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Abstract

We study how learning affects an uninformed monopolist's supply and investment decisions under multiplicative uncertainty in demand. The monopolist is uninformed because it does not know one of the parameters defining the distribution of the random demand. Observing prices reveals this information slowly. We first show how to incorporate Bayesian learning into dynamic programming by focusing on sufficient statistics and conjugate families of distributions. We show their necessity in dynamic programming to be able to solve dynamic programs either analytically or numerically. This is important since it is not true that a solution to the infinite-horizon program can be found either analytically or numerically for any kinds of distributions. We then use specific distributions to study the monopolist's behavior. Specifically, we rely on the fact that the family of normal distributions with an unknown mean is a conjugate family for samples from a normal distribution to obtain closed-form solutions for the optimal supply and investment decisions. This enables us to study the effect of learning on supply and investment decisions, as well as the steady state level of capital. Our findings are as follows. Learning affects the monopolist's behavior. The higher the expected mean of the demand shock given its beliefs, the higher the supply and the lower the investment. Although learning does not affect the steady state level of capital since the uninformed monopolist becomes informed in the limit, it reduces the speed of convergence to the steady state.

1 Introduction

The evolution of capital specific to a firm plays a key role in how it develops over time as well as its optimal supply and investment decisions. Taking the price of capital used by a firm as given, as if the firm was a perfect competitor in the capital market, misses the fact that there is capital specific to a firm that can only be moved between firms at a nonlinear cost. Both human and physical capital have this property. Moreover, how well-informed a firm is about the structure of demand or the production process plays a crucial role in the dynamic analysis of the firm. For instance, a firm might be uninformed about the distribution of the random demand or the random production process. In that case, it has the opportunity to learn through experience. In other words, there is a relationship between the evolution of a firm through capital accumulation and information acquisition about the structure of demand or the production process.

The issue of investment under uncertainty without learning has been studied extensively in optimal growth. In the early literature on optimal growth, the dynamic equation governing capital formation was deterministic, see Cass [5] and Koopmans [21]. This was a natural place to begin the study of optimal growth since growth had already been studied in a deterministic environment by Ramsey [29] and the technology for studying the problem in a more general, stochastic, environment had not yet been fully developed. This was changed by the optimal growth model of Brock and Mirman [4] which built on earlier studies of positive growth under uncertainty [26, 27]. The motivation for studying stochastic rather than deterministic growth models was to reduce the information available to the economic agents in order to provide more realistic results. Indeed, in deterministic models, the economic agents are assumed to have perfect foresight in understanding the effect of their decisions on the future evolution of the dynamic system. Adding uncertainty in the dynamics means that the economic agents need not know precisely every outcome of their investment decisions, i.e., they need not know with certainty the effect of their investment decisions on the future path of the system. Although the assumption of stochastic growth is less restrictive than the original one of deterministic dynamics, it is still quite restrictive since it requires that the economic agents

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know precisely the stochastic effect of their investment decisions on the future evolution of the dynamic system, i.e., perfect foresight is replaced by rational expectations. It would be even more useful and realistic to study models in which the future outcome of present decisions is even murkier by assuming that the economic agents do not have complete knowledge of the distribution of future stochastic outcomes. For instance, suppose that the economic agents do not know about a parameter characterizing the distribution. They would then have to learn about the environment they face. Rational expectations would then be applied not only on the stochastic variables as in the economic growth literature but also on the stochastic learning process. This change in modeling would lead to a better understanding of the effect of optimal decisions on the dynamics of the economy and yield a more precise understanding of optimal saving and consumption than is currently available in economic growth models. It is natural to expect that the ideas from the growth literature play an important role as well in the study of firms faced with dynamic output decisions through the accumulation of specific capital.

There is an emerging literature studying the effect of learning in dynamic models of economic growth as well as more general dynamic models, beginning with the paper of Freixas [15], but also including the works of El-Gamal and Sundaram [12], Bertocchi and Spagat [3], and Datta et al. [7]. These studies are, in turn, based on the models of learning in which the only link between periods is beliefs. See Prescott [28], Grossman et al. [18], Easley and Kiefer [10, 11], Balvers and Cosimano [2], Aghion et al. [1], Fusselman and Mirman [16], Mirman et al. [25], Trefler [33], Creane [6], Fishman and Gandal [13], and Keller and Rady [19]. In these models, there is no *natural* dynamics and thus no possibility to study investment. To the issues studied in these *nondynamic* models, the introduction of a *natural* dynamics adds a rich and complicated set of questions and issues that have either been studied superficially in the literature or have not yet been addressed. In fact, there are many aspects that must be considered when studying the effect of learning and experimentation in dynamic models. For instance, the unknown parameter could be in the objective function, in the dynamic equation, or in both.

Although learning may be studied in the context of economic growth models, we focus

on industrial organization. Specifically, we study how learning affects the behavior of an uninformed monopolist in a dynamic model with capital that is specific to the firm, along the lines studied in Koulovatianos and Mirman [22]. The monopolist faces multiplicative uncertainty in demand and is uninformed because he does not know one of the parameters defining the distribution of the random demand. There is no uncertainty or learning from the production process for capital. Observing prices reveals this information slowly. Both active or passive learning can be studied in our model depending on the parameter unknown to the monopolist. Active learning arises when the monopolist's decisions affect the information used to learn about the unknown parameter while passive learning arises when the monopolist's decisions do not affect the information used to learn about the unknown parameter. It is the purpose of this paper to study the case of passive learning.

Incorporating learning into dynamic program brings another difficulty. It is that of modeling appropriately the distribution of the random demand along with the distribution that characterizes the prior belief about the unknown parameter of the distribution of the random demand. We discuss sufficient statistics and conjugate families of distributions and show their necessity in dynamic programming to be able to solve dynamic programs either analytically or numerically. This is important since it is not true that a solution to the infinite-horizon program can be found either analytically or numerically for any kinds of distributions. See [23] for an exception.

We then use specific distributions to study the monopolist's behavior. Specifically, we rely on the fact the family of normal distributions with an unknown mean is a conjugate family for samples from a normal distribution to obtain closed-form solution for optimal supply and investment decisions. This enables us to study the effect of learning on supply and investment decisions, as well as the steady state level of capital. In fact, we show that learning plays an important role in the optimal supply and investment decision of the firm. In our model, the demand shock is multiplicative in demand so that learning has no effect on the monopolist's behavior if there is no cost function. When there is a cost function, learning affects the monopolist's behavior. The higher the expected mean of the demand shock given its beliefs, the higher the supply and the lower the investment. Although learning does not

affect the steady state level of capital since the uninformed monopolist becomes informed in the limit, it reduces the speed of convergence to the steady state.

The paper is organized as follows. Section 2 introduces the general model. We first discuss the dynamic framework in section 2.1, then the learning framework in section 2.2. The dynamic framework is combined with the passive learning framework in section 2.3. Bayesian statistics and techniques are discussed in section 3. The effect of learning is studied in section 4. All proofs are relegated to section 5.

2 The General Model

2.1 The Dynamic Framework

Consider an infinitely-lived monopolist who makes supply and investment decisions under uncertainty in demand in order to maximize the sum of discounted expected profits subject to a deterministic law of motion for capital. The monopolist supplies $J \geq 1$ exclusive markets. In period t, the monopolist is endowed with a stock of capital k_t yielding output $f(k_t)$, from which $q_{jt} \geq 0$ is supplied to market j, $j = 1, ..., J$, and

$$k_{t+1} = f(k_t) - \zeta \sum_{j=1}^{J} q_{jt} \geq 0 \tag{1}$$

is invested. Here, $\zeta > 0$ characterizes the impact of extraction on the stock of capital in period $t+1$. The total cost of supplying $\sum_{j=1}^{J} q_{jt} \leq f(k_t)$ across J markets is $c\left(k_t, \{q_{jt}\}_{j=1}^{J}\right)$.

The monopolist faces uncertainty in demand: the price P_{jt} is a realization of the random price $\tilde{P}_{jt}$ in market j with $\tilde{P}_{jt} = g(q_{jt}, \gamma, \tilde{\varepsilon}_{jt})$, $\partial g / \partial q_{jt} < 0$, where $\gamma \in \Gamma$ is a parameter (possibly infinitely-dimensioned) and $\tilde{\varepsilon}_{jt}$ is a market and time-specific demand shock. Let $\{\tilde{\varepsilon}_{jt}\}_{j=1}^{J}$ be i.i.d. across markets and periods with p.d.f. $\phi_{\varepsilon}(\varepsilon_{jt} | \theta) > 0$, $\varepsilon_{jt} \in \Omega_{\varepsilon}$, depending on a parameter $\theta \in \Theta$, possibly infinitely-dimensioned. Hence, the distribution of $\tilde{P}_{jt}$ depends on the quantity q_{jt} supplied to market j in period t, but not on the quantity q_{sr} supplied to market s in period r, $s \neq j, r \neq t$. Since $\{\tilde{\varepsilon}_{jt}\}_{j=1}^{J}$ are i.i.d. across markets and periods, $\tilde{P}_{jt} | q_{jt}$, $j = 1, ..., J$, $t = 1, ...,$ are independently and identically distributed across markets

and periods.

