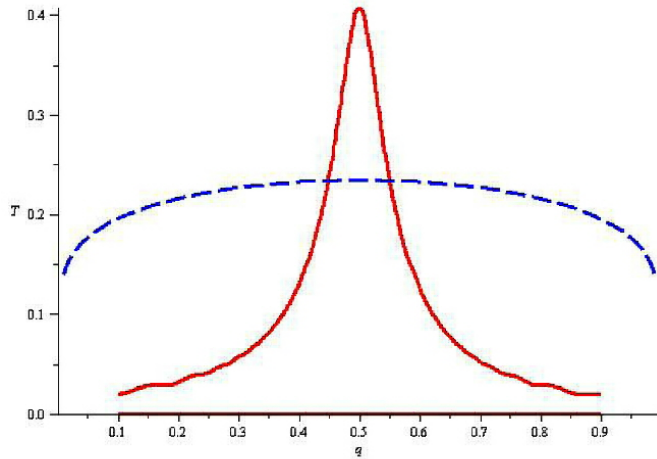


Figure 1: Turnout as a function of q in the M (continuous) and P (dashed) models ($z = 5$, $N = 3000$).

When one party (e.g. party B) has the ex-ante advantage over the other party (A), we have a higher turnout in the P system. Numerically, for instance when $q = 1/3$, we have

$q = 1/3$	α	β	α/β	W	T
P	24.8%	22%	1.27	27.8%	23%
M	7.1%	6.7%	1.06	30.9%	6.8%

Note also in both the M and the P systems the presence of the underdog compensation ($\alpha > \beta$) which is partial ($q\alpha < (1 - q)\beta$). Moreover, note the higher underdog compensation α/β in the P system and consequently the higher relative winning margin W in the M system.

Whereas when the election is close and no party has an ex ante advantage, i.e. $q = 1/2$, turnout T in the M system surpasses the turnout in the P system

$q = 1/2$	α	β	T
P	23.5%	23.5%	23.5%
M	40.9%	40.9%	40.9%

Note for different q 's the much larger variability of turnout numbers T in the M system when compared to the P system.

To compare the underdog compensations in general, the picture below illustrates how the ratio α/β varies with q in the P system (dashed line) and in the M system (continuous line). Contrast these decreasing curves with the steeper one that is obtained in the M system

under homogeneous cost (dotted line) when there is full underdog compensation, the election is expected to be tied and the winning margin is zero regardless of the initial preference split. In sum, this example illustrates how the underdog compensation is higher in a proportional power sharing system, while the turnout is lower in a proportional power sharing system only when the distribution of party supporters is symmetric.

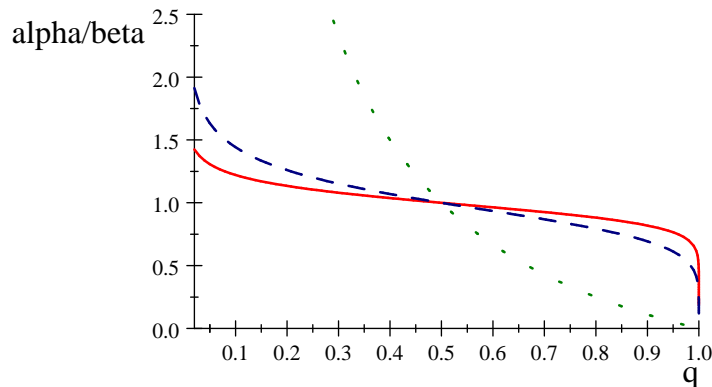


Figure 2: Underdog compensation α/β as a function of q in the P (dashed), M (continuous) and M with homogenous cost (dotted) models ($z = 5$).

3. ROBUSTNESS AND EXTENSIONS

The basic pivotal voter model analyzed in the last section is idealized in several ways. The purpose of this section is to show that many of the results in this idealized version extend to alternative models of turnout that have been proposed, and to show that the basic approach is quite flexible and can be extended to accommodate a much broader array of political systems. We consider a number of generalizations or modifications of the basic model studied in the last section: we first of all generalize the model to allow for intermediate systems between the two extremes of P and M; then, in section 3.2, we analyze the pivotal model without population uncertainty and allowing for such intermediate systems; third, we show in section 3.3 that the main comparative results obtained using the pivotal model (with or without population uncertainty and allowing for intermediate systems) extend even when studying the problem using *group-based* turnout models, like the mobilization model or the ethical voting model; finally, we will show that the main comparative results on turnout are not affected by the number of parties present in a proportional election system.

3.1. Intermediate Power Sharing Rules. The model is simple enough that we can accommodate a wide range of intermediate power sharing rules between P and M using a single parameter in the payoff function. Intermediate systems are plausible and perhaps even more

realistic than either of the extremes. For example, even in a winner take all system like the U.S. Presidential race, a large winning margin carry with it added benefits to the winner due to a "mandate" effect.²⁷ Also, the fact that the legislative branch in a M system has leverage over the executive branch and the presidency will tend to smooth out the winner-take-all payoff function in the direction of a proportional system. On the flip side, in parliamentary systems that require the formation of a coalitional governing cabinet, a party that is fortunate to win a clear majority of seats outright has much less incentive (or in some cases none at all) to compromise with other parties in order to govern effectively. The expected vote shares for party A and B are

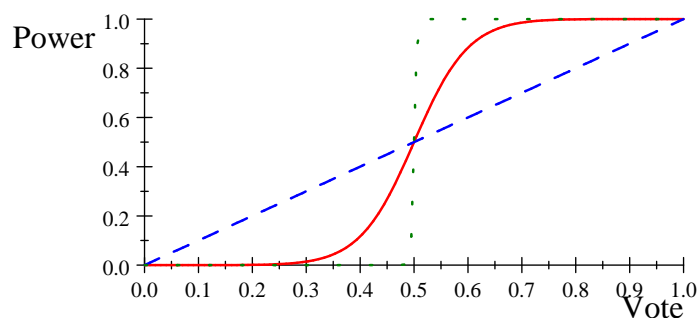
$$V = \frac{q\alpha}{T}, \quad 1 - V = \frac{(1 - q)\beta}{T}$$

In a γ -power sharing system, payoffs as a function of the vote share is represented by standard "contest success function"²⁸, where γ ranges from 1 to ∞ .

$$P_{\gamma}^A(V) = \frac{V^{\gamma}}{V^{\gamma} + (1 - V)^{\gamma}}, \quad P_{\gamma}^B(V) = \frac{(1 - V)^{\gamma}}{V^{\gamma} + (1 - V)^{\gamma}}$$

The two extreme cases correspond to P ($\gamma = 1$) and M ($\gamma = \infty$).²⁹

The figure below illustrates the power share payoff P_{γ}^A as a function of the vote share V for three power sharing parameters γ , namely: $\gamma = 1$ (i.e. the P system, dashed line), $\gamma = 5$ (i.e. an intermediate power sharing system, continuous line), and $\gamma \rightarrow \infty$ (i.e. a pure M system, dotted line).



²⁷See Castanheira (2003) and Razin (2003).for the effects on voting behavior of mandates in a one dimensional policy space.

²⁸See for instance Hirshleifer (1989), among others. When nobody votes ($\alpha = \beta = 0$) assume equal shares ($V = 1/2$).

²⁹There are other ways to introduce a proportionality parameter. In a more recent paper, Faravelli and Sanchez-Pages (2012) model it as a linear combination of PR and SMP.

Figure 3: Power Sharing Functions in the P (dashed), approaching the M system (continuous) and pure M system (dotted).

3.2. The Finite Voter Model with No Population Uncertainty. For the M system, this model reduces to the one studied by Levine and Palfrey (2007) and Palfrey and Rosenthal (1985). The formulation of the equilibrium conditions in that model extend to the P system and also to any arbitrary γ -power sharing system. Let N_A denote the number of voters with a preference for party A and $N_B = N - N_A$ denote the number of voters with a preference for party B , and assume without loss of generality that $N_A \leq N_B$ and both N_A and N_B are common knowledge. As before, a symmetric equilibrium is characterized by two cutoff levels, one for each party, c_α and c_β , with corresponding expected turnout levels equal to $\alpha = F(c_\alpha)$ and $\beta = F(c_\beta)$. The equilibrium conditions for a γ -power sharing system is characterized as follows.³⁰

Given expected turnout rates in the two parties, α and β , the expected marginal benefit of voting (B_P^A, B_P^B) of a party A and party B citizen are equal to respectively:

$$(5) \quad \sum_{j=0}^{N_A-1} \sum_{k=0}^{N_B} \left[\frac{(j+1)^\gamma}{(j+1)^\gamma + k^\gamma} - \frac{j^\gamma}{j^\gamma + k^\gamma} \right] \binom{N_A-1}{j} \binom{N_B}{k} \alpha^j (1-\alpha)^{N_A-1-j} \beta^k (1-\beta)^{N_B-k}$$

$$(6) \quad \sum_{j=0}^{N_B-1} \sum_{k=0}^{N_A} \left[\frac{(j+1)^\gamma}{(j+1)^\gamma + k^\gamma} - \frac{j^\gamma}{j^\gamma + k^\gamma} \right] \binom{N_B-1}{j} \binom{N_A}{k} \beta^j (1-\beta)^{N_B-1-j} \alpha^k (1-\alpha)^{N_A-k}$$

Where the first term in brackets in the summation is the increase in power share, as derived from vote shares, and the remaining terms represent the probability of the vote share being equal to $\frac{j}{j+k}$ without your vote, given turnout rates α and β .³¹ The equilibrium condition for c_α and c_β are given by:

$$B_\gamma^A = c_\alpha, \quad B_\gamma^B = c_\beta$$

While closed form analytical expressions of the equilibria do not exist, they are easily computed numerically. The figures below show the equilibrium solution as a function of γ , for the following parameters.

³⁰In more recent papers, Kartal (2012) and Faravelli and Sanchez-Pages (2012) use this finite rational voter approach to compare turnout in M and PR systems, reporting results similar to ours. Kartal focuses on the welfare comparisons between P and M.

³¹By convention, we denote $\frac{j}{j+k} = .5$ if $j = k = 0$.

