Pricing Interest Rate Derivatives under Multi-Factor GARCH

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(Preliminary work, please do not quote)

Abstract

This paper presents semi-closed-form solutions to a wide range of interest rate derivatives, such as options on discount bonds and on coupon bonds, options on the short rate, options on yield spreads and on a basket of yields. A multi-factor GARCH framework of the short rate and its variance components is considered. We define a generalized zero-coupon bond and derive the moment generating function (MGF) of the discount bond log-price. The solution method relies on Fourier-inverting the MGF to compute the cumulative probabilities. The solution is found very accurate and offers considerable savings in computation time when compared to Monte Carlo simulation.

JEL Classification: G12; G13.

Keywords: characteristic functions; Fourier transform; GARCH models; Gauss-Laguerre quadrature rule; interest rate derivatives.

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I. Introduction

Since the seminal work in Engle (1982) and Bollerslev (1986), GARCH models have been widely used to describe the dynamics of financial time series, especially in equity and foreign exchange markets (see, for example, Bollerslev, Chou, and Kroner (1992) for a literature review). Recently, a number of GARCH processes have also been suggested in the literature to model the dynamics of interest rates (see, among others, Ball and Torous (1999) and Heston and Nandi (1999)). This paper develops a three-factor GARCH model of short term interest rates and presents semi-closed-form solutions to a wide variety of interest rate derivative prices. Our solution method relies on inverting the characteristic functions using Fourier transform and derives the corresponding cumulative probabilities. When compared to some alternative approaches in the literature, our method is found to be both accurate and computationally efficient.

Our choice of a three-factor GARCH model for the interest rate processes can be justified as follows. Firstly, as concluded in Litterman and Scheinkman (1991), we need at least three factors to adequately model the interest rate dynamics. Therefore, our model consists of three factors. Secondly, interest rate volatility is surely stochastic and changes over time (see Fong and Vasicek (1991), Andersen and Lund (1997), Kalimipalli and Susmel (2004), and Trolle and Schwartz (2009) etc.). The stochastic nature of interest rate volatility is also evident from taking a look at Tables 1, 2, and 3. In these tables, we compute the summary statistics and covariance matrix of the daily U.S. Treasury rates as well as their first order differences for a number of maturities over the time period of 2001 to 2008 (source of data: H.15 release at the U.S. Federal Reserve Board). The tables clearly show that interest rate series are heteroskedastic and non-normal. Thus in our model we treat interest rate volatility as stochastic.

[Tables 1, 2, and 3 are about here.]

Thirdly, in Table 4 we conduct a principal component analysis (PCA hereafter) of the daily variances of our Treasury time series. The first two components can explain 90.10% and 9.02% of the movements in daily variances, respectively. As a result, in our model we use two volatility factors (apart from the short rate factor, see Eq. (1) in Section 2 below), in order to better capture the dynamics of interest rate volatility.

[Table 4 is about here.]

Finally, our use of two volatility factors is similar to the approach taken in Christoffersen, Heston, and Jacobs (2009) and Gauthier and Possamai (2009). These authors are concerned about equity market instead. Their results suggest that the use of two volatility factors can better explain the slope and the level of the "volatility smirk" found in the equity option market.

A number of authors have also recently adopted some GARCH processes to model interest rate processes. For example, see the paper by Cvsa and Ritchken (2001). However, our model framework is different from theirs, and more importantly, we are able to derive semi-closed-form solutions to interest derivative prices, whereas they rely on some numerical methods.

The rest of the paper is organized as follows. In Section 2, we present our three-factor GARCH model of interest rates. In Section 3, we derive the pricing formulas for discount bonds, zero-coupon bond options, coupon bond options, short rate options, average rate options, yield spread options, and yield basket options, respectively. Section 4 contains several numerical examples to illustrate the computation of various option prices using our approach. Finally, Section 5 concludes. All technical details are in the Appendices.