The monopolist's dynamic program is

$$\max_{\left\{\{q_{jt}\}_{j=1}^{J}, k_{t+1}\right\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \delta^t \left(\sum_{j=1}^{J} g\left(q_{jt}, \gamma, \tilde{\varepsilon}_{jt}\right) q_{jt} - c\left(k_t, \{q_{jt}\}_{j=1}^{J}\right) \right) \right] \qquad (2)$$

where $\delta \in [0,1]$ is the discount factor, subject to the law of motion (1) for capital.

2.2 The Learning Framework

Including parameters γ and θ in the model allows us to study the effect of learning in economic models. Three cases are distinguished. First, the monopolist is uninformed only about the value of θ. Second, it is uninformed only about the value of γ. Third, it is uninformed about the values of both γ and θ. There is a difference between learning about the value of γ and learning about the value of θ, namely, the difference between active and passive learning.[1] Loosely speaking, under active learning, the monopolist's supply and investment decisions affect the learning process while they do not under passive learning. While it is the purpose of this paper to focus on the case of passive learning, i.e., the monopolist is only uninformed about the value of $\theta = \theta^* \in \Theta$, we first discuss the difference between passive and active learning. We assume throughout this paper that the monopolist is a Bayesian learner, i.e., Bayesian methods are used to learn about the environment. In Bayesian analysis, the monopolist begins with prior knowledge expressed as a distribution on the parameter space and updates its beliefs, given the data.

The monopolist is justified using Bayesian methods if the updated beliefs becomes more accurate and precise as more data points are collected. This property is called consistency of the posterior distribution. Consistency implies that the monopolist eventually learns the true value of the unknown parameter. When the parameter space is finitely-dimensioned, consistency of the posterior distribution is obtained if and only if the value of the unknown parameter lies in the support of the parameter, see Freedman [14] and Schwartz [31]. How-

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[1]The third case in which both values of γ and θ are unknown is a combination of the first two cases.

ever, inconsistency of Bayesian procedures is quite general in non-parametric cases, e.g., if the parameter space is infinitely-dimensioned. A classical example of inconsistency is found in Freedman [14]. The issue of consistency in nonparametrics is far from resolved. See chapter 4 in Gosh and Ramamoorthi [17] for a discussion on the consistency of Bayes procedures in the nonparametrics case. To avoid issues of convergence of the posterior distribution to the true values of the unknown parameters, we assume that Γ and Θ are finitely-dimensioned.

However, incomplete learning can still arise in economic models. For example, incomplete learning occurs if actions and beliefs are intertwined, such as n-armed bandit problems, confounding action problems, and problems in which learning the exact state of the world has no economic value. See Rothschild [30], Kihlstrom et al. [20], McLennan [24], Easley and Kiefer [10], and Smith and Sørensen [32], among others. This type of incomplete learning does not arise in our class of models.

2.2.1 Active Learning

Active learning, or experimentation, arises when the monopolist's decisions affect the information used to learn about the unknown parameter.[2,3] In our model, this is the case when the value of γ is unknown. Let the unknown value of parameter γ be $\gamma^* \in \Gamma$. The monopolist begins period t with prior beliefs about γ^* characterized by the prior p.d.f. ξ_γ^t on Γ. That is, for any $X \subset \Gamma$, the monopolist's prior probability that $\gamma^* \in X$ in period t is

$$\int_X \xi_\gamma^t(\gamma)\, \mathrm{d}\gamma.$$

After supplying $\sum_{j=1}^J q_{jt}$, the monopolist observes a random sample of J prices $\{P_{jt}\}_{j=1}^J$, where P_{jt} is a realization of the random price $\tilde{P}_{jt}$ in market j, $j = 1, ..., J$, in period t. Let

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[2]Active learning has been studied in models in which the only link between periods is beliefs. See Prescott [28], Grossman et al. [18], Easley and Kiefer [10, 11], Balvers and Cosimano [2], Aghion et al. [1], Fusselman and Mirman [16], Mirman et al. [25], Trefler [33], Creane [6], Fishman and Gandal [13], and Keller and Rady [19].

[3]There is an emerging literature on the effect of active learning in dynamic models of economic growth as well as more general dynamic models, beginning with the paper of Freixas [15], but also including the works of El-Gamal and Sundaram [12], Bertocchi and Spagat [3], and Datta et al. [7].

$\phi_P\left(P_{jt} \mid q_{jt}, \gamma\right)$, $P_{jt} \in \Omega_P$, be the p.d.f. of P_{jt}, for each j and t.[4] By Bayes' theorem, the uninformed monopolist's posterior beliefs for period $t+1$ are characterized by the posterior p.d.f.

$$\xi_\gamma^{t+1}\left(\gamma \left| \{P_{jt}\}_{j=1}^J, \{q_{jt}\}_{j=1}^J\right.\right) = \frac{L_U^t\left(\{P_j\}_{j=1}^J \left| \{q_j\}_{j=1}^J, \gamma\right.\right)\xi_\gamma^t(\gamma)}{\int\limits_{y\in\Gamma} L_U^t\left(\{P_{jt}\}_{j=1}^J \left| \{q_{jt}\}_{j=1}^J, y\right.\right)\xi_\gamma^t(y)\,\mathrm{d}y}, \tag{3}$$

where

$$L_U^t\left(\{P_{jt}\}_{j=1}^J \left| \{q_{jt}\}_{j=1}^J, \gamma\right.\right) = \prod_{j=1}^J \phi_P\left(P_{jt} \mid q_{jt}, \gamma\right)$$

is the likelihood function of the random sample $\{P_{jt}\}_{j=1}^J$ in period t, for given $\{q_{jt}\}_{j=1}^J$ and $\gamma \in \Gamma$. Notice, that the monopolist's supply decisions affect the posterior p.d.f. (3) of γ. Hence, active learning or experimentation is implied. Intuitively, the monopolist's supply decisions may be adjusted to spread apart the distributions from which the prices are drawn, thus making the price more informative signals of the true distribution.[5]

2.2.2 Passive Learning

Passive learning arises when the monopolist's decisions do not affect the information used to learn about the unknown parameter.[6] This is the case when the value of θ is unknown. In our model, let the unknown value of parameter θ be $\theta^* \in \Theta$. The monopolist begins period t with prior beliefs about θ^* characterized by the prior p.d.f. ξ_θ^t. That is, for any $X \subset \Theta$, the monopolist's subjective prior probability that $\theta^* \in X$ is

$$\int_X \xi_\theta(\theta)\,\mathrm{d}\theta.$$

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[4] The distribution of $\tilde{P}_{jt}$ is derived from the distribution of $\tilde{\varepsilon}_{jt}$, that is, the p.d.f. of $\tilde{P}_{jt}$ is

$$\phi_P\left(P_{jt} \mid q_{jt}, \gamma\right) = \phi_\varepsilon\left(\varepsilon_{jt} \mid \theta\right)\left|\frac{\partial g\left(q_{jt}, \gamma, \varepsilon_{jt}\right)}{\partial \varepsilon_{jt}}\right|^{-1}$$

for $P_{jt} = q_{jt}^{-\frac{1}{\gamma}}\varepsilon_{jt} \in \Omega_P$.

[5] The effect of active learning is studied in a dynamic monopoly without investment in Mirman et al. [25].

[6] Demers [9] studies the investment decision of a perfectly competitive firm facing a random demand with an unknown mean. The firm is a passive learner not because of the structure of demand as in our class of models, but because the firm is a perfect competitor. The firm has no impact on the demand and, thus, cannot affect the information, regardless of the demand structure.

After supplying $\sum_{j=1}^{J} q_{jt}$, the monopolist observes a random sample of J prices $\{P_{jt}\}_{j=1}^{J}$, where P_{jt} is a realization of the random price $\tilde{P}_{jt}$ in market j, $j = 1, ..., J$, in period t. It then solves for $\{\varepsilon_{jt} = G(P_{jt}, q_{jt}, \gamma)\}_{j=1}^{J}$ in order to form posterior beliefs about θ^*.[7] By Bayes' theorem, the uninformed monopolist's posterior beliefs are characterized by the posterior p.d.f.

$$\xi_\theta^{t+1}\left(\theta \,\middle|\, \{\varepsilon_j\}_{j=1}^{J}\right) = \frac{L_U^t\left(\{\varepsilon_{jt}\}_{j=1}^{J} \,\middle|\, \theta\right) \xi_\theta^t(\theta)}{\int\limits_{y \in \Theta} L_U^t\left(\{\varepsilon_{jt}\}_{j=1}^{J} \,\middle|\, y\right) \xi_\theta^t(y)\, \mathrm{d}y}, \tag{4}$$

where

$$L_U^t\left(\{\varepsilon_{jt}\}_{j=1}^{J} \,\middle|\, \theta\right) = \prod_{j=1}^{J} \phi_\varepsilon\left(\varepsilon_{jt} \,\middle|\, \theta\right)$$

is the likelihood function of random sample $\{\varepsilon_{jt}\}_{j=1}^{J}$ in period t, for $\theta \in \Theta$. Notice that the monopolist's supply decisions cannot affect the posterior p.d.f. (4). That is, there is passive learning.

2.3 Only θ is Unknown

We now concentrate on incorporating the passive learning framework into the dynamic model of the firm. Suppose the monopolist does not know that the value of θ is $\theta^* \in \Theta$, but knows the value of γ. Incorporating passive learning into the monopolist's dynamic program (2) adds a stochastic law of motion for beliefs characterized by the posterior p.d.f. (4), along with the deterministic law of motion for capital (1). Note that the law of motion for beliefs (4) is autonomous, in the sense that no action of the uninformed monopolist can influence its learning.