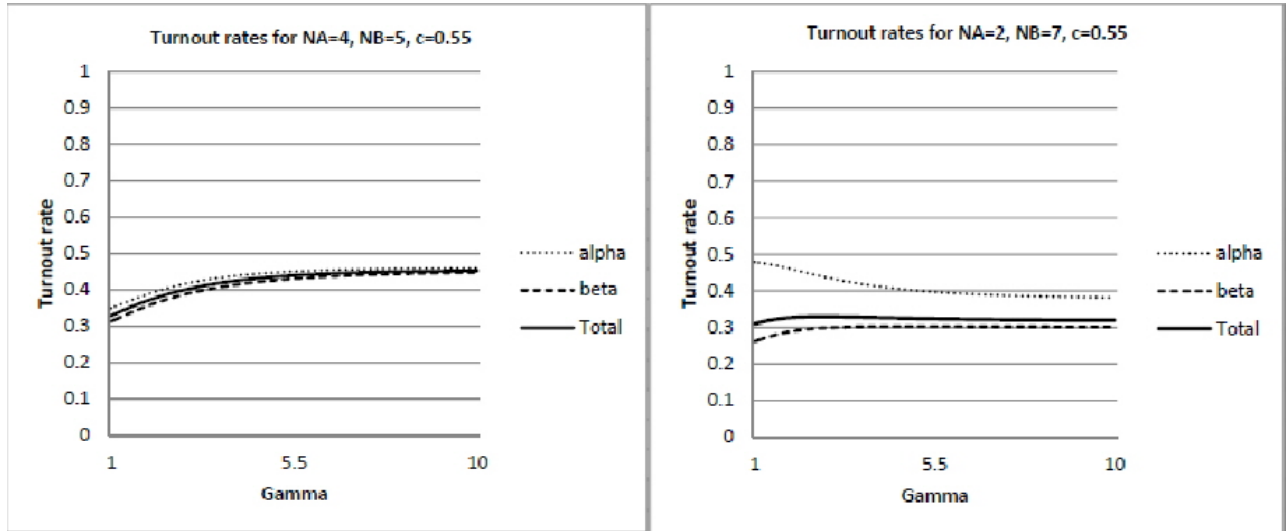


Figure 4: Turnout as γ increases from PR approaching MR.

In the close to even preference split case ($N_A = 4, N_B = 5$), turnout increases as we approach the majoritarian system because winning the election becomes paramount so competition becomes fiercer.

In the large majority case, i.e. an uneven preference split ($N_A = 2, N_B = 7$), turnout is (slightly) decreasing. As we approach the majoritarian system, the incentive to vote is reduced: winning becomes all that matters and the underdog has a small chance of winning when preferences are uneven. We see in both cases the presence of the underdog effect. The magnitude of the underdog effect is gradually decreasing as we approach the majoritarian system, which confirms and extends the previously obtained result (4).

3.3. Group-based Models.

3.3.1. *Mobilization Model.* Morton (1987, 1991) and Shachar and Nalebuff (1999) among others have proposed models based on group mobilization, where parties can mobilize and coordinate citizens to go vote. There is evidence that mobilization effects play some role in turnout variation across elections and across electoral systems. Even though we believe that the analysis conducted so far provides per se many new insights, we want to extend the analysis to other turnout models. The basic idea behind these models is that the positive externality of voting among supporters of the same party is somehow internalized, leading to higher turnout. Moreover, the electoral rules - especially as the rules effect eventual power sharing between the parties - can have a strong effect on turnout, so regardless of the size of the population turnout is high.

A group mobilization model a la Shachar and Nalebuff (1999) where parties' campaign efforts and spending are able to mobilize and coordinate citizens to go vote, is one example that shows how our results can be adapted to this approach. In that model, each group can "purchase" turnout of its party members by engaging in costly get-out-the-vote efforts. Thus, parties trade off mobilization costs for higher expected vote shares, taking as given the mobilization choice of the other party.

3.3.2. Ethical Voter Model. A second approach that is also grounded in group-oriented behaviors is the ethical voter model of Feddersen and Sandroni (2006), which assumes that citizens are "rule utilitarian" so they act as one. This involves an equilibrium between two party-planners on each side A and B. In this solution each planner looks at the total benefit from the outcome of the election considering the total cost of voting incurred by the supporters of his side, taking the other planner's turnout strategy as given. The logic of the ethical voter models is similar to the group mobilization models. They are almost identical on the benefit side but differ slightly on the cost side. In the case of the ethical voter model, one gets almost exactly the same kind of partial compensation as in the pivotal voter model with non-negative costs.

3.3.3. Common analysis. We describe the ethical voter model and mobilization model together because they are operationally similar. As in our basic model, the population is a continuum of measure one, divided into q A supporters and $(1 - q)$ B supporters. For any voting cost thresholds (c_α, c_β) , i.e., given that the voter participation for each side is $(\alpha = F(c_\alpha), \beta = F(c_\beta))$, turnout is again $T = q\alpha + (1 - q)\beta$. We assume F is weakly concave.³²

Both models have identical group benefit. In a γ -power sharing system described above, the marginal group benefits to the two parties, with respect to (c_α, c_β) are, respectively:

$$\frac{dP_\gamma^A}{dc_\alpha} = \frac{dP_\gamma^A}{dV} \left(\frac{(1 - q)\beta}{T^2} \right) q f(c_\alpha), \quad \frac{dP_\gamma^A}{dc_\beta} = -\frac{dP_\gamma^B}{dV} \left(\frac{q\alpha}{T^2} \right) (1 - q) f(c_\beta)$$

where:

$$\frac{dP_\gamma^A}{dV} = -\frac{dP_\gamma^B}{dV} = \frac{\gamma}{V(1 - V)} \frac{\left(\frac{V}{1 - V}\right)^\gamma}{\left[1 + \left(\frac{V}{1 - V}\right)^\gamma\right]^2}$$

³²The same condition was needed to have uniqueness of a solution in the rational voter M-model.

3.3.4. *Solution to the Mobilization Model.* A mobilization model assumes that more campaign spending by a party brings more votes for the party according to an exogenous technology. In major elections, candidates and parties engage in hugely expensive get-out-the-vote drives. Empirical evidence suggests that these drives are effective. We consider a very simple version of group mobilization. We assume the cost for a party of mobilizing to the polls all his supporters with voting cost below c is $l(c)$, where $c \in [0, \bar{c}]$ and l is increasing, convex and twice differentiable. We also assume it is infinitely costly for a party to turn out all its supporters: $l(\bar{c}) = \infty$.

For the mobilization model the first order conditions that characterize the solution are:

$$\begin{aligned} \frac{dP_\gamma^A}{dV} \left(\frac{(1-q)\beta}{T^2} \right) q f(c_\alpha) &= l'(c_\alpha) \\ \frac{dP_\gamma^A}{dV} \left(\frac{q\alpha}{T^2} \right) (1-q) f(c_\beta) &= l'(c_\beta) \end{aligned}$$

which yields the following *zero underdog compensation* condition

$$\frac{\alpha l'(c_\alpha)}{f(c_\alpha)} = \frac{\beta l'(c_\beta)}{f(c_\beta)} \implies T = \alpha = \beta, \quad c_\alpha = c_\beta$$

that is, both parties turn out the same proportion of their supporters.³³

The mobilization model is reduced to one equation in one unknown, equating marginal benefit (MB) and marginal cost

$$(7) \quad MB = \gamma \frac{\left(\frac{V}{1-V}\right)^\gamma}{\left[1 + \left(\frac{V}{1-V}\right)^\gamma\right]^2} = \gamma \frac{\left(\frac{q}{1-q}\right)^\gamma}{\left[1 + \left(\frac{q}{1-q}\right)^\gamma\right]^2} = G(\alpha)$$

where

$$G(\alpha) := \alpha \frac{l'(c_\alpha)}{f(c_\alpha)}$$

is increasing in α . The solution is hence unique and it exists because $l(\bar{c}) = \infty$. The solution has the following properties:

- (1) Turnout $T = \alpha$ increases when the marginal benefit (MB) increases;
- (2) As γ goes to infinity (M model) the marginal benefit goes to infinity when $q = 1/2$ and goes to zero otherwise;
- (3) When $\gamma = 1$ (P model) the marginal benefit $\frac{\left(\frac{q}{1-q}\right)^\gamma}{\left[1 + \left(\frac{q}{1-q}\right)^\gamma\right]^2}$ is positive for all $q \in (0, 1)$ and peaks but stays finite at $q = 1/2$.

³³We need to assume F weakly concave (as in the rational voter M model) to guarantee the LHS expressions above are increasing in their argument.

The picture below shows the marginal benefit as a function of the closeness of the election q for $\gamma = 1$ (i.e. the P system, dashed line), and for $\gamma = 5$ (i.e. approximating the M system, continuous line).

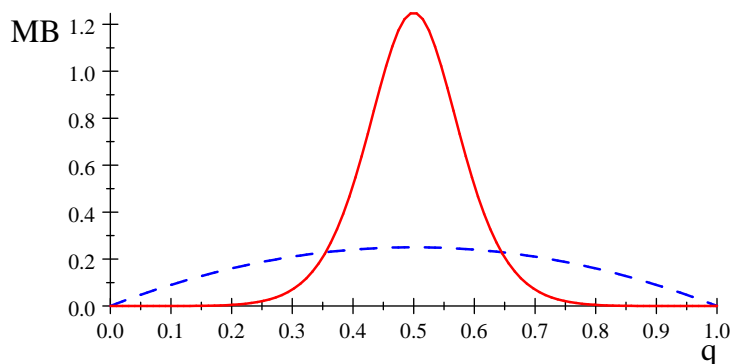


Figure 5: Marginal benefit as a function of q in the P system (dashed) and approaching the M system (continuous).

3.3.5. *Solution to the Ethical Voter Model.* The ethical voter model assumes that citizens are “rule utilitarian” so they act as one. This means that we have to find a party-planner solution on each side A and B. In this solution each planner looks at the total benefit from the outcome of the election considering the total cost of voting incurred by the supporters of his side.³⁴ The cost of turning out the voters for the social planner on side A is the total cost born by all the citizens on side A that vote, namely

$$C(c_\alpha) := q \int_0^{c_\alpha} cf(c) dc$$

The citizens with cost below the planner-chosen cost threshold c_α vote because ethical voter models assume citizens get an exogenous benefit D (larger than their private voting cost $c \leq c_\alpha$) for “doing their part” in following the optimal rule established by the planner. We have as first order conditions

$$\begin{aligned} \frac{dP_\gamma^A}{dV} \left(\frac{(1-q)\beta}{T^2} \right) qf(c_\alpha) &= qc_\alpha f(c_\alpha) \\ \frac{dP_\gamma^A}{dV} \left(\frac{q\alpha}{T^2} \right) (1-q)f(c_\beta) &= (1-q)c_\beta f(c_\beta) \end{aligned}$$

³⁴We assume “collectivism”, so the planner on each side, A and B, only looks at the total cost of voting of the voters on his side. The results would not have changed had we assumed “altruism” as in Feddersen and Sandroni (2006): each planner takes into account the cost of voting of all citizens that vote regardless of their side.

which gives the condition

$$q\alpha F^{-1}(\alpha) = (1 - q)\beta F^{-1}(\beta)$$

The above is a partial underdog compensation condition which happens to be the same as the partial underdog compensation condition (3) obtained in the P system of the rational voter model.