2. Three-Factor GARCH Model for the Short Rate

We consider the following three-factor Heston-Nandi GARCH model (2000) under the physical probability (P):[1]

$$\begin{cases} r_{t+1} = r_t + \kappa(\theta - r_t) + \lambda h_{t+1} + \mu v_{t+1} + \sqrt{h_{t+1}} z_{t+1} + \sqrt{v_{t+1}} w_{t+1} \\ h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 \left(z_t - \tilde{\varphi} \sqrt{h_t} \right)^2 \\ v_{t+1} = \alpha_0 + \alpha_1 v_t + \alpha_2 \left(w_t - \tilde{\gamma} \sqrt{v_t} \right)^2 \end{cases} \tag{1}$$

[1] Note that this setting is also an extension of the credit spread GARCH model proposed in Tahani (2006).

where $\{z_t\ :1 \le t\}$ and $\{w_t\ :1 \le t\}$ are two independent sequences of independent standard normal variables; h_{t+1} and v_{t+1} are the conditional variance components of the short rate r_{t+1} known at time *t*. The parameters $(\tilde{\varphi}, \tilde{\gamma})$ control for the skewness or the asymmetry of the distribution of the short rate. The conditional covariance of the short rate and its variance components is:

$$\begin{cases} Cov_t(r_{t+1}, h_{t+2}) = -2\tilde{\varphi}\beta_2 h_{t+1} \\ Cov_t(r_{t+1}, v_{t+2}) = -2\tilde{\gamma}\alpha_2 v_{t+1} \end{cases} \tag{2}$$

Following Fong and Vasicek (1991) and Cvsa and Ritchken (2001), we assume that the risk premia are given by $\lambda\sqrt{h_{t+1}}$ and $\mu\sqrt{v_{t+1}}$, respectively. We can then rewrite Eq. (1) under a risk-neutral measure (*Q*) as:

$$\begin{cases} r_{t+1} = r_t + \kappa(\theta - r_t) + \sqrt{h_{t+1}} z^*_{t+1} + \sqrt{v_{t+1}} w^*_{t+1} \\ h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 \left(z^*_t - \varphi\sqrt{h_t}\right)^2 \\ v_{t+1} = \alpha_0 + \alpha_1 v_t + \alpha_2 \left(w^*_t - \gamma\sqrt{v_t}\right)^2 \end{cases} \tag{3}$$

where

$$\begin{cases} z^*_t = z_t + \lambda\sqrt{h_t} \quad ; \quad \varphi = \tilde{\varphi} + \lambda \\ w^*_t = w_t + \mu\sqrt{v_t} \quad ; \quad \gamma = \tilde{\gamma} + \mu \end{cases} \tag{4}$$

and (z^*_{t+1}, w^*_{t+1}) are two independent risk-neutral standard normal variables conditional on the information available at time *t*. Figure 1 presents some simulation results of the short rate under the three-factor GARCH model above.

[Figure 1 is about here.]

3. Pricing Formulas

This section illustrates the valuation of a wide variety of bonds and options such as discount bond options, coupon bond options, options on the short rate, options on a yield spread and options on a basket of yields. The generalization of the model to the multi-factor framework is presented in Appendix F.

3.1 Discount bonds

The time-t discount bond with maturity date $t+n$ is given under the risk-neutral measure Q by (see Appendix A for details):

$$P(t,t+n) \equiv E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\right) = \exp\left(-A(n)r_t + B(n)h_{t+1} + D(n)v_{t+1} + C(n)\right) \qquad (5)$$

where the functions A, B, C and D are computed recursively using the initial values $A(0) = B(0) = D(0) = C(0) = 0$ and the recurrence equations presented in Appendix A.

3.2 Discount bond options

A call option with maturity date $t+n$ on the discount bond $P(t+n,t+n+m)$ and strike price K has a price given by:

$$\begin{aligned} Call_{zero}(t,n,m,K) &\equiv E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\left(P(t+n,t+n+m)-K\right)^+\right) \\ &= P(t,t+n+m)\times Q^{t+n+m}\left(\ln P(t+n,t+n+m)\geq \ln K\right) \qquad (6)\\ &\quad - K\times P(t,t+n)\times Q^{t+n}\left(\ln P(t+n,t+n+m)\geq \ln K\right) \end{aligned}$$

where Q^{t+n+s}, $s\in\{0,m\}$, is the forward measure[2] with the following Radon-Nikodym derivative:

$$\frac{dQ^{t+n+s}}{dQ} \equiv \frac{\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}P(t+n,t+n+s)}{P(t,t+n+s)} \quad ; \quad s\in\{0,m\} \qquad (7)$$

The moment generating function (MGF hereafter) of the logarithm of the zero-coupon bond $P(t+n,t+n+m)$ under the forward measure is:

$$MGF_{zero}(\psi;t,n,m,s) \equiv E_t^{Q^{t+n+s}}\left(\exp\{\psi\times\ln P(t+n,t+n+m)\}\right) \qquad (8)$$

Given the expression for the discount bond in Eq. (5) and the Radon-Nikodym derivative in Eq. (7), we have:

[2] See Geman, El Karoui, and Rochet (1995) for the derivation of the forward measure and its use in option pricing.