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[7]We assume that $\partial g/\partial \varepsilon_{jt} > 0$ for $\varepsilon_{jt} \in \Omega_\varepsilon$ so that $P_{jt} = g(q_{jt}, \gamma, \varepsilon_{jt})$ is uniquely solvable for ε_{jt} as a function of P_{jt} and q_{jt}, i.e., there exists a function G such that $\varepsilon_{jt} = G(P_{jt}, q_{jt}, \gamma)$ for each $P_{jt} \in \Omega_P$ and $\gamma \in \Gamma$.

The uninformed monopolist's dynamic program is summarized by the Bellman equation:[8]

$$
\begin{aligned}
V_U\left(k; \xi_\theta\left(\theta\right), \theta \in \Theta\right) &= \max_{\{q_j \geq 0\}_{j=1}^J} \left\{ E_{\{\tilde{\varepsilon}_j\}_{j=1}^J | \xi} \left[\sum_{j=1}^J g\left(q_j, \gamma, \tilde{\varepsilon}_j\right) q_j - c\left(k, \{q_j\}_{j=1}^J\right) \right.\right.\\
&\quad \left.\left. + \delta V_U\left(\hat{k}; \hat{\xi}_\theta\left(\theta \Big| \{\tilde{\varepsilon}_j\}_{j=1}^J\right), \theta \in \Theta\right) \right] \right\}, \\
&= \max_{\{q_j \geq 0\}_{j=1}^J} \left\{ \int \cdots \int_{\Omega_\varepsilon^J} \left[\sum_{j=1}^J g\left(q_j, \gamma, \varepsilon_j\right) q_j - c\left(k, \{q_j\}_{j=1}^J\right) \right.\right.\\
&\quad \left. + \delta V_U\left(f\left(k\right) - \zeta \sum_{j=1}^J q_j; \hat{\xi}_\theta\left(\theta \Big| \{\varepsilon_j\}_{j=1}^J\right), \theta \in \Theta\right) \right] \\
&\quad \left. \cdot L_U^*\left(\{\varepsilon_j\}_{j=1}^J\right) \prod_{j=1}^J \mathrm{d}\varepsilon_j \right\},
\end{aligned}
\tag{5}
$$

where $E_{\{\tilde{\varepsilon}_j\}_{j=1}^J | \xi}$ is the expectation operator over $\{\tilde{\varepsilon}_j\}_{j=1}^J$ conditional on the prior p.d.f. ξ_θ and

$$
L_U^*\left(\{\varepsilon_j\}_{j=1}^J\right) = \int_\Theta L_U\left(\{\varepsilon_j\}_{j=1}^J | \theta\right) \xi\left(\theta\right) \mathrm{d}\theta
$$

is the joint p.d.f. of $\{\tilde{\varepsilon}_j\}_{j=1}^J$, given the prior p.d.f. ξ_θ.[9]

3   Bayesian Statistics and Techniques

In general, dynamic programs with passive learning such as (5) are intractable, i.e., they are not solvable either analytically or numerically.[10]   There are two main issues that need to be addressed. First, the value function V in expression (5) depends on the variable k and the function $\xi_\theta\left(\theta\right)$, $\theta \in \Theta$. Unless the space Θ contains a finite number of elements, the state space $(k; \xi_\theta\left(\theta\right), \theta \in \Theta)$ is infinitely-dimensioned and it is impossible to tract the evolution of beliefs via the p.d.f. $\xi_\theta\left(\theta\right)$. Second, the law of motion for beliefs characterized by the posterior p.d.f. (4) of θ, does not prevent the prior and posterior p.d.f.'s ξ_θ and $\hat{\xi}_\theta$

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[8]We remove the time subscipt t since the model is time-consistent. Variables in the subsequent period are denoted by $\hat{\ }$.

[9]The p.d.f. $L_U^*\left(\{\varepsilon_j\}_{j=1}^J\right)$ is also called the marginal likelihood function of $\{\tilde{\varepsilon}_j\}_{j=1}^J$.

[10]The problem is not whether a solution exists but whether one can characterize the solution and study it.

from belonging to different families of distributions. This would render the dynamic program intractable. As observed in the stochastic law of motion (4) for beliefs, the distributional assumption of $\{\tilde{\varepsilon}_j\}_{j=1}^{J}$ affects the posterior distribution $\hat{\xi}_\theta$. Therefore, ξ_θ and $\hat{\xi}_\theta$ may not belong to the same family of distributions for any likelihood function L of $\{\tilde{\varepsilon}_j\}_{j=1}^{J}$. In fact, most likelihood functions L of $\{\tilde{\varepsilon}_j\}_{j=1}^{J}$ yield a posterior p.d.f. $\hat{\xi}_\theta$ that is not in the family of distributions to which ξ_θ belongs. For instance, suppose that the prior p.d.f. of parameter $\tilde{\theta}$ is normal with mean μ and variance σ^2. For most likelihood functions L of $\{\tilde{\varepsilon}_j\}_{j=1}^{J}$, the posterior p.d.f. of $\tilde{\theta}$ given $\{\varepsilon_j\}_{j=1}^{J}$ is not normally distributed and the characterization of the posterior p.d.f. is, in general, intractable.

We therefore make specific assumptions about the distributions of the random sample $\{\tilde{\varepsilon}_j\}_{j=1}^{J}$ and the prior distribution ξ_θ so that the state space $(k; \xi_\theta(\theta), \theta \in \Theta)$ is finitely-dimensional and the prior and posterior beliefs belong to the same family of distributions. To that end, we focus on the class of distributions of random sample $\{\tilde{\varepsilon}_j\}_{j=1}^{J}$ that have a fixed number $s \geq 1$ of sufficient statistics. We first define the notion of sufficient statistics and present a result that asserts that if L has a sufficient statistic, then there exists a prior p.d.f. ξ_θ such that ξ_θ and $\hat{\xi}_\theta$ belong to the same family. Focusing on the class of distributions of random sample $\{\tilde{\varepsilon}_j\}_{j=1}^{J}$ that have a fixed number $s \geq 1$ of sufficient statistics addresses the issues of tractability as well as ensuring that the prior and posterior p.d.f.'s of θ are in the same family of distributions.

3.1 Sufficient Statistics

The treatment of the data is simplified if a few numerical values, or statistics, summarize the relevant information of the data. Such summaries are known as sufficient statistics. Loosely speaking, a statistic T_J is called a sufficient statistic if, for any prior distribution of $\tilde{\theta}$, its posterior distribution depends on the random sample $\{\tilde{\varepsilon}_j\}_{j=1}^{J}$, only through the statistic $T_J\left(\{\tilde{\varepsilon}_j\}_{j=1}^{J}\right)$. Formally,

Definition 1 *A statistic T_J is a sufficient statistic for the family of likelihood functions*

$\left\{ L\left(\{\tilde{\varepsilon}_j\}_{j=1}^J \,|\theta\right), \theta \in \Theta\right\}$ *if*

$$\hat{\xi}_\theta\left(\theta \,\middle|\, \{\varepsilon_j^1\}_{j=1}^J\right) = \hat{\xi}_\theta\left(\theta \,\middle|\, \{\varepsilon_j^2\}_{j=1}^J\right)$$

for any prior $\xi_\theta(\theta)$ and any two data sets $\{\varepsilon_j^1\}_{j=1}^J$ and $\{\varepsilon_j^2\}_{j=1}^J$, such that $T_J\left(\{\varepsilon_j^1\}_{j=1}^J\right) = T_J\left(\{\varepsilon_j^2\}_{j=1}^J\right)$.

Thus, $T_J\left(\{\varepsilon_j\}_{j=1}^J\right)$ is sufficient to compute the posterior distribution of $\tilde{\theta}$ from any prior distribution and any data set $\{\varepsilon_j\}_{j=1}^J$.

3.2   Conjugate Prior Distributions

If the random sample $\{\tilde{\varepsilon}_j\}_{j=1}^J$ is drawn from a family of distributions for which there is a sufficient statistic of fixed dimension, then there exists a family of distributions of $\tilde{\theta}$ that is closed under sampling. That is, if the prior distribution of $\tilde{\theta}$ belongs to a family of distributions, then for any sample size J and any values of the observations $\{\tilde{\varepsilon}_j\}_{j=1}^J$, the posterior distribution of $\tilde{\theta}$ also belongs to the same family. This family of distributions is also called a conjugate family of distributions because of the special relationship that exists between this family of distributions of the unknown parameter $\tilde{\theta}$ and the family of distributions of the observations $\{\tilde{\varepsilon}_j\}_{j=1}^J$. Formally,

Remark 2 *Whenever a family of likelihood functions $\left\{ L\left(\{\tilde{\varepsilon}_j\}_{j=1}^J \,|\theta\right), \theta \in \Theta\right\}$ has a sufficient statistic $T_J\left(\{\tilde{\varepsilon}_j\}_{j=1}^J\right)$ of fixed dimension $s \geq 1$ for every sample size J, there exists a conjugate family of distributions for the unknown parameter $\tilde{\theta}$.*

Proof. See DeGroot [8], p. 163. ∎

Many families of likelihood functions have a sufficient statistic of fixed dimensions $s \geq 1$, for every sample size J. For instance, the exponential family of distributions has a sufficient statistic of fixed dimension $s \geq 1$, independent of the size J.[11]

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[11] See chapter 9 "Conjugate Prior Distributions" in DeGroot [8] for a detailed discussion on conjugate families of distributions.