The solution for the ethical voter model for general γ is more complicated than the group mobilization model in γ -power sharing system, as the underdog compensation is strictly partial (not zero), so $\alpha \neq \beta$ and we maintain the two equations in two unknowns, that is

$$q\alpha F^{-1}(\alpha) = (1 - q)\beta F^{-1}(\beta) = \gamma \frac{\left(\frac{q\alpha}{(1-q)\beta}\right)^\gamma}{\left[1 + \left(\frac{q\alpha}{(1-q)\beta}\right)^\gamma\right]^2}$$

However, given that the underdog compensation is not full the comparative statics is similar to the case of zero compensation obtained in the mobilization model. Namely if a solution (α, β) exists,³⁵ then α and β , and hence T , increase when the marginal benefit increases. Taking limits, as γ goes to infinity (M model) the marginal benefit on the RHS goes to infinity when $q = 1/2$ and to zero otherwise. When $\gamma = 1$ (P model) the marginal benefit is positive for all $q \in (0, 1)$ and peaks but stays finite at $q = 1/2$.

Note that if the underdog compensation were full (which happens for instance with homogeneous costs) the marginal benefit would become $\frac{\gamma}{4}$, so the result would be different: regardless of the initial preference split q , turnout would increase with the intensity of the contest γ . As explained, the rational voter model with homogenous cost gives an equivalent result.

3.4. Extending the Basic Model to k-Parties. This section proposes one possible way to extend the turnout model for P systems to k parties. The extension is meant to be illustrative of possible directions the model can be generalized. All the analysis in the paper up to this point is conducted by altering the mapping from vote shares to power shares but keeping, for simplicity, the two-party assumption. However, as we show in this extension, the comparative results in terms of turnout *do not necessarily depend on the number of parties under the P system*. Below we explicitly compute the equilibrium for any number of parties in the P system

³⁵Coate and Conlin (2004) and Feddersen and Sandroni (2006) provide specific conditions on the voting cost distributions that guarantee existence.

with sincere voting. This allows us to obtain a simple comparative statics result within the proportional power sharing system: turnout increases in the number of parties.³⁶

To keep notation simple, we illustrate only the three party case. Define

$$A := \alpha q_A N, \quad B := \beta q_B N, \quad C := \gamma q_C N, \quad \text{with: } q_A + q_B + q_C = 1$$

The marginal benefit³⁷ for party A is

$$B_P^A = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \left(\frac{e^{-A} A^a}{a!} \right) \left(\frac{e^{-B} B^b}{b!} \right) \left(\frac{e^{-C} C^c}{c!} \right) \left(\frac{a+1}{a+b+c+1} - \frac{a}{a+b+c} \right)$$

Lemma 8. *The marginal benefit has the closed form*

$$B_P^A = \left(1 - \frac{A}{A+B+C} \right) \frac{1 - e^{-(A+B+C)}}{A+B+C} + \left(\frac{A}{A+B+C} - \frac{1}{3} \right) e^{-(A+B+C)}$$

By symmetry the expressions B_P^B and B_P^C for parties B or C are straightforward.

For any number of parties the following comparative statics result holds.

Proposition 9. .

- *The comparison between turnout in the P system and the M system continues to hold even when there are multiple parties in the P system.*
- *If parties are symmetric, turnout in the P system increases as the number of parties increases.*

The turnout comparison result remains unchanged with more parties because the marginal benefit of voting in the P system always declines asymptotically at the intermediate rate $1/N$, as was the case for the P system with two parties.

Within the $1/N$ order of magnitude of the size effect, turnout increases when there are more symmetric parties. This is consistent with the fact that smaller parties obtain a higher turnout

³⁶The extension to multiple parties presented here could be useful especially for future research, because it could help to open a bit the reduced form proportionality of influence parameter. With many parties the reduced form linear mapping from vote shares to power shares a la Lizzeri and Persico (2001) can be explicitly obtained from a standard post election legislative bargaining model of alternating offers a la Baron and Ferejohn (1989). Snyder, Ting and Ansolabehere (2005) analyze the conditions under which the expected power shares are proportional to the vote shares.

³⁷Assume again that if nobody votes, power is shared equally, namely

$$\frac{a}{a+b+c} = 1/3 \quad \text{for } a = b = c = 0$$

in the P system. The intuition for the latter follows from the following two observations. First, fixing the number of votes z for all other parties, the vote share increase for party A is

$$\left(\frac{a+1}{a+z+1} - \frac{a}{a+z} \right) = \left(\frac{a^2+a}{z} + 2a+1+z \right)^{-1}$$

which is larger for smaller values of the random variable a , i.e. the number of votes for party A . Second, for a given a , in the marginal benefit B_P^A (see (2) and expressions above) a smaller party (i.e. a party with a smaller q_A) assigns larger probability weight $\left(\frac{e^{-A} A^a}{a!} \right)$ to small values of a .

4. EXPERIMENTAL ANALYSIS

Even though the point estimates of turnout when using the costly voting rational model are much lower than real turnout levels in elections, as is well known, there is no reason to believe that the comparative predictions of the rational model shouldn't be of guidance. To verify that indeed the comparative results of the paper correspond to actual voting behavior, we bring the model presented in section 2 to the laboratory. In laboratory elections we can have only a finite number of voters, but the model is easily adapted to this case. The equilibrium conditions for our laboratory implementation of the model, with finite electorates and no population uncertainty,³⁸ are given below. It is straightforward to exactly characterize symmetric Bayesian equilibrium for these finite environments, and comparative statics that are similar to the Poisson model can be computed directly from these exact equilibrium solutions.

In what follows, let N_A denote the number of voters with a preference for party A and $N_B = N - N_A$ denote the number of voters with a preference for party B , and assume without loss of generality that $N_A \leq N_B$. As before, a symmetric equilibrium is characterized by two cutoff levels, one for each party, c_α and c_β , with corresponding expected turnout levels equal to $\alpha = F(c_\alpha)$ and $\beta = F(c_\beta)$. The equilibrium conditions are slightly different for the M and P systems, and these are derived next.

4.0.1. *Equilibrium conditions for M.* Given expected turnout rates in the two parties, α and β the expected marginal benefit of voting for a party A citizen corresponds to the limit of

³⁸The reason to consider *known* population size is that the analytical computations with the Poisson game approach apply only to the limiting case of very large electorates, which is not feasible in the laboratory.

expressions (5) and (6) when γ goes to infinity. This reduces to:

$$B_M^A = \frac{1}{2} \left[\begin{array}{l} \sum_{k=0}^{N_A-1} \binom{N_A-1}{k} \binom{N_B}{k} \alpha^k (1-\alpha)^{N_A-1-k} \beta^k (1-\beta)^{N_B-k} + \\ \sum_{k=0}^{N_A-1} \binom{N_A-1}{k} \binom{N_B}{k+1} \alpha^k (1-\alpha)^{N_A-1-k} \beta^{k+1} (1-\beta)^{N_B-1-k} \end{array} \right]$$

$$B_M^B = \frac{1}{2} \left[\begin{array}{l} \sum_{k=0}^{\min\{N_A, N_B-1\}} \binom{N_A}{k} \binom{N_B-1}{k} \alpha^k (1-\alpha)^{N_A-k} \beta^k (1-\beta)^{N_B-1-k} + \\ \sum_{k=0}^{N_A-1} \binom{N_A}{k+1} \binom{N_B-1}{k} \alpha^{k+1} (1-\alpha)^{N_A-1-k} \beta^k (1-\beta)^{N_B-1-k} \end{array} \right]$$

where $\frac{1}{2}$ is the value of creating or breaking a tie. In each expression, the first summation is the probability of your vote breaking a tie, and the second summation is the probability of your vote creating a tie, given turnout rates α and β . The equilibrium conditions are given by:

$$B_M^A = F^{-1}(\alpha), \quad B_M^B = F^{-1}(\beta)$$

4.0.2. *Equilibrium conditions for P.* Given expected turnout rates in the two parties, α and β the expected marginal benefit of voting for a party A citizen is obtained from expressions (5) and (6) by setting $\gamma = 1$:

$$B_P^A = \sum_{j=0}^{N_A-1} \sum_{k=0}^{N_B} \left[\frac{j+1}{j+1+k} - \frac{j}{j+k} \right] \binom{N_A-1}{j} \binom{N_B}{k} \alpha^j (1-\alpha)^{N_A-1-j} \beta^k (1-\beta)^{N_B-k}$$

$$B_P^B = \sum_{j=0}^{N_B-1} \sum_{k=0}^{N_A} \left[\frac{j+1}{j+1+k} - \frac{j}{j+k} \right] \binom{N_B-1}{j} \binom{N_A}{k} \beta^j (1-\beta)^{N_B-1-j} \alpha^k (1-\alpha)^{N_A-k}$$

Where the first term in the summation is the increase in vote share and the second term is the probability of the vote share being equal to $\frac{j}{j+k}$ without your vote, given turnout rates α and β .³⁹ The equilibrium conditions are given by:

$$B_P^A = F^{-1}(\alpha), \quad B_P^B = F^{-1}(\beta)$$

4.0.3. *Experimental Design and Parameters.* All our electorates in the experiment have exactly $N = 9$ voters, with three different N_A treatments: $N_A = 2, 3, 4$. We consider two different distributions of voter costs. In our *low cost* (or, equivalently, *high benefit*) elections c_i is uniformly distributed on the interval $[0, .3]$. In our *high cost* (or, equivalently, *low benefit*) elections c_i is uniformly distributed on the interval $[0, .55]$. Table 1 below gives the symmetric equilibrium expected turnout levels (by party and total turnout) for each treatment, rounded

³⁹By convention, we denote $\frac{j}{j+k} = .5$ if $j = k = 0$.

to two decimal places.

N_A	N_B	c_{\max}	Rule	α^*	β^*	$\frac{\alpha^*N_A + \beta^*N_B}{N}$
4	5	.3	M	.60	.72	.67
3	6	.3	M	.51	.52	.52
2	7	.3	M	.45	.40	.41
4	5	.55	M	.46	.45	.46
3	6	.55	M	.41	.37	.39
2	7	.55	M	.38	.30	.32
4	5	.3	P	.48	.43	.45
3	6	.3	P	.55	.39	.45
2	7	.3	P	.67	.36	.43
4	5	.55	P	.35	.31	.33
3	6	.55	P	.40	.29	.32
2	7	.55	P	.48	.26	.31

Table 1. Equilibrium turnout rates by treatment.

There are five main theoretical hypotheses comparing turnout in the M and P voting systems in the elections we study. We state these below:

H1 For the larger party, turnout is higher in M than in P.

H2 Total expected turnout is higher under M than under P.

H3 The competition effect is reversed for the smaller party in the proportional vote system.