$$MGF_{zero}(\psi;t,n,m,s)=\frac{1}{P(t,t+n+s)}E_t^Q\begin{pmatrix}\exp\left\{-\sum_{i=t}^{t+n-1}r_i\right\}\times\exp\left\{-\left(A(s)+\psi A(m)\right)r_{t+n}\right\}\\ \times\exp\left\{\left(B(s)+\psi B(m)\right)h_{t+n+1}\right\}\\ \times\exp\left\{\left(D(s)+\psi D(m)\right)v_{t+n+1}\right\}\\ \times\exp\left\{\left(C(s)+\psi C(m)\right)\right\}\end{pmatrix} \quad (9)$$

Following Tahani and Li (2011), let us define the *generalized zero-coupon* as:[3]

$$\Pi(t,t+n)\equiv E_t^Q\left(\exp\left\{-\overline{R}\sum_{i=t}^{t+n-1}r_i\right\}\times\exp\left\{-\overline{A}r_{t+n}+\overline{B}h_{t+n+1}+\overline{D}v_{t+n+1}+\overline{C}\right\}\right) \quad (10)$$

The generalized zero-coupon $\Pi(t,t+n)$ will prove very useful in the pricing of a multitude of derivative securities on interest rates. As previously derived for the discount bond formula, it can be shown that (see Appendix B for details):

$$\Pi(t,t+n;\overline{A},\overline{B},\overline{D},\overline{C},\overline{R})=\exp\left(-A(n)r_t+B(n)h_{t+1}+D(n)v_{t+1}+C(n)\right) \quad (11)$$

where the initial values are $A(0)=\overline{A}, B(0)=\overline{B}, D(0)=\overline{D}, C(0)=\overline{C}$. The MGF in Eq. (9) can therefore be computed using Eq. (11) where:[4]

$$\begin{cases}\overline{R}=1 \quad ; \quad \overline{A}=A(s)+\psi A(m) \quad ; \quad \overline{B}=B(s)+\psi B(m)\\ \overline{D}=D(s)+\psi D(m) \quad ; \quad \overline{C}=C(s)+\psi C(m)\end{cases}$$

The probabilities in Eq. (6) for $s\in\{0,m\}$ can now be recovered as inverse Fourier transforms of the characteristic function[5]:

$$Q^{t+n+s}\left(\ln P(t+n,t+n+m)\geq\ln K\right)=\frac{1}{2}+\frac{1}{\pi}\int_0^{+\infty}\mathrm{Re}\left\{K^{-i\psi}\frac{MGF_{zero}(i\psi,t,n,m,s)}{i\psi}\right\}d\psi \quad (12)$$

The call price is then computed as in Eq. (6). The put price can be computed using the call-put parity relationship.[6]

3.3 Coupon bond options

We define the price at time *t* of an *M*-coupon bond with maturity date *t*+*m* by:

[3] The generalized zero-coupon should be understood as $\Pi(t,t+n;\overline{A},\overline{B},\overline{D},\overline{C},\overline{R})$.

[4] Note that the discount bond price $P(t,t+n+s)$ is equivalent to $\Pi(t,t+n+s;0,0,0,0,1)$.

[5] The cumulative probabilities obtained by inverse Fourier transforms of the MGFs are computed using the Gauss-Laguerre quadrature rule. Please refer to Tahani and Li (2011) for the details.

[6] $Put_{zero}(t,n,m,K)=Call_{zero}(t,n,m,K)+KP(t,t+n)-P(t,t+n+m)$.

$$H(t,t+m,M)=\sum_{i=1}^{M}a_i P(t,t+i\tfrac{m}{M}) \tag{13}$$

where M is the number of coupons and $(a_i)_{1\le i\le M}$ are the cash flows. Following Munk (1999) and Tahani and Li (2011), we define the *stochastic duration* of the coupon bond as the time to maturity of the zero-coupon bond having the same *instantaneous variance of relative price changes*. The price of an option on the coupon bond is therefore approximated using the option on the corresponding zero-coupon bond with the same stochastic duration. More specifically, the instantaneous variance of the relative price changes of the zero-coupon bond $P(t,t+q)$ is defined as:[7]

$$\begin{aligned}
Var_t^Q\left(\frac{\Delta P}{P}\right) &\equiv E_t^Q\left(\left\{\frac{\Delta P}{P}-E_t^Q\left(\frac{\Delta P}{P}\right)\right\}^2\right)\\
&=\frac{1}{P^2(t,t+q)}Var_t^Q\big(P(t+1,t+q)\big)\\
&=\frac{1}{P^2(t,t+q)}\begin{bmatrix}\Pi\big(t,t+1;2A(q-1),2B(q-1),2D(q-1),2C(q-1),0\big)\\ -\Pi^2\big(t,t+1;A(q-1),B(q-1),D(q-1),C(q-1),0\big)\end{bmatrix}
\end{aligned} \tag{14}$$