Focusing on the family of likelihood functions that have a sufficient statistic of fixed dimensions $s \geq 1$ for every sample size J ensures not only that there exists a prior p.d.f. ξ_θ such that ξ_θ and $\hat{\xi}_\theta$ belong the same family but also that ξ_θ and $\hat{\xi}_\theta$ can be characterized by a finite number $N \geq 1$ of variables that are functions of the $s \geq 1$ statistics T_J. Specifically, let $\{b_n\}_{n=1}^N$ and $\left\{ \hat{b}_n \left(\{\varepsilon_j\}_{j=1}^J \right) \right\}_{n=1}^N$ be N variables characterizing the prior and posterior p.d.f. of $\tilde{\theta}$, respectively.[12] Assuming a family of likelihood functions that have a sufficient statistic of fixed dimensions $s \geq 1$ for every sample size J allows us to rewrite dynamic program (5) for the uninformed monopolist as

$$
\begin{aligned}
V_U \left(k, \{b_n\}_{n=1}^N \right) &= \max_{\{q_j \geq 0\}_{j=1}^J} \left\{ \int \cdots \int_{\Omega_\varepsilon^J} \left[\sum_{j=1}^J g\left(q_j, \gamma, \varepsilon_j \right) q_j - c\left(k, \{q_j\}_{j=1}^J \right) \right. \right. \\
&\quad \left. + \delta V_U \left(f(k) - \zeta \sum_{j=1}^J q_j, \left\{ \hat{b}_n \left(\{\varepsilon_j\}_{j=1}^J \right) \right\}_{n=1}^N \right) \right] \\
&\quad \left. \cdot L_U^* \left(\{\varepsilon_j\}_{j=1}^J \right) \prod_{j=1}^J \mathrm{d}\varepsilon_j \right\}.
\end{aligned}
\tag{6}
$$

4 The Effect of Learning

4.1 Assumptions

In order to study the maximization problem of the monopolist, we postulate the demand function $\tilde{P}_j = q_j^{-\frac{1}{\gamma}} \tilde{\varepsilon}_j$ in market j, where $\gamma > 1$ is the elasticity of demand and $\ln \tilde{\varepsilon}_j \sim N(\theta, 1/r)$. Since the family of lognormal distributions for the random sample $\{\tilde{\varepsilon}_j\}_{j=1}^J$ belongs to the exponential family of distributions, there exists a conjugate family of distributions for the unknown parameter $\tilde{\theta}$. The conjugate family of distributions for $\tilde{\theta}$ is normal with mean ρ and precision $\tau > 0$. Using (4), the posterior beliefs about $\tilde{\theta}$ are normally distributed, i.e.,

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[12]For instance, if the prior p.d.f. of $\tilde{\theta}$ is normal with mean μ and variance σ^2, then $\{b_j\}_{j=1}^2 = \{\mu, \sigma^2\}$ and

$$
\left\{ \hat{b}_n \left(\{\varepsilon_j\}_{j=1}^J \right) \right\}_{n=1}^2 = \left\{ \hat{\mu} \left(\{\varepsilon_j\}_{j=1}^J \right), \hat{\sigma}^2 \left(\{\varepsilon_j\}_{j=1}^J \right) \right\}.
$$

$\tilde{\theta} \sim N\left(\hat{\rho}, 1/\left(\tau + Jr\right)\right)$, where

$$\hat{\rho} \equiv \hat{\rho}\left(\{\varepsilon_j\}_{j=1}^J\right) = \frac{\tau\rho + Jr\overline{\varepsilon}_J}{\tau + Jr}, \tag{7}$$

is the posterior mean. The posterior mean (7) is a weighted average of the prior mean ρ and the sample mean $\overline{\varepsilon}_J = (1/J)\sum_{j=1}^J \ln \varepsilon_j$. The weights of ρ and $\overline{\varepsilon}_J$ are proportional to τ and Jr, respectively. The higher the precision of the prior distribution of $\tilde{\theta}$, the greater the weight that is given to the prior mean ρ, while the greater the size of the data set J or the higher the precision of the data-generating process r, the greater the weight that is given to the sample mean $\overline{\varepsilon}_J$. Note also that the variance of the posterior distribution of $\tilde{\theta}$ is decreasing in J. More price observations reveal more information about the unknown parameter θ. Note also that under our distributional assumptions, the posterior mean (7) is a consistent estimate of $E\left[\ln \tilde{\varepsilon}_{jt}\right] = \theta^*$, implying that the Bayesian estimate of $E\left[\tilde{\varepsilon}_{jt}\right]$ is also consistent. Formally,

Proposition 3 *Under our distributional assumptions, $\hat{\rho}\left(\{\tilde{\varepsilon}_j\}_{j=1}^J\right)$ converges in probability to θ^*, i.e., $\hat{\rho}\left(\{\tilde{\varepsilon}_j\}_{j=1}^J\right) \xrightarrow{P} \theta^*$.*

Proof. Use Kolmogorov's strong law of large number on the posterior mean (7). ∎

Therefore, the uninformed monopolist's beliefs about parameter θ converges in probability to the true value θ^*, implying that the Bayesian estimate on $\tilde{\varepsilon}_j$ also converges to $e^{\theta^* + 1/2r}$. Without consistency of the Bayes procedure, using Bayesian methods does not lead to learning about the environment.

Further assumptions are needed to study the effect of learning in this model. The monopolist faces the production function $f(k) = \left(\alpha k^{1-\frac{1}{\gamma}} + (1-\alpha)\chi\right)^{\frac{\gamma}{\gamma-1}}$, $\alpha \in (0,1]$, $\chi \geq 0$,[13] and incurs a total cost of

$$c\left(k, \{q_j\}_{j=1}^J\right) = \nu f(k)^{-\beta}\left(\sum_{j=1}^J q_j^\phi\right)^\eta,$$

---

[13]If $\alpha = 1$, then $f(k) = k$, which corresponds to a market trading a nonrenewable resource.

$\nu \geq 0$, $\beta > 0$, $\phi, \eta \geq 1$, and

$$\phi\eta - \beta = 1 - 1/\gamma > 0, \tag{8}$$

$\gamma > 1$.[14,15]

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[14]The restriction $\phi\eta - \beta = 1 - 1/\gamma$ serves two purposes. First, it yields closed-form solutions of the monopolist's optimal supply and investment decisions. Second, it is a sufficient condition ensuring that an interior solution always exists with strictly positive profits. That is, the restriction rules out any exit strategy. Suppose that

$$\phi\eta < 1 - 1/\gamma,$$

then it is possible that the monopolist exits permanently.

[15]The cost function we use is very general and admits different scenarios. Consider three of them.

1. The case of $\eta = 1$ and $\phi > 1$. Here, the monopolist employs l_j workers to produce q_j for market j, e.g., the monopolist employs l_j fishermen in market j to extract the stock of fish k. The final-output production is of the form

$$q_j = f(k)^\alpha l_j^\sigma$$

and the cost function for labor l_j is νl_j^ψ, then

$$
\begin{aligned}
c\left(k, \{q_j\}_{j=1}^J\right) &= \nu \sum_{j=1}^J f(k)^{-\frac{\alpha\psi}{\sigma}} q_j^{\frac{\psi}{\sigma}}, \\
&\equiv \nu f(k)^{-\beta} \sum_{j=1}^J q_j^\phi,
\end{aligned}
$$

where $\beta \equiv \alpha\psi/\sigma$ and $\phi \equiv \psi/\sigma$.

2. The case of $\eta > 1$ and $\phi = 1$. The monopolist centralizes production. The final-output production is of the form

$$q = f(k)^\alpha \left(\sum_{j=1}^J l_j\right)^\sigma$$

and the cost function for labor l is νl, then

$$
\begin{aligned}
c\left(k, \{q_j\}_{j=1}^J\right) &= \nu f(k)^{-\frac{\alpha\psi}{\sigma}} \left(\sum_{j=1}^J q_j\right)^{\frac{\psi}{\sigma}}, \\
&\equiv \nu f(k)^{-\beta} \left(\sum_{j=1}^J q_j\right)^\eta,
\end{aligned}
$$

where $\beta \equiv \alpha\psi/\sigma$ and $\eta \equiv \psi/\sigma$.

3. The case of $\eta > 1, \phi > 1$. The monopolist hires workers in market j but assembles the final output in one central location.