That is, for the smaller party, turnout decreases as their share of the electorate increases.

Under M, the usual competition effect applies to both parties: elections that are expected to be closer lead to higher turnout.⁴⁰

H4 The competition effect on total expected turnout is negligible in P elections.

H5 In all P elections we study, there is an underdog effect. There is an underdog effect in all M elections, except for reverse underdog effects in the low cost 5-4 and 6-3 M elections.

In the experimental section we will return to these five predictions of the known population model.

4.1. Procedures. A total of 153 subjects participated in 1700 elections across 17 sessions. Each session consisted of two parts with 50 nine-voter elections in each part. The parameters

⁴⁰In the general model with population uncertainty we were not able to obtain general results on the competition effects, whereas the numerical analysis of the known population case allows for these additional predictions.

were the same in all elections within a part, but in each session exactly one parameter was changed between part I and part II. In all sessions the same voting rule (M or P) was used in all 100 elections. For all of the treatments except for the 7-2 elections, the distribution of voting costs were the same for all 100 elections. Half of these sessions were conducted with part I having 5-4 elections and part II having 6-3 elections. The other half of the sessions reversed the order so the 6-3 elections were in part I and the 5-4 elections in part II. For the 7-2 elections, half the elections in a session were conducted with $c_{\max} = 55$ and half with $c_{\max} = 30$, in both orders. Subjects were informed of the exact parameters (N_A , N_B , C_{\max} and the voting rule) at the beginning of each part. Before each election, each subject was randomly assigned to either group A or group B and assigned a voting cost, drawn independently from the uniform distribution between 0 and c_{\max} , in integer increments. Therefore, each subject gained experience as a member of the majority and minority party in both parts of the session. Instructions were read aloud so everyone could hear, and Powerpoint slides were projected in front of the room to help explain the rules. After the instructions were read, subjects were walked through two practice rounds and then were required to correctly answer all the questions on a computerized comprehension quiz before the experiment began. After the first 50 rounds, a very short set of new instructions were read aloud to explain the change of parameters.

The wording in the instructions was written so as to induce as neutral an environment as possible.⁴¹ There was no mention of voting or winning or losing or costs. The labels were abstract. The smaller group was referred to the alpha group (A) and the larger group was referred to as the beta group (B). Individuals were asked in each round to choose X or Y. For the M treatment, if more members of $A(B)$ chose X than members of $B(A)$ chose X, then each member of $A(B)$ received 100 and each member of group $B(A)$ received 0. In case of a tie, each member of each group received the expected value of a fair coin toss, 50. For the P treatment, each voter received a share of 100 proportional to the number of voters in their party that chose X compared to the number of voters in the other party that chose X. The voting cost was implemented as an opportunity cost and was referred to as a "Y bonus". It was added to a player's earnings if that player chose Y instead of X. If a player chose X, that player did not receive their Y bonus in that election. Y-bonuses were randomly redrawn in every election, independently for each subject, and subjects were only told their own Y bonus. Bonuses were integer valued and took on values from 0 to 30 in the low cost treatment and 0 to 55 in the

⁴¹A sample of the instructions from one of the sessions is in Appendix E (not for publication).

high cost treatment. Payoffs were denominated in points that were converted to US dollars at a pre-announced rate.⁴² Each subject earned the sum of their earnings across all elections. All decisions took place through computers, using the Multistage experimental software program. The experiments were conducted in January and April of 2011, and subjects were registered students at Caltech.⁴³ Each session lasted about forty five minutes and subjects earned between eleven and seventeen dollars, in addition to a fixed payment for showing up on time.

4.2. Experimental Results. Table 2 summarizes the observed turnout rates by treatment. The table reports turnout by party and also total turnout for each experimental treatment. The last three columns give the equilibrium turnout levels. The table also reports standard errors clustered at the individual voter level.

N_A	N_B	c_{\max}	Rule	$\hat{\alpha}$	$\hat{\beta}$	\hat{T}	α^*	β^*	T^*
4	5	.3	M	0.622 (.042)	0.636 (.056)	0.630 (.048)	.60	.72	.67
3	6	.3	M	0.513 (.050)	0.520 (.073)	0.52 (.063)	.51	.52	.52
2	7	.3	M	0.490 (.063)	0.360 (.071)	0.39 (.060)	.45	.40	.41
4	5	.55	M	0.479 (.024)	0.451 (.034)	0.464 (.020)	.46	.45	.46
3	6	.55	M	0.436 (.021)	0.398 (.031)	0.411 (.019)	.41	.37	.39
2	7	.55	M	0.330 (.046)	0.284 (.037)	0.294 (.028)	.38	.30	.32
4	5	.3	P	0.547 (.025)	0.486 (.020)	0.51 (.016)	.48	.43	.45
3	6	.3	P	0.547 (.054)	0.465 (.061)	0.49 (.042)	.55	.39	.45
2	7	.3	P	0.600 (.040)	0.421 (.049)	0.461 (.041)	.67	.36	.43
4	5	.55	P	0.362 (.026)	0.370 (.039)	0.367 (.020)	.35	.31	.33
3	6	.55	P	0.477 (.037)	0.305 (.033)	0.362 (.027)	.40	.29	.32
2	7	.55	P	0.515 (.029)	0.320 (.037)	0.363 (.032)	.48	.26	.31

Table 2. Observed turnout rates. Clustered standard errors in parenthesis.

Using these turnout data, we turn to the five hypotheses generated by the theoretical equilibrium turnout levels. Recall that there are five main theoretical predictions from the pivotal voter model about differences between turnout in the M and P voting systems in the elections we study. We go through each of these briefly below.

⁴²Each point was equal to \$.01.

⁴³Data for the high cost M 5-4 and 6-3 elections are from Levine & Palfrey (2007).

- H1 **For the larger party, turnout is higher in M than P.** We find support for this hypothesis except for the extreme landslide elections ($N_A = 2$) where turnout rates are slightly higher in P than M. Thus the theory is supported in four out of six paired comparisons.
- H2 **Total turnout is higher under M than under P.** We find support for this hypothesis except for the extreme landslide elections ($N_A = 2$) where turnout rates are slightly higher in P than M. However, the low cost elections are the one exception where turnout is predicted to be higher in P than M. Thus the theory is supported in five out of six paired comparisons.
- H3 **The competition effect is reversed for the smaller party under P. That is, for the smaller party, turnout decreases as their share of the electorate increases. Under P for the majority party, as well as under M, the usual competition effect applies to both parties: elections that are expected to be closer lead to higher turnout.** We measure the competition effect as the difference in turnout between the 5-4 and 6-3 elections and the difference in turnout between the 6-3 and 7-2 elections. The sign is correctly predicted in all cases (sixteen out of sixteen comparisons).
- H4 **The competition effect on total expected turnout is larger in the M elections than the P elections.** This is exactly what we find in the data. The sign is correctly predicted in all four cases (four out of four comparisons). The competition effect differences are reported in Table 3. For example, for the low cost elections ($c_{\max} = .33$) turnout declines by 38% under the M rule, (from 0.63 to 0.39) and only 10% under the P rule (0.51 to 0.46) in toss-up elections (5-4) compared to the landslide elections (7-2). In the high cost elections ($c_{\max} = .33$) the contrast is even sharper: 37% decline under the M rule, (from 0.46 to 0.29) and less than 3% decline under the P rule (0.37 to 0.36).

<i>Comparison</i>	c_{\max}	M	P
$\hat{T}_{5/4} - \hat{T}_{6/3}$	0.30	0.11	0.02
$\hat{T}_{6/3} - \hat{T}_{7/2}$	0.30	0.13	0.03
$\hat{T}_{5/4} - \hat{T}_{7/2}$	0.30	0.24	0.05
$\hat{T}_{5/4} - \hat{T}_{6/3}$	0.55	0.04	0.01
$\hat{T}_{6/3} - \hat{T}_{7/2}$	0.55	0.13	0.00
$\hat{T}_{5/4} - \hat{T}_{7/2}$	0.55	0.17	0.01

Table 3. Competition Effect M vs. P

H5 In all P elections we study, there is an underdog effect. There is an underdog effect in all M elections, except for the predicted reverse underdog effects in the low cost 5-4 and 6-3 M elections. Thus, all of our underdog hypotheses have support in the data. We find that in all P elections there is an underdog effect, with one exception where the difference is less than one percentage point ($\hat{\alpha} = .362, \hat{\beta} = .370$). That one exception is the 5-4 high cost treatment, where theory predicts the smallest effect (less than four percentage points). In the M elections, all predicted underdog and reverse underdog effects are observed in the data. (eleven of twelve comparisons)

Thus, the comparative statics are correctly predicted in 40 out of 44 paired comparisons, with many of these differences significant at the 5% or 10% level using individual-level clustered standard errors. To illustrate how close the equilibrium turnout rates are to the equilibrium turnout rates, the figure below presents a scatter plot of the observed vs. equilibrium turnout rates. A perfect fit of the data to the theory would have all the points lined up along the 45% degree line. A simple OLS regression of the observed turnout on equilibrium turnout, using the 36 points in the graph gives a slope of .815, an intercept of .097 and an R-squared equal to .871. The theoretical model slightly underestimates turnout when the model prediction is below 50% and over-estimates turnout when the model prediction is over 50%, consistent with the findings of Levine and Palfrey (2007) on their much larger data set for plurality elections.

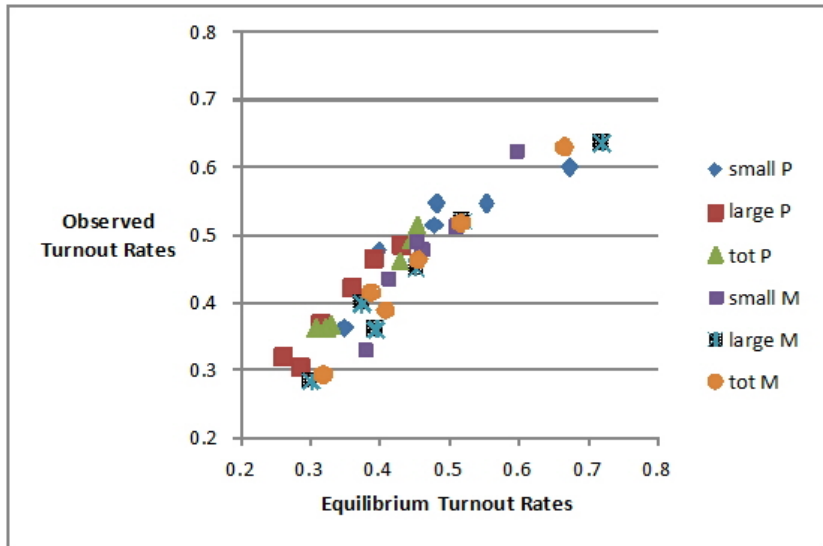


Figure 6: Scatter plot of observed vs. equilibrium turnout rates.