Similarly, the instantaneous variance of the relative price changes of the coupon bond $H(t,t+m,M)$ is given by:

$$Var_t^Q\left(\frac{\Delta H}{H}\right)=\frac{1}{H^2(t,t+m,M)}\sum_{i=1}^{M}a_i^{\,2}Var_t^Q\big(P(t+1,t+i\tfrac{m}{M})\big) \tag{15}$$

where $Var_t^Q\big(P(t+1,t+i\tfrac{m}{M})\big)$ is given as in Eq. (14). The value of q that equates the variances in Eqs. (14-15), denoted by $\bar{q}$ hereafter, is the stochastic duration of the coupon bond. Note that $\bar{q}$ must be an integer and hence it must be solved for recursively. The price of the call option with maturity *t+n* on the coupon bond $H(t+n,t+n+m,M)$ can be approximated by a multiple of the price of the call option on the zero-coupon bond $P(t+n,t+n+\bar{q})$. More formally, we have:

$$Call_{bond}(t,n,m,M,K)\cong \zeta\, Call_{zero}(t,n,\bar{q},\tfrac{K}{\zeta}) \tag{16}$$

where $\zeta=\frac{H(t,t+m,M)}{P(t,t+\bar{q})}$. The put option can be approximated similarly.

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[7] In our discrete-time framework, we define the relative price change by $\frac{\Delta P}{P}=\frac{P(t+1,t+q)}{P(t,t+q)}-1$.

3.4 Short rate options

A call option with maturity date t+n on the short rate with strike K has a price equal to:

$$\begin{aligned} Call_{rate}(t,n,K) &\equiv E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\left(r_{t+n}-K\right)^+\right) \\ &= \left.\frac{\partial\Pi(t,t+n;-\psi,0,0,0,1)}{\partial\psi}\right|_{\psi=0}\times Q^r\left(r_{t+n}\geq K\right)-K\times P(t,t+n)\times Q^{t+n}\left(r_{t+n}\geq K\right) \end{aligned} \tag{17}$$

where the probabilities and their corresponding MGFs are derived in Appendix C.

3.5 Average rate Options

A call option with maturity date t+n on the average rate, defined as $\frac{1}{n+m}\sum_{i=t-m}^{t+n-1} r_i$, where t-$m \leq t \leq t$+n, and a strike K has a price equal to:

$$\begin{aligned} Call_{avg}(t,n,m,K) &\equiv E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\left(\tfrac{1}{n+m}\sum_{i=t-m}^{t+n-1} r_i - K\right)^+\right) \\ &= \left.\frac{\partial\Pi(t,t+n;0,0,0,0,-\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=-1}\times Q^{Avg}\left(L\right)-\left(K-\tfrac{1}{n+m}\sum_{i=t-m}^{t-1} r_i\right)\times P(t,t+n)\times Q^{t+n}\left(L\right) \end{aligned} \tag{18}$$

where $L=\left\{\tfrac{1}{n+m}\sum_{i=t}^{t+n-1} r_i \geq K-\tfrac{1}{n+m}\sum_{i=t-m}^{t-1} r_i\right\}$. The details of the derivation are provided in Appendix D.

3.6 Yield spread options

The continuously-compounded yield for the period $(t,t+n)$ is defined as:

$$Y(t,t+n)=\frac{A(n)r_t - B(n)h_{t+1} - D(n)v_{t+1} - C(n)}{n} \tag{19}$$

A call option with maturity date t+n and a strike K on a spread between two yields of different maturities can be priced as follows:

$$
\begin{aligned}
&Call_{spread}(t,n,m_1,m_2,K) \\
&\equiv E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\left(Y(t+n,t+n+m_2)-Y(t+n,t+n+m_1)-K\right)^+\right) \\
&=\left.\frac{\partial\Pi(t,t+n;-\psi\tilde{A},-\psi\tilde{B},-\psi\tilde{D},-\psi\tilde{C},1)}{\partial\psi}\right|_{\psi=0}\times Q^{\Delta Y}(L)-P(t,t+n)\times K\times Q^{t+n}(L)
\end{aligned}
\tag{20}
$$

where $L=\{Y(t+n,t+n+m_2)-Y(t+n,t+n+m_1)\geq K\}$. The details of the derivation are given in Appendix E. A special case of this type is the *exchange option* obtained by simply setting $K = 0$.