Therefore, the dynamic program (6) for the uninformed monopolist is rewritten as

$$
V_U(k, \rho, \tau) = \max_{\{q_j > 0\}_{j=1}^J} \left\{ \int \cdots \int_{\Omega_\varepsilon^J} \left[\sum_{j=1}^J \varepsilon_j q_j^{1-\frac{1}{\gamma}} - \nu f(k)^{-\beta} \left(\sum_{j=1}^J q_j^\phi \right)^\eta + \right. \right.
$$
$$
\left. \left. \delta V_U \left(f(k) - \zeta \sum_{j=1}^J q_j, \hat{\rho}\left(\{\varepsilon_j\}_{j=1}^J\right), \tau + Jr \right) \right] L_U^* \left(\{\varepsilon_j\}_{j=1}^J\right) \prod_{j=1}^J d\varepsilon_j \right\}, \tag{9}
$$

$$
= \max_{\{q_j > 0\}_{j=1}^J} \left\{ \left[\sum_{j=1}^J e^{\rho + \frac{r+\tau}{2r\tau}} q_j^{1-\frac{1}{\gamma}} - \nu f(k)^{-\beta} \left(\sum_{j=1}^J q_j^\phi \right)^\eta + \right. \right.
$$
$$
\delta \int \cdots \int_{\Omega_\varepsilon^J} V_U \left(f(k) - \zeta \sum_{j=1}^J q_j, \hat{\rho}\left(\{\varepsilon_j\}_{j=1}^J\right), \tau + Jr \right) \Bigg]
$$
$$
\left. \cdot L_U^* \left(\{\varepsilon_j\}_{j=1}^J\right) \prod_{j=1}^J d\varepsilon_j \right\}, \tag{10}
$$

where the pair $\{b_n\}_{n=1}^2 \equiv (\rho, \tau)$ and

$$
\left\{ \hat{b}_n \left(\{\varepsilon_j\}_{j=1}^J\right) \right\}_{n=1}^2 \equiv \left\{ \hat{\rho}\left(\{\varepsilon_j\}_{j=1}^J\right), \tau + Jr \right\}
$$

characterize the evolution of beliefs. Here, $L_U^* \left(\{\varepsilon_j\}_{j=1}^J\right)$ is the joint p.d.f. of $\{\tilde{\varepsilon}_j\}_{j=1}^J$ given the uninformed monopolist's beliefs about θ.

4.2 The Benchmark Model

In order to measure the effect of learning on the monopolist's behavior, we solve the dynamic program for the informed monopolist, i.e., the value of θ is $\theta^* \in \Theta$. Then, the informed monopolist's dynamic program is

$$
V_I(k) = \max_{\{q_j > 0\}_{j=1}^J} \left\{ \int \cdots \int_{\Omega_\varepsilon^J} \left[\sum_{j=1}^J \varepsilon_j q_j^{1-\frac{1}{\gamma}} - \nu f(k)^{-\beta} \left(\sum_{j=1}^J q_j^\phi \right)^\eta + \delta V_I \left(f(k) - \zeta \sum_{j=1}^J q_j \right) \right] \right.
$$
$$
\left. \cdot L_I^* \left(\{\varepsilon_j\}_{j=1}^J\right) \prod_{j=1}^J d\varepsilon_j \right\},
$$

$$
= \max_{\{q_j > 0\}_{j=1}^J} \left\{ \sum_{j=1}^J e^{\theta^* + \frac{1}{2r}} q_j^{1-\frac{1}{\gamma}} - \nu f(k)^{-\beta} \left(\sum_{j=1}^J q_j^\phi \right)^\eta + \delta V_I \left(f(k) - \zeta \sum_{j=1}^J q_j \right) \right\}, \tag{11}
$$

where $\ln \tilde{\varepsilon}_j \sim iidN\left(\theta^*, 1/r\right)$. Here, $L_I^*\left(\{\varepsilon_j\}_{j=1}^J\right)$ is the joint p.d.f. of $\{\tilde{\varepsilon}_j\}_{j=1}^J$ given the information of the informed monopolist about θ. Note that the stock of capital is the only state variable for the informed monopolist's dynamic program.

4.3 The Supply and Investment Strategies

The next two propositions present the optimal supply and investment strategies of the uninformed and informed monopolists, respectively.

Proposition 4 *If the monopolist is uninformed, i.e., dynamic program* (10)*, then* $Q_U = \omega_U\left(\rho, \tau\right)f\left(k\right)$ *is supplied to market j and* $\hat{K}_U = \left(1 - \zeta J \omega_U\left(\rho, \tau\right)\right)f\left(k\right)$ *is invested. Here,*

$$\omega_U\left(\rho, \tau\right) \in \left(0, \min\left\{\left(\zeta J\right)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}}\right)^{1/\beta}\right\}\right)$$

is unique and characterized implicitly by

$$\frac{\left(1 - \frac{1}{\gamma}\right)e^{\rho + \frac{r+\tau}{2r\tau}} - \eta\phi v J^{\eta-1}\omega_U^\beta}{e^{\rho + \frac{r+\tau}{2r\tau}} - vJ^{\eta-1}\omega_U^\beta} = \frac{\alpha\delta\left(1 - \frac{1}{\gamma}\right)\zeta J\omega_U}{\left(1 - \zeta J\omega_U\right)^{\frac{1}{\gamma}} - \alpha\delta\left(1 - \zeta J\omega_U\right)}, \tag{12}$$

where $\omega_U \equiv \omega_U\left(\rho, \tau\right)$.

Proposition 5 *If the monopolist is informed, i.e., dynamic program* (11)*, then* $Q_I = \omega_I f\left(k\right)$ *is supplied to market j and* $\hat{K}_I = \left(1 - \zeta J \omega_I\right)f\left(k\right)$ *is invested. Here,*

$$\omega_I \in \left(0, \min\left\{\left(\zeta J\right)^{-1}, \left(\frac{e^{\theta^* + \frac{1}{2r}}}{vJ^{\eta-1}}\right)^{1/\beta}\right\}\right)$$

is unique and characterized implicitly by

$$\frac{\left(1 - \frac{1}{\gamma}\right)e^{\theta^* + \frac{1}{2r}} - \eta\phi v J^{\eta-1}\omega_I^\beta}{e^{\theta^* + \frac{1}{2r}} - vJ^{\eta-1}\omega_I^\beta} = \frac{\alpha\delta\left(1 - \frac{1}{\gamma}\right)\zeta J\omega_I}{\left(1 - \zeta J\omega_I\right)^{\frac{1}{\gamma}} - \alpha\delta\left(1 - \zeta J\omega_I\right)}. \tag{13}$$

Note that $\omega_U\left(\rho, \tau\right)$ and ω_I are similar since expressions (12) and (13) have the same structure. The only difference between expressions (12) and (13) is the expectation of the

16

demand shock given the information available to the monopolist. The uninformed monopolist's expected mean of the demand shock is $e^{\rho + \frac{r+\tau}{2r\tau}}$ while the informed monopolist's is $e^{\theta^* + \frac{1}{2r}}$. Therefore, the other structural parameters γ, η, ϕ, ν, J, β, α, δ, and ζ affect the supply and investment decisions in the same direction, whether the monopolist is uninformed or informed. They only differ in magnitude because of differences on the expectation of the demand shock.

Note also that the uninformed monopolist's share of output to each market $\omega_U(\rho, \tau)$ evolves over time as beliefs are updated after each period since ρ and τ are two state variables with the autonomous laws of motion (7) and $\hat{\tau} = \tau + Jr$. This does not happen with the informed monopolist since ω_I is fixed over time.

The Cost Function. The presence of a cost function is essential in our model for learning to affect behavior. Formally,

Proposition 6 *If there is no cost, i.e., $\nu = 0$, then learning does not affect the monopolist's behavior, i.e., whether or not the monopolist knows that the value of θ is θ^*,*

$$Q_U = Q_I = \frac{1 - \alpha^\gamma \delta^\gamma}{\zeta J} f(k)$$

is supplied to market j and

$$\hat{K}_U = \hat{K}_I = \alpha^\gamma \delta^\gamma f(k)$$

is invested.

When there is no cost, $\nu = 0$, the uncertainty in demand is multiplicative. Therefore, the information about the distribution of the demand shock does not affect the monopolist's behavior. When $\nu > 0$, the uncertainty is no longer multiplicative and it follows that

Proposition 7 *If there is a cost, i.e., $\nu > 0$, then learning affects the monopolist's behavior.*

1. *If $e^{\rho + \frac{r+\tau}{2r\tau}} > e^{\theta^* + \frac{1}{2r}}$, then $Q_U > Q_I$ and $\hat{K}_U < \hat{K}_I$.*

2. *If $e^{\rho + \frac{r+\tau}{2r\tau}} < e^{\theta^* + \frac{1}{2r}}$, then $Q_U < Q_I$ and $\hat{K}_U > \hat{K}_I$.*

3. If $e^{\rho + \frac{r+\tau}{2r\tau}} = e^{\theta^* + \frac{1}{2r}}$, then $Q_U = Q_I$ and $\hat{K}_U = \hat{K}_I$.

In other words, from Proposition 7, beliefs about the demand shock affect the firm's behavior. When the uninformed monopolist has more pessimistic beliefs about the mean of the demand shock than the informed monopolist, i.e., $e^{\rho + \frac{r+\tau}{2r\tau}} < e^{\theta^* + \frac{1}{2r}}$, then learning decreases supply and increases investment. When the uninformed monopolist has more optimistic beliefs about the mean of the demand shock than the informed monopolist, i.e., $e^{\rho + \frac{r+\tau}{2r\tau}} > e^{\theta^* + \frac{1}{2r}}$, then learning increases supply and decreases investment.

It is worth noting that a higher precision of the beliefs, i.e., a higher τ, decreases the expectation of the demand shock, i.e., $e^{\rho + \frac{r+\tau}{2r\tau}}$ is negatively related to the precision τ of beliefs. This means that $\partial Q_U / \partial \tau < 0$. This is due to the lognormality of the demand shock that makes the precision τ part of the mean of the demand shock.