5. CONCLUDING REMARKS

For any distributions of partisan preferences and voting costs, we have shown that turnout (of rational voters as well as of ethical voters and of mobilized voters) depends on the degree of proportionality of influence in the institutional system in a clear way: higher turnout in a winner-take-all system than in a proportional power sharing system when the population is evenly split in terms of partisan preferences, and vice versa when one party's position has a clear majority of support.

In all of the different models we analyze and in all of the range of power sharing systems we consider $\gamma \in [1, \infty)$, only partial (or zero) underdog compensation occurs, which guarantees that the ex-ante favorite party obtains the higher vote share in a large election. Hence, in a winner-take-all system it is quite clear that the underdog cannot win a large election, which greatly discourages the contest, but with power sharing there is no absolute winner and some competition remains.

The theoretical results are robust to a wide range of alternative assumptions about the voting game and about the rationality of the voters. The common feature of all the various models considered in this paper is that with heterogeneity of voting costs full underdog compensation is not possible, and hence the majority party is expected to maintain a considerable advantage and winning margin in the election. The small probability of victory for the minority, i.e. the low competitiveness of the electoral race, depresses significantly the incentives to turn out in the winner take all system except in the special case in which parties have equal support. Whereas

in a power sharing system the incentives to vote or to mobilize voters are affected to a much lesser extent by the competitiveness or the expected closeness of the electoral race.

Our prediction that for the larger party, turnout is higher in a winner-take-all system than in a proportional power sharing system was confirmed by the experimental analysis, as well as most of the other predictions concerning differences in the competition and underdog effects: in particular, it is interesting that the competition effect is reversed for the smaller party in the proportional system. That is, for the smaller party, turnout decreases as their share of the electorate increases. With a winner-take-all system, the usual competition effect applies to both parties: elections that are expected to be closer lead to higher turnout. The prediction that competition effect on total expected turnout is negligible in a proportional system also found strong support.

6. APPENDIX A: PROOFS

Proof of Lemma. 1 We first show that $\lim_{N \rightarrow \infty} \alpha_N = \lim_{N \rightarrow \infty} \beta_N = 0$. Define the modified Bessel functions of the first kind (see Abramowitz and Stegun, 1965), as

$$I_0(z) := \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!} \frac{\left(\frac{z}{2}\right)^k}{k!}, \quad I_1(z) := \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!} \frac{\left(\frac{z}{2}\right)^{k+1}}{(k+1)!}$$

Defining $x := qN\alpha$, $y := (1-q)N\beta$, $z := 2\sqrt{xy}$, the benefits of voting (B_M^A, B_M^B) can be written as

$$B_M^A = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{e^{-x} x^k}{k!} \right) \left(\frac{e^{-y} y^k}{k!} \right) \left(1 + \frac{y}{k+1} \right) = \frac{e^{-x} e^{-y}}{2} \left(I_0(2\sqrt{xy}) + \sqrt{\frac{y}{x}} I_1(2\sqrt{xy}) \right)$$

$$B_M^B = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{e^{-x} x^k}{k!} \right) \left(\frac{e^{-y} y^k}{k!} \right) \left(1 + \frac{x}{k+1} \right) = \frac{e^{-x} e^{-y}}{2} \left(I_0(2\sqrt{xy}) + \sqrt{\frac{x}{y}} I_1(2\sqrt{xy}) \right)$$

For large z the modified Bessel functions are asymptotically equivalent and approximate to (Abramowitz and Stegun, 1965)⁴⁴

$$I_0(z) \simeq I_1(z) \simeq \frac{e^z}{2\pi z}$$

For any exogenously fixed $(\alpha, \beta) \in (0, 1]^2$ x and y go to infinity as N goes to infinity, so we can approximate the benefits of voting for large N as

$$B_M^A \simeq e^{-x-y+2\sqrt{xy}} \frac{\sqrt{x} + \sqrt{y}}{4\sqrt{\pi} \sqrt{\sqrt{xy}} \sqrt{x}} \frac{1}{\sqrt{x}}, \quad B_M^B \simeq e^{-x-y+2\sqrt{xy}} \frac{\sqrt{x} + \sqrt{y}}{4\sqrt{\pi} \sqrt{\sqrt{xy}} \sqrt{y}} \frac{1}{\sqrt{y}}$$

As a consequence for any given $(\alpha, \beta) \in (0, 1]^2$ the benefits of voting vanish as N grows, namely $\lim_{N \rightarrow \infty} B_M^A(\alpha, \beta) = 0$, $\lim_{N \rightarrow \infty} B_M^B(\alpha, \beta) = 0$. Now consider (α, β) as endogenous, i.e. solutions to the pair of equations, $B_M^A(\alpha, \beta) = F^{-1}(\alpha)$, $B_M^B(\alpha, \beta) = F^{-1}(\beta)$. Since F and F^{-1} are increasing and continuous with $F(0) = 0$, then $B_M^A(\alpha, \beta) = F^{-1}(\alpha)$ implies $\lim_{N \rightarrow \infty} \alpha_N = 0$. Likewise, we have $\lim_{N \rightarrow \infty} \beta_N = 0$. Next, we show that $\lim_{N \rightarrow \infty} N\alpha_N = \lim_{N \rightarrow \infty} N\beta_N = \infty$, $\lim_{N \rightarrow \infty} \frac{\alpha_N}{\beta_N} \in (0, \infty)$. Suppose $\lim_{N \rightarrow \infty} N\alpha_N < \infty$ and $\lim_{N \rightarrow \infty} N\beta_N < \infty$, then $\lim_{N \rightarrow \infty} B_M^A(\alpha_N, \beta_N) > 0$ and any solution to $B_M^A(\alpha, \beta) = F^{-1}(\alpha)$ would imply $\lim_{N \rightarrow \infty} \alpha_N > 0$, which contradicts $\lim_{N \rightarrow \infty} \alpha_N = 0$. Suppose $\lim_{N \rightarrow \infty} N\alpha_N = \infty$ and $\lim_{N \rightarrow \infty} N\beta_N < \infty$, then $\lim_{N \rightarrow \infty} \frac{\alpha_N}{\beta_N} = \infty$ which implies (using a Taylor expansion of F^{-1} on the numerator and the denominator around zero) that $\lim_{N \rightarrow \infty} \frac{F^{-1}(\alpha_N)}{F^{-1}(\beta_N)} = \infty$.

For all N we have

$$\frac{B_M^A(\alpha_N, \beta_N)}{B_M^B(\alpha_N, \beta_N)} = \frac{F^{-1}(\alpha_N)}{F^{-1}(\beta_N)}$$

⁴⁴ $X(z) \simeq Y(z)$ means that $\lim_{z \rightarrow \infty} \frac{X(z)}{Y(z)} = 1$.

Taking the limit on one side we have

$$L := \lim_{N \rightarrow \infty} \frac{B_M^A(\alpha_N, \beta_N)}{B_M^B(\alpha_N, \beta_N)} = \lim_{\frac{x}{y} \rightarrow \infty} \frac{I_0(2\sqrt{xy}) + I_1(2\sqrt{xy}) \sqrt{\frac{y}{x}}}{I_0(2\sqrt{xy}) + I_1(2\sqrt{xy}) \sqrt{\frac{x}{y}}} \leq 1$$

In fact, $L \leq 1$ if $\lim_{\frac{x}{y} \rightarrow \infty} \frac{I_1(2\sqrt{xy})}{I_0(2\sqrt{xy})} = 0$ and $L = 0$ if $\lim_{\frac{x}{y} \rightarrow \infty} \frac{I_1(2\sqrt{xy})}{I_0(2\sqrt{xy})} \in (0, +\infty]$. So we have a contradiction as $L \leq 1$ cannot be equal to $\lim_{N \rightarrow \infty} \frac{F^{-1}(\alpha_N)}{F^{-1}(\beta_N)} = \infty$. The same argument shows that it cannot be the case that $\lim_{N \rightarrow \infty} N\alpha_N < \infty$ and $\lim_{N \rightarrow \infty} N\beta_N = \infty$. The above arguments also imply that we cannot have either $\lim_{N \rightarrow \infty} \frac{\alpha_N}{\beta_N} = 0$, $\lim_{N \rightarrow \infty} \frac{\alpha_N}{\beta_N} = \infty$. \square

Proof of Lemma 2. For N large, since $\lim_{N \rightarrow \infty} N\alpha_N = \lim_{N \rightarrow \infty} N\beta_N = \infty$ we can use the asymptotic expression for the modified Bessel functions, so the system becomes

$$\begin{aligned} B_M^A &= \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{g+h}{4\sqrt{\pi}\sqrt{hg}} \frac{1}{g} + o(e^{-N}N^{-1/2}) = F^{-1}(\alpha) \\ B_M^B &= \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{g+h}{4\sqrt{\pi}\sqrt{hg}} \frac{1}{h} + o(e^{-N}N^{-1/2}) = F^{-1}(\beta) \end{aligned}$$

where $o(x)$ groups terms of higher order than x , namely for which $\lim_{N \rightarrow \infty} \frac{o(x)}{x} = 0$, and where we defined $g := \sqrt{q\alpha}$, $h := \sqrt{(1-q)\beta_M(\alpha)}$. The above system yields $\sqrt{q\alpha}F^{-1}(\alpha) = \sqrt{(1-q)\beta}F^{-1}(\beta) + o(1)$. For ease of notation in what follows we omit the term $o(1)$ and all similar o terms. But it should be clear that where needed we can use the following property: $\forall \varepsilon > 0, \exists \bar{N} : N > \bar{N} \implies o(1) < \varepsilon$. Since the function $\sqrt{\alpha}F^{-1}(\alpha)$ is increasing we can define the function $\beta := \beta_M(\alpha)$ where $\beta_M : [0, 1] \rightarrow [0, 1]$ is increasing and differentiable, with $\beta_M(0) = 0$. The system is reduced to a single equation $B_M^A(\alpha, \beta_M(\alpha)) = F^{-1}(\alpha)$.

We now show existence of a solution to the above equation by showing that the two continuous functions on either side must cross at least once.

Assume wlog $q < 1/2$. We have $\alpha \in (0, 1] \implies g < h$, and for any fixed N we have

$$\lim_{\alpha \rightarrow 0} \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{g+h}{4\sqrt{\pi}\sqrt{hg}} \frac{1}{g} > \lim_{\alpha \rightarrow 0} \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{2}{4\sqrt{\pi}\sqrt{hg}} = \infty$$

For $\alpha = 1$ we have $h > g = \sqrt{q}$, so for all N above a certain value we have

$$\frac{e^{-N(h-g)^2}}{\sqrt{N}} \left(\frac{g+h}{4\sqrt{\pi}\sqrt{hg}} \frac{1}{g} \right) < 1$$

which proves existence of a solution, because $F^{-1}(\alpha)$ is increasing and $F^{-1}(1) = 1$.