3.7 Yield basket options

We can generalize the previous calculation to price an option on a basket of yields. The call on the basket can be written as:

$$
\begin{aligned}
&Call_{basket}(t,n,\{m_1,...,m_l\},K) \\
&\equiv E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\left(\sum_{j=1}^{l}\omega_j Y(t+n,t+n+m_j)-K\right)^+\right) \\
&=\left.\frac{\partial\Pi(t,t+n;-\psi\Sigma_A,-\psi\Sigma_B,-\psi\Sigma_D,-\psi\Sigma_C,1)}{\partial\psi}\right|_{\psi=0}\times Q^{Basket}(L)-P(t,t+n)\times K\times Q^{t+n}(L)
\end{aligned}
\tag{21}
$$

where $(\omega_j)_{1\leq j\leq l}$ are the basket weights and $L=\left\{\sum_{j=1}^{l}\omega_j Y(t+n,t+n+m_j)\geq K\right\}$. $\Sigma_A,\Sigma_B,\Sigma_C,\Sigma_D$ and the details of the derivation are provided in Appendix E.

4. Numerical Examples

This section presents some numerical examples to assess the accuracy and the efficiency of our proposed solution method. The benchmark option prices are given by a Monte Carlo simulation based on 10^5 paths repeated 50 times. The parameter values chosen are similar to those parameters in Heston and Nandi (2000) and have been adjusted for our model as per the PCA analysis in Table 4.

First, we price a three-month at-the-money forward call option on a three-month zero-coupon bond with a face value of \$100. The strike price K of this call option is \$98.5229. Table 5 shows how our model price converges to the benchmark price given by the Monte Carlo simulations for different quadrature orders as well as the associated standard deviation. Note that an order of 28 is sufficient to obtain a very accurate option price. The second example is a three-month call on a six-month zero-coupon bond. The strike price of this call option is $96.9252. The third example is a three-month call on a one-year discount bond. Here the strike price is $93.6139. Finally, the fourth example examines a six-month call option with a strike price of $93.3474 on a one-year zero-coupon bond. It is shown in the table that a high degree of precision can be achieved at a quadrature order of (as low as) 18.

[Table 5 is about here.]

Next we value some options on coupon bonds as an illustration. The first example in Table 6 is a three-month call option on a one-year coupon bond that pays a quarterly coupon of $4 and has a stochastic duration of 205 days. The call has a strike price of $108.99. The second example is a three-month call on a one-year step-up bond, which has a quarterly step-up coupon of $4, $6, $8, and $10, respectively. The stochastic duration of the coupon bond is 181 days and the call has a strike price of $120.35. Again, our solution method can value the call option in a very accurate and efficient manner. Indeed, it takes about a quadrature order of 23 to achieve an accurate price for both calls.

[Table 6 is about here.]

5. Conclusion

This paper derives semi-closed-form pricing formulas for various interest rate derivatives under a three-factor GARCH model. Our solution method consists of deriving the moment generating function of the logarithm of the zero-coupon bond under the forward measure and Fourier-inverting the corresponding characteristic function using the Gauss-Laguerre quadrature rule. The numerical analysis in this paper shows that our approach is very accurate and fast and compares favorably to some alternative methods in the literature.

Our valuation approach can be extended to price more complex fixed income derivatives, such as credit risk derivatives. Extending our solution method to these applications is an interesting venue for our future research.

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Appendix A: Discount bond formula

It is obvious that since $P(t,t)=1$, the initial values of functions *A*, *B*, *D* and *C* are $A(0)=0, B(0)=0, D(0)=0, C(0)=0$. We also have $P(t,t+1)=\exp(-r_t)$, that means $A(1)=1, B(1)=0,\ \ D(1)=0, C(1)=0$. We now show that $P(t,t+2)\equiv E_t^Q\left(\exp(-r_t-r_{t+1})\right)$ is given by Eq. (5):

$$\begin{aligned} P(t,t+2) &\equiv E_t^Q\left(\exp(-r_t-r_{t+1})\right) \\ &= \exp(-r_t)\times E_t^Q\left(\exp(-r_{t+1})\right) \\ &= \exp(-r_t)\times E_t^Q\left(\exp\left\{-r_t-\kappa(\theta-r_t)-\sqrt{h_{t+1}}z_{t+1}^*-\sqrt{v_{t+1}}w_{t+1}^*\right\}\right) \qquad (A.1) \\ &= \exp\left\{-r_t-(1-\kappa)r_t-\kappa\theta\right\}\times E_t^Q\left(\exp\left\{-\sqrt{h_{t+1}}z_{t+1}^*-\sqrt{v_{t+1}}w_{t+1}^*\right\}\right) \\ &= \exp\left\{-(2-\kappa)r_t-\kappa\theta+\tfrac{1}{2}h_{t+1}+\tfrac{1}{2}v_{t+1}\right\} \end{aligned}$$