Correct Beliefs. Suppose now that the uninformed monopolist has correct beliefs about the value of θ but remains uninformed. Two cases are studied. First, suppose that $\rho = \theta^*$, then

Proposition 8 *If there is a cost, i.e., $\nu > 0$, and $\rho = \theta^*$, then $e^{\rho + \frac{r+\tau}{2r\tau}} > e^{\theta^* + \frac{1}{2r}}$ and $Q_U > Q_I$ and $\hat{K}_U < \hat{K}_I$.*

In other words, from Proposition 8, the uninformed monopolist supplie s more than the informed monopolist when $\rho = \theta^*$. Although the uninformed monopolist faces more uncertainty than the informed monopolist since $\tau < +\infty$ and, has a higher expectation of the demand shock, since $e^{\rho + \frac{r+\tau}{2r\tau}} > e^{\theta^* + \frac{1}{2r}}$. Therefore, the uninformed monopolist supplies more and invests less.

Second, suppose that the uninformed monopolist incorrect beliefs about the value of θ, but has correct beliefs about the expectation of the demand shocks, i.e., $\rho = \theta^* - 1/\tau$ and thus $e^{\rho + \frac{r+\tau}{2r\tau}} = e^{\theta^* + \frac{1}{2r}}$, then

Proposition 9 *If there is a cost, i.e., $\nu > 0$, and $\rho = \theta^* - 1/\tau$, then $e^{\rho + \frac{r+\tau}{2r\tau}} = e^{\theta^* + \frac{1}{2r}}$ and $Q_U = Q_I$ and $\hat{K}_U = \hat{K}_I$.*

In other words, from Proposition 9, learning has no effect on the monopolist's supply and investment decisions when the expectation on the demand shock is the same for both the uninformed and informed monopolists. In our class of models, if $e^{\rho + \frac{r+\tau}{2r\tau}} = e^{\theta^* + \frac{1}{2r}}$, then $Q_U = Q_I$. We now investigate whether this is always true in a more general setting.

Consider the expression

$$
\int_{\Omega_\varepsilon} \left[\sum_{j=1}^J g\left(q_j, \gamma, \varepsilon_j\right) q_j \right] L_U^* \left(\{\varepsilon_j\}_{j=1}^J\right) \prod_{j=1}^J \mathrm{d}\varepsilon_j = \int_{\Omega_\varepsilon} \left[\sum_{j=1}^J g\left(q_j, \gamma, \varepsilon_j\right) q_j \right] L_I^* \left(\{\varepsilon_j\}_{j=1}^J\right) \prod_{j=1}^J \mathrm{d}\varepsilon_j,
$$

(14)

where L_U^* is the likelihood function of the uninformed monopolist, given its beliefs about θ while L_I^* is the likelihood function of the informed monopolist, given that $\theta = \theta^*$ is known. In the static case, under expression (14), $Q_U = Q_I$, since the uninformed and informed monopolists maximize the same objective function. In our class of models, expression (14) is equivalent to $e^{\rho + \frac{r+\tau}{2r\tau}} = e^{\theta^* + \frac{1}{2r}}$, resulting in $Q_U = Q_I$.

In general, the uninformed monopolist's dynamic program is

$$
\begin{aligned}
V_U\left(k, \{b_n\}_{n=1}^N\right) = \max_{\{q_j \geq 0\}_{j=1}^J} & \left\{ \int \cdots \int_{\Omega_\varepsilon^J} \left[\sum_{j=1}^J g\left(q_j, \gamma, \varepsilon_j\right) q_j \right] L_U^*\left(\{\varepsilon_j\}_{j=1}^J\right) \prod_{j=1}^J \mathrm{d}\varepsilon_j - c\left(k, \{q_j\}_{j=1}^J\right) \right. \\
& \left. + \delta \int \cdots \int_{\Omega_\varepsilon^J} V_U\left(\hat{k}, \left\{\hat{b}_n\left(\{\varepsilon_j\}_{j=1}^J\right)\right\}_{n=1}^N\right) L_U^*\left(\{\varepsilon_j\}_{j=1}^J\right) \prod_{j=1}^J \mathrm{d}\varepsilon_j \right\}.
\end{aligned}
$$

while the informed monopolist's dynamic program is

$$
V_I\left(k\right) = \max_{\{q_j > 0\}_{j=1}^J} \left\{ \int_{\Omega_\varepsilon} \left[\sum_{j=1}^J g\left(q_j, \gamma, \varepsilon_j\right) q_j \right] L_I^*\left(\{\varepsilon_j\}_{j=1}^J\right) \prod_{j=1}^J \mathrm{d}\varepsilon_j - c\left(k, \{q_j\}_{j=1}^J\right) + \delta V_I\left(\hat{k}\right) \right\}.
$$

When the uncertainty is multiplicative in demand, expression (14) holds if and only if $e^{\rho + \frac{r+\tau}{2r\tau}} = e^{\theta^* + \frac{1}{2r}}$, and $Q_U = Q_I$ and the value function is of the form

$$
V_U\left(k, \{b_n\}_{n=1}^N\right) = Z_1\left(\{b_n\}_{n=1}^N\right) W\left(k\right) + Z_2\left(\{b_n\}_{n=1}^N\right),
$$

(15)

where Z_1, W, and Z_2 are functions, so that the updated value function is

$$V_U\left(k, \hat{b}_n\left(\{\varepsilon_j\}_{j=1}^J\right)\right) = Z_1\left(\hat{b}_n\left(\{\varepsilon_j\}_{j=1}^J\right)\right) W\left(\hat{k}\right) + Z_3\left(\hat{b}_n\left(\{\varepsilon_j\}_{j=1}^J\right)\right)$$

and its expectation is

$$
\begin{aligned}
EV_U\left(\hat{k}, \hat{b}_n\left(\{\varepsilon_j\}_{j=1}^J\right)\right) &= \int_{\Omega_\varepsilon} Z_1\left(\left\{\hat{b}_n\left(\{\varepsilon_j\}_{j=1}^J\right)\right\}_{n=1}^N\right) W\left(\hat{k}\right) + Z_2\left(\left\{\hat{b}_n\left(\{\varepsilon_j\}_{j=1}^J\right)\right\}_{n=1}^N\right) \\
&\qquad \left[\int_\Theta \prod_{j=1}^J \phi\left(\varepsilon_j\,|y\right)\xi\left(y\right)\mathrm{d}y\right]\prod_{j=1}^J \mathrm{d}\varepsilon_j, \\
&= z_1 W\left(\hat{k}\right) + z_2,
\end{aligned}
\tag{16}
$$

where

$$z_1 = \int_{\Omega_\varepsilon} Z_1\left(\left\{\hat{b}_n\left(\{\varepsilon_j\}_{j=1}^J\right)\right\}_{n=1}^N\right)\left[\int_\Theta \prod_{j=1}^J \phi\left(\varepsilon_j\,|y\right)\xi\left(y\right)\mathrm{d}y\right]\prod_{j=1}^J \mathrm{d}\varepsilon_j,$$

$$z_2 = \int_{\Omega_\varepsilon} Z_2\left(\left\{\hat{b}_n\left(\{\varepsilon_j\}_{j=1}^J\right)\right\}_{n=1}^N\right)\left[\int_\Theta \prod_{j=1}^J \phi\left(\varepsilon_j\,|y\right)\xi\left(y\right)\mathrm{d}y\right]\prod_{j=1}^J \mathrm{d}\varepsilon_j.$$

Otherwise, it is not necessarily the case that the expected continuation of the value function is of the form (16) and $Q_U = Q_I$ when expression (14) holds regardless of the demand structure.

4.4 The Steady State Levels of Supply and Capital

Although learning affects the monopolist's supply and investment decisions, it does not change the steady state levels of supply and capital because the uninformed monopolist becomes informed about the value of the unknown parameter θ in the limit. Incomplete learning cannot occur in our model since the Bayes procedure is consistent. In other words,

$$e^{\rho + \frac{r+\tau}{2r\tau}} \xrightarrow{P} e^{\theta^* + \frac{1}{2r}}$$

since $\rho \xrightarrow{P} \theta^*$ and $\tau \to \infty$. However, learning affects the rate of convergence to the steady state. In particular, if the uninformed monopolist is more optimistic than the informed monopolist, i.e., $e^{\rho + \frac{r+\tau}{2r\tau}} > e^{\theta^* + \frac{1}{2r}}$, then $Q_U > Q_I$. As a result, learning slows down the convergence rate toward the steady state if and only if $k < \overline{K}$, where $\overline{K}$ the steady state level of capital. On the other hand, if the uninformed monopolist is less optimistic than the informed monopolist, i.e., $e^{\rho + \frac{r+\tau}{2r\tau}} < e^{\theta^* + \frac{1}{2r}}$, then $Q_U < Q_I$. As a result, learning speeds up the convergence rate toward the steady state if and only of $k < \overline{K}$.