For uniqueness we need to show that the B_M^A is decreasing in α , namely that the following quantity is negative

$$\frac{d}{dg} \left(\frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{g+h}{4\sqrt{\pi}g\sqrt{hb}} \right) = \frac{e^{-N(h-g)^2}}{\sqrt{N}} \left(-2N(h-g) \frac{d(h-g)}{dh} \frac{g+h}{4\sqrt{\pi}g\sqrt{gh}} + \frac{d}{dh} \left(\frac{g+h}{4\sqrt{\pi}g\sqrt{gh}} \right) \right)$$

For large N this derivative will be negative if and only if

$$\frac{d(h-g)}{da} = \frac{\sqrt{1-q} d\beta'}{\sqrt{q} d\alpha'} - 1 > 0.$$

Denoting $\beta' := \sqrt{\beta}$, $\alpha' := \sqrt{\alpha}$, and $G(\alpha') := \alpha' F^{-1}((\alpha')^2) = \sqrt{\alpha} F^{-1}(\alpha)$, we have

$$\left(\sqrt{1-q} \right) G(\beta') = (\sqrt{q}) G(\alpha') \implies \frac{\sqrt{1-q} d\beta'}{\sqrt{q} d\alpha'} = \frac{G'(\alpha')}{G'(\beta')}$$

So we need G' to be increasing

$$G'(\alpha') = \frac{d}{d\alpha} (\sqrt{\alpha} F^{-1}(\alpha)) \frac{d\alpha}{d\alpha'} = 2 \frac{d}{d\alpha} (\alpha F^{-1}(\alpha))$$

so it suffices for $\alpha F^{-1}(\alpha)$ to be weakly convex, which is implied by $F(\alpha)$ weakly concave.

As for the size effect, note that the marginal benefit side B_P^A decreases with N for all α while the cost side remains unchanged. Hence by the implicit function theorem as we increase N we have lower α which implies lower β and in turn lower turnout, formally

$$\begin{aligned} 0 &= \frac{d(B_M^A - F^{-1})}{d\alpha} \frac{d\alpha}{dN} + \frac{d(B_M^A - F^{-1})}{dN} \\ \frac{d\alpha}{dN} &= - \frac{\frac{dB_M^A}{dN}}{\frac{d(B_M^A - F^{-1})}{d\alpha}} < 0 \implies \frac{d\beta}{dN} < 0 \implies \frac{dT_M}{dN} < 0 \end{aligned}$$

The underdog compensation is a consequence of F^{-1} being increasing, namely

$$\begin{aligned} q\alpha (F^{-1}(\alpha))^2 &= (1-q)\beta (F^{-1}(\beta))^2 \\ q < 1/2 &\iff \alpha > \beta, \quad q\alpha < (1-q)\beta \end{aligned}$$

□

Proof of Lemma 4. For given (α, β) call the expected number of voters for each party $R := qN\alpha$, $S := (1-q)N\beta$, we have

$$B_P^A = e^{-R-S} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left(\frac{R^a}{a!} \right) \left(\frac{S^b}{b!} \right) \left(\frac{a+1}{a+b+1} - \frac{a}{a+b} \right)$$

By differentiating and integrating the summands and inverting the series and integral operators we have

$$\begin{aligned} \sum_{b=0}^{\infty} \frac{S^b}{b!} \frac{a}{a+b} &= \frac{a}{S^a} \sum_{b=0}^{\infty} \int_0^S \frac{d}{dr} \left(\frac{1}{b!} r^{a+b} \right) dr = \\ &= \frac{a}{S^a} \int_0^S \sum_{b=0}^{\infty} \left(\frac{1}{b!} r^{a+b-1} \right) dr = \begin{cases} \frac{a}{S^a} \int_0^S r^{a-1} e^r dr & \text{for } a \geq 1 \\ 1/2 & \text{for } a = 0 \end{cases} \end{aligned}$$

and

$$\sum_{b=0}^{\infty} \frac{S^b}{b!} \frac{a+1}{a+b+1} = \frac{a+1}{S^{a+1}} \int_0^S r^a e^r dr$$

By inverting the series and integral operators again in the series over a , we have

$$\begin{aligned} B_P^A &= e^{-R-S} \left(\sum_{a=0}^{\infty} \frac{R^a}{a!} \left(\frac{a+1}{S^{a+1}} \int_0^S r^a e^r dr \right) - \sum_{a=1}^{\infty} \frac{R^a}{a!} \left(\frac{a}{S^a} \int_0^S r^{a-1} e^r dr \right) - \frac{1}{2} \right) \\ &= e^{-R-S} \left(\int_0^S \left(\frac{1}{S} \left(\sum_{a=0}^{\infty} \frac{\left(\frac{R}{S}r\right)^a}{a!} + \sum_{a=1}^{\infty} \frac{\left(\frac{R}{S}r\right)^a}{(a-1)!} \right) - \frac{R}{S} \sum_{a=1}^{\infty} \frac{\left(\frac{R}{S}r\right)^{a-1}}{(a-1)!} \right) e^r dr - \frac{1}{2} \right) \\ &= e^{-R-S} \left(\frac{1}{S^2} \int_0^S e^{(1+\frac{R}{S})r} (S - RS + Rr) dr - \frac{1}{2} \right) \\ &= \frac{S}{(R+S)^2} - \frac{e^{-(R+S)}}{(R+S)^2} \frac{S^2 - R^2 + S}{2} \end{aligned}$$

and by symmetry

$$B_P^B(R, S) = B_P^A(S, R)$$

□

Proof of Lemma 5. We first show that

$$\lim_{N \rightarrow \infty} \alpha_N = \lim_{N \rightarrow \infty} \beta_N = 0$$

For any fixed $\alpha > 0$ and $\beta > 0$, by inspection of the closed form expression (2) we see that $\lim_{N \rightarrow \infty} B_P^A(\alpha, \beta) = \lim_{N \rightarrow \infty} B_P^B(\alpha, \beta) = 0$, so the same argument obtained in Lemma (1) for the M system applies. Next, we show that $\lim_{N \rightarrow \infty} N\alpha_N = \lim_{N \rightarrow \infty} N\beta_N = \infty$, $\lim_{N \rightarrow \infty} \frac{\alpha_N}{\beta_N} \in (0, \infty)$. Summing the two P system equations we have

$$\frac{1}{NT} \left(1 - \frac{e^{-NT}}{2} \right) = F^{-1}(\alpha) + F^{-1}(\beta)$$

Since the RHS goes to zero the LHS will too, which means that NT must go to infinity so we cannot have both $\lim_{N \rightarrow \infty} N\alpha_N < \infty$ and $\lim_{N \rightarrow \infty} N\beta_N < \infty$. For N large, since the exponential terms e^{-NT} in (2) vanish faster than the hyperbolic terms, the system approximates to

$$(8) \quad \frac{(1-q)\beta}{NT^2} = F^{-1}(\alpha), \quad \frac{q\alpha}{NT^2} = F^{-1}(\beta)$$

Suppose $\lim_{N \rightarrow \infty} N\alpha_N = \infty$ and $\lim_{N \rightarrow \infty} N\beta_N < \infty$, then $\lim_{N \rightarrow \infty} \frac{\alpha_N}{\beta_N} = \infty$ which implies (using a Taylor expansion of F^{-1} on the numerator and the denominator around zero) that $\lim_{N \rightarrow \infty} \frac{F^{-1}(\alpha_N)}{F^{-1}(\beta_N)} = \infty$. From (8) we have

$$\frac{1-q}{q} \frac{\beta_N}{\alpha_N} = \frac{F^{-1}(\alpha_N)}{F^{-1}(\beta_N)}$$

so we reach a contradiction as the above equality cannot hold as $N \rightarrow \infty$. The same argument shows that it cannot be the case that $\lim_{N \rightarrow \infty} N\alpha_N < \infty$ and $\lim_{N \rightarrow \infty} N\beta_N = \infty$.

The above arguments also imply that we cannot have either $\lim_{N \rightarrow \infty} \frac{\alpha_N}{\beta_N} = 0$ or $\lim_{N \rightarrow \infty} \frac{\alpha_N}{\beta_N} = \infty$. □

Proof of Lemma 6. The approximated system (8) yields

$$\begin{aligned} q\alpha F^{-1}(\alpha) &= (1-q)\beta F^{-1}(\beta) + o(1) \\ q < 1/2 &\iff \alpha > \beta \end{aligned}$$

For ease of notation in what follows we omit the term $o(1)$ and all similar terms. But it should be clear that where necessary we will use the following property: $\forall \varepsilon > 0, \exists \bar{N} : N > \bar{N} \implies o(1) < \varepsilon$. Since the function $\alpha F^{-1}(\alpha)$ is increasing we can define the function $\beta_P(\alpha) : [0, 1] \rightarrow [0, 1]$ to be an increasing differentiable function with $\beta_P(0) = 0$. We now reduced the P system to one equation

$$B_P^A := \frac{(1-q)\beta_P(\alpha)}{NT^2} = F^{-1}(\alpha)$$

which we now show has one and only one solution.

The cost side $F^{-1}(\alpha)$ is increasing from 0 to 1. Uniqueness comes from the fact that the benefit side decreases in α as its derivative is proportional to

$$\begin{aligned} \frac{\partial B_P^A}{\partial \alpha} &\propto [\beta'_P(\alpha)(q\alpha + (1-q)\beta_P(\alpha)) - 2\beta_P(\alpha)(q + (1-q)\beta'_P(\alpha))] \\ &= -[(1-q)\beta - q\alpha]\beta'_P(\alpha) + 2q\beta_P(\alpha) < 0 \end{aligned}$$

as

$$\alpha > \beta \implies q\alpha < q\alpha \frac{F^{-1}(\alpha)}{F^{-1}(\beta)} = (1-q)\beta$$

Existence comes from the fact that for α approaching zero the benefit diverges as for any fixed N we have

$$\lim_{\alpha \rightarrow 0} \frac{1}{N} \frac{(1-q)\beta_P(\alpha)}{(q\alpha + (1-q)\beta_P(\alpha))^2} > \lim_{\alpha \rightarrow 0} \frac{1}{N} \frac{(1-q)\beta_P}{\alpha} = \infty$$

because

$$\lim_{\alpha \rightarrow 0} \frac{\beta_P}{\alpha} = \lim_{\alpha \rightarrow 0} \frac{q}{1-q} \frac{F^{-1}(\alpha)}{F^{-1}(\beta_P)} > \frac{q}{1-q} > 0$$

and for $\alpha = 1$ we have eventually (i.e. for all N above a certain value),

$$\frac{1}{N} \left(\frac{(1-q)\beta_P(1)}{(q + (1-q)\beta_P(1))^2} \right) < F^{-1}(1) = 1$$

Hence a unique solution $(\alpha_P, \beta_P(\alpha_P))$ exists for the equilibrium problem.