Note that z_{t+1}^* and w_{t+1}^* are two independent standard normal variables conditional on time *t*. This shows that that:

$$P(t,t+2)=\exp\left(-A(2)r_t+B(2)h_{t+1}+D(2)v_{t+1}+C(2)\right) \qquad (A.2)$$

Let us now compute the following general expectation where η and ω are constant:

$$\begin{aligned} &E_t^Q\left(\exp\left\{-\eta\sqrt{h_{t+1}}z_{t+1}^*+\omega h_{t+2}\right\}\right)=E_t^Q\left(\exp\left\{-\eta\sqrt{h_{t+1}}z_{t+1}+\omega\beta_0+\omega\beta_1h_{t+1}+\omega\beta_2\left(z_{t+1}^*-\varphi\sqrt{h_{t+1}}\right)^2\right\}\right) \\ &=\exp\left\{\omega\beta_0+\omega\beta_1h_{t+1}+\omega\beta_2\varphi^2h_{t+1}\right\}\times E_t^Q\left(\exp\left\{-\eta\sqrt{h_{t+1}}z_{t+1}^*+\omega\beta_2\left(z_{t+1}^{*\,2}-2\varphi\sqrt{h_{t+1}}z_{t+1}^*\right)\right\}\right) \\ &=\exp\left\{\omega\beta_0+\omega\beta_1h_{t+1}+\omega\beta_2\varphi^2h_{t+1}\right\}\times E_t^Q\left(\exp\left\{\omega\beta_2\left(z_{t+1}^{*\,2}-2\left(\varphi+\frac{\eta}{2\omega\beta_2}\right)\sqrt{h_{t+1}}z_{t+1}^*\right)\right\}\right) \\ &=\exp\left\{\omega\beta_0+\omega\beta_1h_{t+1}+\omega\beta_2\varphi^2h_{t+1}-\omega\beta_2\left(\varphi+\frac{\eta}{2\omega\beta_2}\right)^2h_{t+1}\right\}\times E_t^Q\left(\exp\left\{\omega\beta_2\left[z_{t+1}^*-\left(\varphi+\frac{\eta}{2\omega\beta_2}\right)\sqrt{h_{t+1}}\right]^2\right\}\right) \\ &=\exp\left\{-\tfrac{1}{2}\ln(1-2\omega\beta_2)+\omega\beta_0+\omega\beta_1h_{t+1}+\omega\beta_2\varphi^2h_{t+1}+\frac{1}{2(1-2\omega\beta_2)}\left(2\omega\varphi\beta_2+\eta\right)^2h_{t+1}\right\} \end{aligned}$$

(A.3)

In deriving Eq. (A.3) above, we have used the fact that for a standard normal variable *z* and constants *a* and *b*, we have:

$$E\left(\exp\left\{a(b+z)^2\right\}\right)=\exp\left\{-\tfrac{1}{2}\ln(1-2a)+\frac{ab^2}{1-2a}\right\} \qquad (A.4)$$

We assume that the result is now true for a general maturity *n* and show that it is also true for a maturity *n* + 1:

$$
\begin{aligned}
P(t,t+n+1) &\equiv E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n} r_i\right\}\right) = \exp(-r_t)\times E_t^Q\left(E_{t+1}^Q\left(\exp\left\{-\sum_{i=t+1}^{t+n} r_i\right\}\right)\right) \\
&= \exp(-r_t)\times E_t^Q\left(P(t+1,t+n+1)\right) \qquad \text{(A.5)} \\
&= \exp(-r_t)\times E_t^Q\left(\exp\left\{-A(n)r_{t+1}+B(n)h_{t+2}+D(n)v_{t+2}+C(n)\right\}\right) \\
&= \exp\left\{-r_t+C(n)\right\}\times E_t^Q\left(\exp\left\{-A(n)r_{t+1}+B(n)h_{t+2}+D(n)v_{t+2}\right\}\right) \\
&= \exp\left\{-r_t+C(n)\right\}\times E_t^Q\left(\exp\left\{\begin{array}{l}-A(n)\left(r_t+\kappa(\theta-r_t)+\sqrt{h_{t+1}}z_{t+1}^*+\sqrt{v_{t+1}}w_{t+1}^*\right) \\ +B(n)h_{t+2}+D(n)v_{t+2}\end{array}\right\}\right) \\
&= \exp\left\{-r_t+C(n)-A(n)\left(r_t+\kappa(\theta-r_t)\right)\right\} \\
&\qquad \times E_t^Q\left(\exp\left\{-A(n)\sqrt{h_{t+1}}z_{t+1}^*+B(n)h_{t+2}\right\}\right)\times E_t^Q\left(\exp\left\{-A(n)\sqrt{v_{t+1}}z_{t+1}^*+D(n)v_{t+2}\right\}\right)
\end{aligned}
$$