5 Appendix

5.1 Proof of Proposition 4

We first show that the instantaneous profit function

$$\sum_{j=1}^{J} e^{\rho + \frac{r+\tau}{2r\tau}} q_j^{1-\frac{1}{\gamma}} - \nu f(k)^{-\beta} \left(\sum_{j=1}^{J} q_j^{\phi} \right)^{\eta} \tag{17}$$

is strictly concave. Then, we show that the value function is of the form

$$V_U(k, \rho, \tau) = Z_1(\rho, \tau) k^{1-\frac{1}{\eta}} + Z_2(\rho, \tau), \tag{18}$$

where $Z_1(\rho, \tau), Z_2(\rho, \tau) > 0$ are functions of ρ and τ, but not k, and that the optimal quantity supplied is symmetric across markets, unique, and of the form $Q_U = \omega_U(\rho, \tau) f(k)$, where

$$\omega_U(\rho, \tau) \in \left(0, \min \left\{ (\zeta J)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{v J^{\eta - 1}} \right)^{1/\beta} \right\} \right).$$

Note that $\omega_U(\rho, \tau) < (\zeta J)^{-1}$ implies that some output is invested for next period, i.e., $\hat{k} = f(k) - \zeta J Q_U > 0$. Note also that

$$\omega_U(\rho, \tau) < \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{v J^{\eta - 1}} \right)^{1/\beta}$$

21

implies that the instantaneous profit function (17) is strictly positive in equilibrium.

1. Recall that the restrictions on the values of the parameters are $\gamma > 1$, $\nu > 0$, $\beta > 0$, $\eta, \phi \geq 1$, $\delta \in [0, 1]$, $\zeta > 0$, and

$$\phi\eta - \beta = 1 - 1/\gamma.$$

The Hessian $\mathcal{H}$ of expression (17) evaluated at the symmetric optimal quantity supplied, $q_j = Q_U \in \left(0, \min\left\{(\zeta J)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{\nu J^{\eta-1}}\right)^{1/\beta}\right\} f(k)\right)$ for all j, has elements

$$\mathcal{H}_{jj} = -\left(\frac{1}{\gamma}\left(1 - \frac{1}{\gamma}\right) e^{\rho + \frac{r+\tau}{2r\tau}} Q_U^{-\frac{1}{\gamma}-1} + \nu\phi\eta J^{\eta-2}\left((\eta - 1)\phi + (\phi - 1)J\right) Q_U^{\phi\eta-2}\right),$$

and

$$\mathcal{H}_{jm} = -\nu\phi\eta J^{\eta-2}(\eta - 1)\phi Q_U^{\phi\eta-2},$$

$j \neq m$, $j, m = 1, ..., J$. Given the restrictions on the values of the parameters, $\mathcal{H}_{jj} < \mathcal{H}_{jm} < 0$, and $\mathcal{H}_{jm} = \mathcal{H}_{\hat{j}\hat{m}}$, for all $j, m, \hat{j}, \hat{m} = 1, ..., J$. Therefore, the determinants of the principal minors are of the right sign and $\mathcal{H}$ is negative definite. It follows that the instantaneous profit function (17) is strictly concave for $Q_U \in \left(0, \min\left\{(\zeta J)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{\nu J^{\eta-1}}\right)^{1/\beta}\right\} f(k)\right)$.

2. Updating the value function (18) and plugging it into value function (10) yields

$$V_U(k, \rho, \tau) = \max_{\{q_j > 0\}_{j=1}^J} \left\{ \sum_{j=1}^J \left(e^{\rho + \frac{r+\tau}{2r\tau}} q_j^{1-\frac{1}{\gamma}} - \nu f(k)^{-\beta} \left(\sum_{j=1}^J q_j^\phi\right)^\eta \right) \right.$$
$$\left. + \delta z_1 \left(f(k) - \zeta \sum_{j=1}^J q_j \right)^{1-\frac{1}{\gamma}} + \delta z_2 \right\}, \tag{19}$$

where

$$z_1 = \int \cdots \int_{\Omega_\varepsilon^J} Z_1\left(\hat{\rho}\left(\{\varepsilon_j\}_{j=1}^J\right), \tau + Jr\right) L_U^*\left(\{\varepsilon_j\}_{j=1}^J\right) \prod_{j=1}^J d\varepsilon_j,$$

$$z_2 = \int \cdots \int_{\Omega_\varepsilon^J} Z_2\left(\hat{\rho}\left(\{\varepsilon_j\}_{j=1}^J\right), \tau + Jr\right) L_U^*\left(\{\varepsilon_j\}_{j=1}^J\right) \prod_{j=1}^J d\varepsilon_j \qquad (20)$$

are functions of ρ and τ. The first-order conditions are

$$q_j : e^{\rho + \frac{r+\tau}{2r\tau}}\left(1 - \frac{1}{\gamma}\right) q^{-\frac{1}{\gamma}} - \phi\eta\nu f(k)^{-\beta}\left(\sum_{j=1}^J q_j^\phi\right)^{\eta-1} q_j^{\phi-1} - \delta\zeta\left(1 - \frac{1}{\gamma}\right) z_1\left(f(k) - \zeta\sum_{j=1}^J q_j\right)^{-\frac{1}{\gamma}} = 0,$$
$$(21)$$

$j = 1, ..., J$. Considering a symmetric optimal supply function of the form $Q_U = \omega_U(\rho, \tau) f(k)$, where $\omega_U(\rho, \tau)$ is a function of the state variables ρ and τ, the first-order conditions (21) become

$$q_j : e^{\rho + \frac{r+\tau}{2r\tau}}\left(1 - \frac{1}{\gamma}\right) \omega_U^{-\frac{1}{\gamma}} - \eta\phi\nu J^{\eta-1}\omega_U^{\phi\eta-1} - \delta\zeta\left(1 - \frac{1}{\gamma}\right) z_1\left(1 - \zeta J\omega_U\right)^{-\frac{1}{\gamma}} = 0, \quad (22)$$

for all j, since $\phi\eta - \beta = 1 - 1/\gamma$.[16] Using the fact that $Q_U = \omega_U f(k)$, the value function for the uninformed monopolist (19) is

$$
\begin{aligned}
V_U(k, \rho, \tau) &= \left[e^{\rho + \frac{r+\tau}{2r\tau}} J\omega_U^{1-\frac{1}{\gamma}} - \nu J^\eta \omega_U^{\phi\eta} + \delta z_1\left(1 - \zeta J\omega_U\right)^{1-\frac{1}{\gamma}}\right] f(k)^{1-\frac{1}{\gamma}} + \delta z_2, \\
&= \left[e^{\rho + \frac{r+\tau}{2r\tau}} J\omega_U^{1-\frac{1}{\gamma}} - \nu J^\eta \omega_U^{\phi\eta} + \delta z_1\left(1 - \zeta J\omega_U\right)^{1-\frac{1}{\gamma}}\right] \alpha k^{1-\frac{1}{\gamma}} \\
&\quad + \left[e^{\rho + \frac{r+\tau}{2r\tau}} J\omega_U^{1-\frac{1}{\gamma}} - \nu J^\eta \omega_U^{\phi\eta} + \delta z_1\left(1 - \zeta J\omega_U\right)^{1-\frac{1}{\gamma}}\right] (1-\alpha)\chi + \delta z_2, \\
&\equiv Z_1(\rho, \tau) k^{1-\frac{1}{\eta}} + Z_2(\rho, \tau),
\end{aligned}
$$

so that

$$Z_1(\rho, \tau) \equiv \frac{\alpha\left(e^{\rho + \frac{r+\tau}{2r\tau}} J\omega_U^{1-\frac{1}{\gamma}} - \nu J^\eta \omega_U^{\phi\eta}\right)}{1 - \alpha\delta\left(1 - \zeta J\omega_U\right)^{1-\frac{1}{\gamma}}} \qquad (23)$$

---

[16]We write ω_U instead of $\omega_U(\rho, \tau)$ to simplify notation.

and

$$Z_2(\rho, \tau) = \frac{Z_1(\rho, \tau)(1-\alpha)\chi}{\alpha} + \delta z_2,$$

for $\omega_U \in \left(0, \min\left\{(\zeta J)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}}\right)^{1/\beta}\right\}\right)$, and where z_2 is defined in expression (20). Plugging expression (23) into the first-order condition (22), using assumption (8), and rearranging terms yields the implicit characterization of ω_U,

$$g\left(\omega_U, e^{\rho + \frac{r+\tau}{2r\tau}}\right) \equiv \frac{\left(1 - \frac{1}{\gamma}\right)e^{\rho + \frac{r+\tau}{2r\tau}} - \phi\eta\nu J^{\eta-1}\omega_U^\beta}{e^{\rho + \frac{r+\tau}{2r\tau}} - \nu J^{\eta-1}\omega_U^\beta} = \frac{\alpha\delta\left(1 - \frac{1}{\gamma}\right)\zeta J\omega_U}{(1 - \zeta J\omega_U)^{\frac{1}{\gamma}} - \alpha\delta(1 - \zeta J\omega_U)} \equiv h(\omega_U). \tag{24}$$

In expression (24), the left-hand side is $g\left(\omega_U, e^{\rho + \frac{r+\tau}{2r\tau}}\right)$ while the right-hand side is $h(\omega_U)$. We now show that $\omega_U \in \left(0, \min\left\{(\zeta J)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}}\right)^{1/\beta}\right\}\right)$.