The proofs for the size effect and the underdog compensation effect are analogous to the ones obtained in the M system. \square

Proof of Proposition 7. First, we compare turnouts. Assuming the cost side $F^{-1}(\alpha)$ is the same in the two systems, it suffices to show that the benefit sides of the equations determining the equilibrium α are ranked.

For any $q \neq 1/2$ we need to show that eventually (i.e. for any N above a given \bar{N}) we have

$$B_M^A(\alpha, \beta_M(\alpha)) < B_P^A(\alpha, \beta_P(\alpha)), \quad \text{for all } \alpha \in (0, 1]$$

namely

$$e^{-N(\sqrt{q\alpha} - \sqrt{(1-q)\beta_M})^2} \sqrt{N} < \frac{(1-q)\beta_P}{(q\alpha + (1-q)\beta_P)^2} \left(\frac{\sqrt{q\alpha} + \sqrt{(1-q)\beta_M}}{4\sqrt{\pi}(q(1-q)\alpha\beta_M)^{1/4}} \frac{1}{\sqrt{q\alpha}} \right)^{-1}$$

which is satisfied as LHS above converges to zero, whereas the RHS is a positive constant for all $\alpha \in (0, 1]$ because

$$\begin{aligned} \alpha \in (0, 1] &\implies \beta_P \in (0, 1], \quad \beta_M \in (0, 1] \\ q \neq 1/2 &\implies \sqrt{q\alpha} \neq \sqrt{(1-q)\beta_M(\alpha)} \end{aligned}$$

Hence, for any eventually we have

$$q \neq 1/2 \implies \alpha_M < \alpha_P$$

The symmetry property $\beta(q) = \alpha(1 - q)$ (which holds in both the M and P systems) implies

$$q \neq 1/2 \implies \beta_M < \beta_P$$

hence

$$q \neq 1/2 \implies T_M < T_P$$

For $q = 1/2$ we have $\alpha = \beta$ in both P and M systems. We need to show that eventually

$$B_M^A > B_P^A, \quad \alpha \in (0, 1]$$

namely

$$\frac{1}{\sqrt{N}} \left(\frac{2\sqrt{q\alpha}}{4\sqrt{\pi}} \right) \frac{1}{q\alpha} > \frac{1}{N} \left(\frac{q\alpha}{2(2q\alpha)^2} \right)$$

Rearranging we have

$$\sqrt{N} \left(\frac{1}{2\sqrt{\pi}\sqrt{q\alpha}} \right) > \left(\frac{1}{8q\alpha} \right)$$

which is satisfied as the RHS is a positive constant and the LHS increases to infinity. Hence

$$q = 1/2 \implies \alpha_M > \alpha_P \implies T_M > T_P$$

Next, we compare underdog compensation effects. Given that for the M system we have

$$q\alpha_M (F^{-1}(\alpha_M))^2 = (1 - q)\beta_M (F^{-1}(\beta_M))^2$$

and for the P system we have

$$q\alpha_P (F^{-1}(\alpha_P)) = (1 - q)\beta_P (F^{-1}(\beta_P))$$

then

$$\frac{1 - q}{q} = \left(\frac{\alpha_P}{\beta_P} \right)^2 \left(\frac{F^{-1}(\alpha_P)}{\frac{\alpha_P}{F^{-1}(\beta_P)}} \right) = \left(\frac{\alpha_M}{\beta_M} \right)^3 \left(\frac{F^{-1}(\alpha_M)}{\frac{\alpha_M}{F^{-1}(\beta_M)}} \right)^2$$

By definition of derivative at zero we have

$$\frac{dF^{-1}}{dx} \Big|_{x=0} = \lim_{x \rightarrow 0} \frac{F^{-1}(x)}{x} \in (0, \infty)$$

For N large, α and β converge to zero both in the M and in the P system so

$$\lim_{N \rightarrow \infty} \left(\frac{F^{-1}(\alpha)}{\frac{\alpha}{F^{-1}(\beta)}} \right) = 1$$

and the result follows. If $\frac{dF^{-1}}{dx} \Big|_{x=0} \in \{0, \infty\}$ then the above limit is indeterminate and the result need not be true. If the function F^{-1} is infinitely differentiable and n is the lowest integer for which

$$\frac{d^n F^{-1}}{dx^n} \Big|_{x=0} \in (0, \infty)$$

then by iterating the procedure we have

$$\lim_{N \rightarrow \infty} \left(\frac{\frac{d^{n-1} F^{-1}(\alpha)}{d\alpha^{n-1}}}{\frac{d^{n-1} F^{-1}(\beta)}{d\beta^{n-1}}} \right) = 1$$

so the underdog compensation comparison generalizes to

$$\frac{1-q}{q} = \left(\frac{\alpha_P}{\beta_P} \right)^{n+1} = \left(\frac{\alpha_M}{\beta_M} \right)^{2n+1}$$

□

Proof of Lemma 8. Express the following series by differentiating and integrating the summands and inverting the series and integral operators

$$\begin{aligned} \sum_{b=0}^{\infty} \frac{B^b}{b!} \frac{a}{a+b+c} &= \frac{a}{B^{a+c}} \sum_{b=0}^{\infty} \int_0^B \frac{d}{dr} \left(\frac{1}{b!} \frac{r^{a+b+c}}{a+b+c} \right) dr \\ &= \frac{a}{B^{a+c}} \int_0^B \sum_{b=0}^{\infty} \left(\frac{1}{b!} r^{a+b+c-1} \right) dr = \begin{cases} \frac{a}{B^{a+c}} \int_0^B r^{a+c-1} e^r dr & \text{for } a \geq 1 \\ 1/3 & \text{for } a = c = 0 \end{cases} \end{aligned}$$

and likewise

$$\sum_{b=0}^{\infty} \frac{B^b}{b!} \frac{a+1}{a+b+1} = \frac{a+1}{B^{a+c+1}} \int_0^B r^{a+c} e^r dr$$

We compute the marginal benefit for party A by inverting the series and integral operators again over the series over a .

$$\begin{aligned} B_P^A &= e^{-(A+B+C)} \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \left(\begin{array}{l} \sum_{a=0}^{\infty} \frac{A^a}{a!} \left(\frac{a+1}{B^{a+c+1}} \int_0^B r^{a+c} e^r dr \right) \\ - \sum_{a=1}^{\infty} \frac{A^a}{a!} \left(\frac{a}{B^{a+c}} \int_0^B r^{a+c-1} e^r dr \right) \end{array} \right) - \frac{1}{3} \right) \\ &= e^{-(A+B+C)} \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \left(\begin{array}{l} \int_0^B \frac{r^c}{B^{c+1}} \left((Ar/B) e^{(Ar/B)} + e^{(Ar/B)} \right) e^r dr \\ - \int_0^B \frac{r^{c-1}}{B^c} \left((Ar/B) e^{(Ar/B)} \right) e^r dr \end{array} \right) - \frac{1}{3} \right) \end{aligned}$$

Inverting the series and integral operators again over the series over c .

$$\begin{aligned} B_P^A &= e^{-(A+B+C)} \left(\begin{array}{l} \int_0^B \left((Ar/B) e^{(Ar/B)} + e^{(Ar/B)} \right) \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \frac{r^c}{B^{c+1}} \right) e^r dr \\ - \int_0^B \left((Ar/B) e^{(Ar/B)} \right) \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \frac{r^{c-1}}{B^c} \right) e^r dr - \frac{1}{3} \end{array} \right) \\ &= e^{-(A+B+C)} \left(\int_0^B \left(\left(\frac{A}{B} r e^{\frac{A+B+C}{B} r} + e^{\frac{A+B+C}{B} r} \right) \frac{1}{B} - \frac{A}{B} e^{\frac{A+B+C}{B} r} \right) dr - \frac{1}{3} \right) \end{aligned}$$

Computing the integral and simplifying, we have

$$\begin{aligned}
B_P^A &= e^{-(A+B+C)} \left(\left(\left(AB \left(\frac{1-e^{A+B+C}}{(A+B+C)^2} + \frac{e^{A+B+C}}{A+B+C} \right) + \left(B \frac{e^{A+B+C}-1}{A+B+C} \right) \right) \frac{1}{B} \right) - \frac{1}{3} \right) \\
&= \left(1 - \frac{A}{A+B+C} \right) \frac{1 - e^{-(A+B+C)}}{A+B+C} + \left(\frac{A}{A+B+C} - \frac{1}{3} \right) e^{-(A+B+C)}
\end{aligned}$$

□

Proof of Proposition 9. A similar calculation gives the analogous result for r parties:

$$B_P^A(r) = \left(\left(1 - \frac{A}{A+B+C+\dots+r} \right) \frac{1 - e^{-(A+B+C+\dots+r)}}{A+B+C+\dots+r} + \left(\frac{A}{A+B+C+\dots+r} - \frac{1}{r} \right) e^{-(A+B+C+\dots+r)} \right)$$

For large enough N , B_P^A approximates to

$$\begin{aligned}
B_P^A &\simeq \left(1 - \frac{A}{A+B+C+\dots+r} \right) \frac{1}{A+B+C+\dots+r} \\
&= \left(\frac{\beta q_B + \gamma q_C + \dots}{(\alpha q_A + \beta q_B + \gamma q_C + \dots)^2} \right) \frac{1}{N}
\end{aligned}$$

so the benefit still decreases as N^{-1} , which implies a higher turnout than in M except in the case when the two parties in M have the same ex-ante support: $q = 1/2$.

For r parties with equal ex-ante support we have

$$q_A = q_B = q_C = \dots = q_r = 1/r \quad \implies \quad \alpha = \beta = \gamma = \dots$$

the first order condition for a party becomes

$$\left(1 - \frac{1}{r} \right) \frac{1 - e^{-\alpha_r N}}{\alpha_r N} \approx \left(1 - \frac{1}{r} \right) \frac{1}{\alpha_r N} = F^{-1}(\alpha_r)$$

so the turnout for that party α_r increases in r . Overall turnout increases too as in this symmetric case we have.

$$T_r = \alpha_r$$

□

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Table E1 displays the estimated logit QRE turnout rates. The estimated value of $\hat{\lambda}$ is 7 for the the M data and 17 for the P data. There is essentially no change in the estimated QRE turnout rates if $\hat{\lambda}$ is constrained to be equal in both treatments.