Using the previous result, it is straightforward to show that:

$$
E_t^Q\left(\exp\left\{-A(n)\sqrt{h_{t+1}}z_{t+1}^*+B(n)h_{t+2}\right\}\right) = \exp\left\{\begin{array}{l}-\tfrac{1}{2}\ln(1-2\beta_2 B(n))+\beta_0 B(n) \\ +\left(\beta_1 B(n)+\beta_2\varphi^2 B(n)+\dfrac{\left[2\varphi\beta_2 B(n)+A(n)\right]^2}{2(1-2\beta_2 B(n))}\right)h_{t+1}\end{array}\right\}
$$

The same can be shown for the expectation involving v_{t+2}. After collecting and rearranging the terms, we obtain the recurrence equations below:

$$
\begin{cases}
A(n+1) = 1+(1-\kappa)A(n) \\
B(n+1) = \left(\beta_1+\beta_2\varphi^2\right)B(n)+\dfrac{1}{2(1-2\beta_2 B(n))}\left(A(n)+2\varphi\beta_2 B(n)\right)^2 \\
D(n+1) = \left(\alpha_1+\alpha_2\gamma^2\right)D(n)+\dfrac{1}{2(1-2\alpha_2 D(n))}\left(A(n)+2\gamma\alpha_2 D(n)\right)^2 \\
C(n+1) = C(n)-\kappa\theta A(n)+\beta_0 B(n)-\tfrac{1}{2}\ln\left(1-2\beta_2 B(n)\right) \\
\qquad\qquad +\alpha_0 D(n)-\tfrac{1}{2}\ln\left(1-2\alpha_2 D(n)\right)
\end{cases}
\qquad \text{(A.6)}
$$

Appendix B : Generalized zero-coupon

Since $\Pi(t,t)=\exp\left(-\overline{A}r_t+\overline{B}h_{t+1}+\overline{D}v_{t+1}+\overline{C}\right)$, the initial values are $\overline{A},\overline{B},\overline{D}$ and $\overline{C}$ respectively. We now show that $\Pi(t,t+1)$ is given by Eq. (11):

$$\Pi(t,t+1) \equiv E_t^Q\left(\exp\left\{-\overline{R}r_t\right\}\times\exp\left\{-\overline{A}r_{t+1}+\overline{B}h_{t+2}+\overline{D}v_{t+2}+\overline{C}\right\}\right)$$
$$=\exp\left\{-\overline{R}r_t\right\}\times E_t^Q\left(\exp\left\{-\overline{A}r_{t+1}+\overline{B}h_{t+2}+\overline{D}v_{t+2}+\overline{C}\right\}\right) \qquad \text{(B.1)}$$

Using the derivation of the discount bond pricing formula in Appendix A, we only need some substitutions to obtain:

$$\Pi(t,t+1)=\exp\left\{-\overline{R}r_t+\overline{C}-\overline{A}\left(r_t+\kappa(\theta-r_t)\right)\right\}$$
$$\times\exp\left\{\beta_0\overline{B}-\tfrac{1}{2}\ln\left(1-2\beta_2\overline{B}\right)+\left(\beta_1\overline{B}+\beta_2\varphi^2\overline{B}+\frac{1}{2(1-2\beta_2\overline{B})}\left(\overline{A}+2\varphi\beta_2\overline{B}\right)^2\right)h_{t+1}\right\}$$
$$\times\exp\left\{\alpha_0\overline{D}-\tfrac{1}{2}\ln\left(1-2\alpha_2\overline{D}\right)+\left(\alpha_1\overline{D}+\alpha_2\gamma^2\overline{D}+\frac{1}{2(1-2\alpha_2\overline{D})}\left(\overline{A}+2\gamma\alpha_2\overline{D}\right)^2\right)v_{t+1}\right\}$$