(a) The properties of $g\left(\omega_U, e^{\rho + \frac{r+\tau}{2r\tau}}\right)$ are

    i.

$$g\left(0, e^{\rho + \frac{r+\tau}{2r\tau}}\right) = 1 - \frac{1}{\gamma} > 0. \tag{25}$$

    ii.

$$\frac{\partial g\left(\omega_U, e^{\rho + \frac{r+\tau}{2r\tau}}\right)}{\partial \omega_U} = \frac{-\beta e^{\rho + \frac{r+\tau}{2r\tau}}\nu J^{\eta-1}\omega_U^{\beta-1}\left(1 - \frac{1}{\gamma} - \eta\phi\right)}{\left(e^{\rho + \frac{r+\tau}{2r\tau}} - \nu J^{\eta-1}\omega_U^\beta\right)^2} < 0, \tag{26}$$

which is strictly negative for $\omega_U \in \left(0, \min\left\{(\zeta J)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}}\right)^{1/\beta}\right\}\right)$, since assumption (8) and $\eta > 1$ implies that

$$1 - \frac{1}{\gamma} - \eta\phi = -\beta < 0.$$

    iii. If $(\zeta J)^{-1} < \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}}\right)^{1/\beta}$, then

$$g\left((\zeta J)^{-1/\beta}, e^{\rho + \frac{r+\tau}{2r\tau}}\right) = \frac{-\beta e^{\rho + \frac{r+\tau}{2r\tau}}\nu J^{\eta-\beta}\zeta^{-\beta+1}\left(1 - \frac{1}{\gamma} - \eta\phi\right)}{\left(e^{\rho + \frac{r+\tau}{2r\tau}} - \nu J^{\eta-1}(\zeta J)^{-\beta}\right)^2} > 0 \tag{27}$$

If $(\zeta J)^{-1} > \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}} \right)^{1/\beta}$, then

$$g \left(\left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}} \right)^{1/\beta}, e^{\rho + \frac{r+\tau}{2r\tau}} \right) = -\infty. \tag{28}$$

(b) The properties of $h(\omega_U)$ are

i.

$$h(0) = 0. \tag{29}$$

ii.

$$h'(\omega_U) = \alpha\delta \left(1 - \frac{1}{\gamma}\right) \zeta J \frac{\frac{1 - \left(1 - \frac{1}{\gamma}\right)\zeta J \omega_U}{(1 - \zeta J \omega_U)^{1 - \frac{1}{\gamma}}} - \alpha\delta}{(1 - \zeta J \omega_U)^{\frac{1}{\gamma}} - \alpha\delta (1 - \zeta J \omega_U)} > 0, \tag{30}$$

which is strictly positive for $\omega_U \in \left(0, \min\left\{ (\zeta J)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}} \right)^{1/\beta} \right\} \right)$, since

$$\frac{1 - \left(1 - \frac{1}{\gamma}\right)\zeta J \omega_U}{(1 - \zeta J \omega_U)^{1 - \frac{1}{\gamma}}} > 1$$

and $(1 - \zeta J \omega_U)^{\frac{1}{\gamma}} > \alpha\delta (1 - \zeta J \omega_U)$ for $\omega_U \in \left(0, \min\left\{ (\zeta J)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}} \right)^{1/\beta} \right\} \right)$.

iii. If $(\zeta J)^{-1} < \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}} \right)^{1/\beta}$, then

$$h\left((\zeta J)^{-1}\right) = \infty. \tag{31}$$

If $(\zeta J)^{-1} > \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}} \right)^{1/\beta}$, then

$$h\left(\left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}} \right)^{1/\beta} \right) = \frac{\alpha\delta \left(1 - \frac{1}{\gamma}\right) \zeta J \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}} \right)^{1/\beta}}{\left(1 - \zeta J \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}} \right)^{1/\beta}\right)^{\frac{1}{\gamma}} - \alpha\delta \left(1 - \zeta J \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}} \right)^{1/\beta}\right)} > 0. \tag{32}$$

3. Therefore, given the parametric assumptions and combining properties (25), (26), (27),

(28), (29), (30), (31), and (32), $g\left(\omega_U, e^{\rho + \frac{r+\tau}{2r\tau}}\right)$ and $h\left(\omega_U\right)$ cross at most once on

$$\omega_U \in \left(0, \min\left\{(\zeta J)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}}\right)^{1/\beta}\right\}\right).$$

It follows that ω_U exists and is unique on $\left(0, \min\left\{(\zeta J)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}}\right)^{1/\beta}\right\}\right)$. Therefore, $0 < Z_1\left(\rho, \tau\right), Z_2\left(\rho, \tau\right), z_1, z_2 < \infty$ and are functions of ρ and τ, but not k, and the value function is bounded and of the form

$$V_U\left(k, \rho, \tau\right) = Z_1\left(\rho, \tau\right) k^{1-\frac{1}{\gamma}} + Z_2\left(\rho, \tau\right) > 0.$$

Moreover, the instantaneous profit function (17) is strictly concave and the objective function

$$\sum_{j=1}^{J} e^{\rho + \frac{r+\tau}{2r\tau}} q_j^{1-\frac{1}{\gamma}} - vf\left(k\right)^{-\beta} \left(\sum_{j=1}^{J} q_j^{\phi}\right)^{\eta} + \delta z_1 \left(f\left(k\right) - \zeta \sum_{j=1}^{J} q_j\right)^{1-\frac{1}{\gamma}} + \delta z_2$$

in (19) is also strictly concave. Therefore, $Q_U = \omega_U\left(\rho, \tau\right) f\left(k\right)$ is the unique maximizer on $\left(0, \min\left\{(\zeta J)^{-1}, \left(\frac{e^{\rho + \frac{r+\tau}{2r\tau}}}{vJ^{\eta-1}}\right)^{1/\beta}\right\} f\left(k\right)\right)$ for the dynamic program (10).

5.2   Proof of Proposition 5

Proving proposition 5 involves similar steps to those in the proof of proposition 4. The informed monopolist supplies $Q_I = \omega_I f\left(k\right)$, where $\omega_I \in \left(0, \min\left\{(\zeta J)^{-1}, \left(\frac{e^{\theta^* + \frac{1}{2r}}}{vJ^{\eta-1}}\right)^{1/\beta}\right\}\right)$ is unique and implicitly characterized by

$$g\left(\omega_I, e^{\theta^* + \frac{1}{2r}}\right) \equiv \frac{\left(1 - \frac{1}{\gamma}\right) e^{\theta^* + \frac{1}{2r}} - \phi\eta vJ^{\eta-1}\omega_I^{\beta}}{e^{\theta^* + \frac{1}{2r}} - vJ^{\eta-1}\omega_I^{\beta}} = \frac{\alpha\delta\left(1 - \frac{1}{\gamma}\right)\zeta J\omega_I}{(1 - \zeta J\omega_I)^{\frac{1}{\gamma}} - \alpha\delta\left(1 - \zeta J\omega_I\right)} \equiv h\left(\omega_I\right).$$
(33)

5.3   Proof of Propositions 7, 8, and 9

Recall that the uninformed monopolist supplies $Q_U = \omega_U f(k)$, where ω_U is implicitly characterized by

$$g\left(\omega_U, e^{\rho+\frac{r+\tau}{2r\tau}}\right) \equiv \frac{\left(1-\frac{1}{\gamma}\right)e^{\rho+\frac{r+\tau}{2r\tau}} - \phi\eta\nu J^{\eta-1}\omega_U^{\beta}}{e^{\rho+\frac{r+\tau}{2r\tau}} - \nu J^{\eta-1}\omega_U^{\beta}} = \frac{\alpha\delta\left(1-\frac{1}{\gamma}\right)\zeta J\omega_U}{(1-\zeta J\omega_U)^{\frac{1}{\gamma}} - \alpha\delta(1-\zeta J\omega_U)} \equiv h(\omega_U),$$

while the informed monopolist supplies $Q_I = \omega_I f(k)$, where ω_I is implicitly characterized by

$$g\left(\omega_I, e^{\theta^*+\frac{1}{2r}}\right) \equiv \frac{\left(1-\frac{1}{\gamma}\right)e^{\theta^*+\frac{1}{2r}} - \phi\eta\nu J^{\eta-1}\omega_I^{\beta}}{e^{\theta^*+\frac{1}{2r}} - \nu J^{\eta-1}\omega_I^{\beta}} = \frac{\alpha\delta\left(1-\frac{1}{\gamma}\right)\zeta J\omega_I}{(1-\zeta J\omega_I)^{\frac{1}{\gamma}} - \alpha\delta(1-\zeta J\omega_I)} \equiv h(\omega_I).$$

We know that
$$\frac{\partial g(\omega, x)}{\partial x} = \frac{\phi\eta - \left(1-\frac{1}{\gamma}\right)}{(x - \nu J^{\eta-1}\omega^{\beta})^2}\nu J^{\eta-1}\omega^{\beta} > 0$$

for $\omega \in \left(0, \left(\frac{x}{\nu J^{\eta-1}}\right)^{1/\beta}\right)$ since $\phi\eta - \beta = 1 - 1/\gamma > 0$ and $\beta > 0$. It follows that

$$e^{\rho+\frac{r+\tau}{2r\tau}} \lesseqgtr e^{\theta^*+\frac{1}{2r}} \Leftrightarrow g\left(\omega, e^{\rho+\frac{r+\tau}{2r\tau}}\right) \lesseqgtr g\left(\omega, e^{\theta^*+\frac{1}{2r}}\right) \Leftrightarrow \omega_U \lesseqgtr \omega_I \Leftrightarrow Q_U \lesseqgtr Q_I.$$

Propositions 7, 8, and 9 follow.

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