N_A	N_B	c_{\max}	Rule	$\hat{\alpha}$	$\hat{\beta}$	\hat{T}	$\alpha_{\hat{\lambda}}^*$	$\beta_{\hat{\lambda}}^*$	$T_{\hat{\lambda}}^*$
4	5	.3	M	0.622 (.042)	0.636 (.056)	0.630 (.048)	0.61	0.65	0.63
3	6	.3	M	0.513 (.050)	0.520 (.073)	0.52 (.063)	0.51	0.52	0.52
2	7	.3	M	0.490 (.063)	0.360 (.071)	0.39 (.060)	0.44	0.42	0.42
4	5	.55	M	0.479 (.024)	0.451 (.034)	0.464 (.020)	0.48	0.45	0.46
3	6	.55	M	0.436 (.021)	0.398 (.031)	0.411 (.019)	0.44	0.40	0.41
2	7	.55	M	0.330 (.046)	0.284 (.037)	0.294 (.028)	0.33	0.28	0.29
4	5	.3	P	0.547 (.025)	0.486 (.020)	0.51 (.016)	0.48	0.44	0.46
3	6	.3	P	0.547 (.054)	0.465 (.061)	0.49 (.042)	0.55	0.40	0.45
2	7	.3	P	0.600 (.040)	0.421 (.049)	0.461 (.041)	0.65	0.37	0.43
4	5	.55	P	0.362 (.026)	0.370 (.039)	0.367 (.020)	0.35	0.32	0.33
3	6	.55	P	0.477 (.037)	0.305 (.033)	0.362 (.027)	0.40	0.29	0.33
2	7	.55	P	0.515 (.029)	0.320 (.037)	0.363 (.032)	0.48	0.27	0.32

Table 4. Observed turnout rates. Clustered standard errors in parenthesis.

The scatter plot of the QRE turnout rates against the observed turnout rates is given below. Note that the slope has increased from 0.82 to 0.89, the constant term has decreased from 0.10 to 0.07 and the R^2 has increased from .87 to .91.

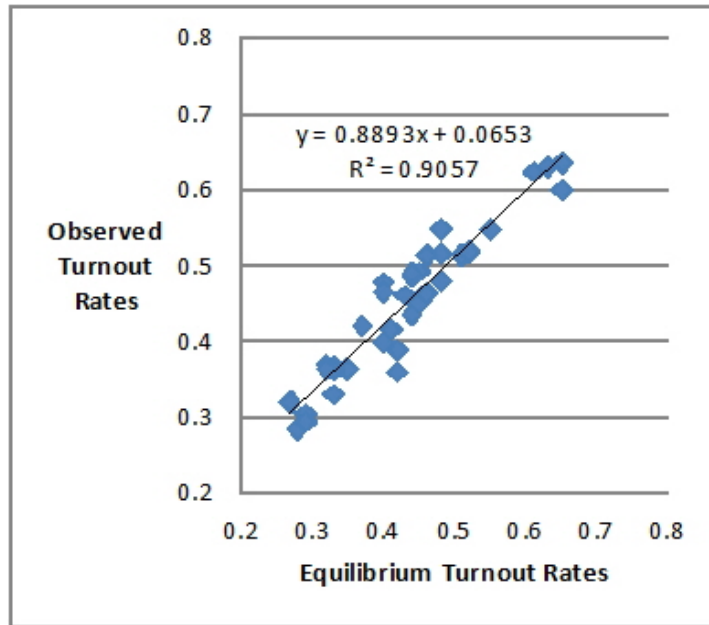


Figure 7: Turnout Rates

7. APPENDIX C: ONLINE SUPPLEMENTARY MATERIAL. NOT FOR PUBLICATION.

EXPERIMENT INSTRUCTIONS

Thank you for agreeing to participate in this decision making experiment. During the experiment we require your complete, undistracted attention, and ask that you follow instructions carefully. You may not open other applications on your computer, chat with other students, or engage in other distracting activities, such as using your phone, reading books, etc. You will be paid for your participation in cash, at the end of the experiment. Different participants may earn different amounts. What you earn depends partly on your decisions, partly on the decisions of others, and partly on chance. The entire experiment will take place through computer terminals, and all interaction between you will take place through the computers. It is important that you not talk or in any way try to communicate with other participants during the experiments. During the instruction period, you will be given a complete description of the experiment and will be shown how to use the computers. If you have any questions during the instruction period, raise your hand and your question will be answered out loud so everyone can hear. If you have any questions after the experiment has begun, raise your hand, and an experimenter will come and assist you.

We will begin with a brief practice session to help familiarize you with the computer interface. The practice rounds will be followed by 2 different paid sessions. Each paid session will consist of 50 rounds. At the end of the last paid session, you will be paid the sum of what

you have earned in all rounds of the two paid sessions, plus the show-up fee of \$5.00. Everyone will be paid in private and you are under no obligation to tell others how much you earned. Your earnings during the experiment are denominated in POINTS. Your DOLLAR earnings are determined by multiplying your earnings in POINTS by a conversion rate. In this experiment, the conversion rate is 0.002, meaning that 100 POINTS is worth 20 cents.

We will now go through two practice rounds to explain the rules for the first part of the experiment, and will explain the screen display. During the practice rounds, please do not hit any keys until I tell you, and when you are prompted by the computer to enter information, please wait for me to tell you exactly what to enter. You are not paid for these practice rounds.

[AUTHENTICATE CLIENTS]

Please pull out your dividers. Please double click on the icon on your desktop that says MULTISTAGE CLIENT. When the computer prompts you for your name, type your First and Last name. Then click SUBMIT and wait for further instructions.

SCREEN 1 (user interface)

[Point out while reading the following.]

You now see the first screen of the experiment on your computer. It should look similar to this screen. Please do not do anything with your mouse yet, until I have finished explaining the screen. [POINT TO PPT SLIDE DISPLAYED ON SCREEN IN FRONT OF ROOM]

Here are the instructions for the first part of the experiment. At the top of the screen will be your id number. Each of you has been assigned to one of two groups, called the ALPHA GROUP and the BETA GROUP. The ALPHA group always has 2 members and the BETA group always has 7 members. The screen informs you which group you will be in and reminds you how many members are in each group.

Each of you will be asked to choose either “X” or “Y” by clicking on a button with the mouse. Please wait and don’t do anything yet.

The sample display in front of the room shows you what the screen looks like for a member of the Alpha group. The screen also tells you what your “Y bonus” is. This is an extra bonus you earn if you choose Y instead of X, independent of what other participants choose.

Your earnings are computed in the following way. It is very important that you understand this, so please listen carefully.

SCREEN 2

[Point while reading.] First suppose you choose X. To compute your earnings, we compare the number of members of your group choosing “X” to the number of members of the other

group choosing X. Your payoff is 105 if the number of members in your group choosing X is greater than the number of members of the other group who choose X. Your payoff is 55 if the number of members in your group choosing X is equal to the number of members of the other group who choose X. Your payoff is 5 if the number of members in your group choosing X is fewer than the number of members of the other group who choose X.

Your earnings are computed slightly differently if you choose Y. Specifically, in addition to the above earnings (either 105, 55, or 5) you also earn your Y bonus. This payoff information is displayed in a table on your screen.

The amount of each participant's Y-bonus is assigned completely randomly by the computer at the beginning of each round and is shown in the second line down from the top of the screen. Y-bonuses are assigned separately for each participant, so different participants will typically have different Y-bonuses. What you see up the front is just an example of one participant's Y-bonus. In any given round you will have an equal chance of being assigned any Y-bonus between 0 and 30 points. Your Y-bonus in each round will not depend on your Y-bonus or decisions in previous rounds, or on the Y-bonuses and decisions of other participants. While you will be told your own Y-bonus in each round before making a decision, you will never be told the Y-bonuses of other participants. You will only know that each of the other participants has a Y-bonus that is some number between 0 and 30.

At this time, if your ID number is even, please click on row label Y; if your ID number is odd, please click on the row label X. Once everyone has made their selection, the results from this first practice round are displayed on your screen. It will look like

SCREEN 3

if your choice was X, and

SCREEN 4

if your choice was Y.

This completes the first practice round, and you now see a screen like this. The bottom of the screen contains a history panel. This panel will be updated to reflect the history of all previous rounds. [go over columns of history screen]

At the beginning of every new round you will be randomly re-assigned to new groups, and will have the opportunity to choose between "X" and "Y." In other words, you will not necessarily be in the same group during each round. You will also be randomly reassigned a new Y-bonus at the beginning of each round.

We will now go to a second practice round. When this practice round is over, an online quiz will appear on your screen. Everyone must answer all the questions correctly before we can proceed to the paid rounds. Does anyone have any question?

Please take note of your new group assignment, alpha or beta, since the group assignments are shuffled randomly between each round. Also, please take note of your new Y-bonus, which has been randomly redrawn between the values of 0 and 30.

[SLIDE 4]

GO TO NEXT MATCH

Please make your decision now by clicking on the row label X or Y.

A quiz is now displayed on your screen. Please read each question carefully and select the correct answer. Once everyone has answered all the questions correctly, you may all go on to the second page of the quiz. After everyone has correctly answered the second page of questions, we will begin the first paid session. If you have any questions as you are completing the quiz, please feel free to raise your hand and I will go to your workstation to answer your question.

The first paid session will follow the same instructions as the practice session. There will be a total of 50 rounds in the first paid session. Let me summarize those instructions before we start.

[Go over summary slide.] Are there any questions before we begin the first paid session? [Answer questions.] Please begin. There will be 50 rounds, and then you will receive new instructions. (Play rounds 1 – 50) The first session is now over.

SESSION 2

We will now begin session 2.

[SLIDE 5]

The second paid session will be slightly different from the first session. Let me summarize those rules before we start. Please listen carefully. The rules are the same as before with only one exception. In each round of this session, you will have an equal chance of being assigned a Y-bonus between 0 and 55 points. Again, our Y-bonus in each round will not depend on your Y-bonus or decisions in previous rounds, or on the Y-bonuses and decisions of other participants. While you will be told your own Y-bonus in each round before making a decision, you will never be told the Y-bonuses of the other participants. You will only know that each of the other participants has a Y-bonus that is some number between 0 and 55. You may choose “X” or “Y”.

There will be 50 rounds in this second session. After each round, group assignments will be randomly reshuffled and everyone will be reassigned a new Y-bonus. Therefore, some rounds you will be in the Alpha group and other rounds you will be in the Beta group. In either case, everyone is told which group they are in and what their private Y-bonus is, before making a choice of X or Y.

Are there any questions before we begin the second paid session? (no quiz) Please Begin.
(Play rounds 1 – 50)

Session 2 is now over. Please record your total earnings in dollars for the experiment on your record sheet. After you have recorded your earnings, click the 'ok' button. We cannot pay anyone until everyone has recorded their earnings AND clicked the ok button. Please remain seated and you will be called up one by one according to your ID number to have your recorded earnings amount checked against our own record. Please wait patiently and do not talk or use the computers.