Collecting and rearranging the terms yields the result for $\Pi(t,t+1)$. We assume that the result is now true for a maturity *n* and show that it is also true for a maturity *n* + 1:

$$\Pi(t,t+n+1)\equiv E_t^Q\left(\exp\left\{-\overline{R}\sum_{i=t}^{t+n}r_i\right\}\times\exp\left\{-\overline{A}r_{t+n+1}+\overline{B}h_{t+n+2}+\overline{D}v_{t+n+2}+\overline{C}\right\}\right)$$
$$=\exp\left\{-\overline{R}r_t\right\}\times E_t^Q\left(\Pi(t+1,t+n+1)\right) \qquad \text{(B.2)}$$
$$=\exp\left\{-\overline{R}r_t\right\}\times E_t^Q\left(\exp\left(-A(n)r_{t+1}+B(n)h_{t+2}+D(n)v_{t+2}+C(n)\right)\right)$$

Since this expectation is similar to the ones in Eqs. (A.5) and (B.1), we can show that we obtain the same recurrence equations as in Eq. (A.6) with this small adjustment:

$$A(n+1)=\overline{R}+(1-\kappa)A(n) \qquad \text{(B.3)}$$

and $A(0)=\overline{A}, B(0)=\overline{B}, D(0)=\overline{D}, C(0)=\overline{C}$.

Appendix C: Short rate options

The call option on the short rate is priced as:

$$Call_{rate}(t,n,K)\equiv E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1}r_i\right\}\times\left(r_{t+n}-K\right)^+\right)$$
$$=E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1}r_i\right\}r_{t+n}\right)\times E_t^{Q^r}\left(1_{\{r_{t+n}\geq K\}}\right)-K\times P(t,t+n)\times E_t^{Q^{t+n}}\left(1_{\{r_{t+n}\geq K\}}\right) \qquad \text{(C.1)}$$

where Q^r has the following Radon-Nikodym derivative:

$$\frac{dQ^r}{dQ} \equiv \frac{\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\} r_{t+n}}{E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\} r_{t+n}\right)} \tag{C.2}$$

The MGF of the short rate at time *t*+*n* under the forward measure Q^r is given by:

$$\begin{aligned} MGF_{rate}^{I}(\psi;t,n) &\equiv E_t^{Q^r}\left(\exp\{\psi r_{t+n}\}\right) \\ &= \frac{1}{E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\} r_{t+n}\right)} \times E_t^Q\left(\exp\left\{\psi r_{t+n} - \sum_{i=t}^{t+n-1} r_i\right\} r_{t+n}\right) \\ &= \frac{1}{\left.\frac{\partial \Pi(t,t+n;-\psi,0,0,0,1)}{\partial \psi}\right|_{\psi=0}} \times \frac{\partial \Pi(t,t+n;-\psi,0,0,0,1)}{\partial \psi} \end{aligned} \tag{C.3}$$

which yields the first part of Eq. (C.1) after the use of inverse Fourier transform to get the probability:

$$Q^r\left(r_{t+n} \geq K\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \operatorname{Re}\left\{e^{-i\psi K}\,\frac{MGF_{rate}^{I}(i\psi;t,n)}{i\psi}\right\} d\psi \tag{C.4}$$

Note that $\Pi(t,t+n;-\psi,0,0,0,1)$ can be computed as:

$$\begin{cases} \Pi(t,t+n;-\psi,0,0,0,1) = \exp\left(-A(n)r_t + B(n)h_{t+1} + D(n)v_{t+1} + C(n)\right) \\ A(0) = -\psi, B(0) = 0, D(0) = 0, C(0) = 0, \overline{R} = 1 \end{cases} \tag{C.5}$$

and its partial derivative can be computed recursively using Eq. (A.6) as:

$$\frac{\partial \Pi(t,t+n;-\psi,0,0,0,1)}{\partial \psi} = \left(-\frac{\partial A(n)}{\partial \psi} r_t + \frac{\partial B(n)}{\partial \psi} h_{t+1} + \frac{\partial D(n)}{\partial \psi} v_{t+1} + \frac{\partial C(n)}{\partial \psi}\right) \times \Pi(t,t+n;-\psi,0,0,0,1) \tag{C.6}$$

where

$$
\begin{cases}
\dfrac{\partial A(n+1)}{\partial \psi} = -(1-\kappa)^{n+1} \\
\dfrac{\partial B(n+1)}{\partial \psi} = \left(\beta_1 + \beta_2 \varphi^2\right)\dfrac{\partial B(n)}{\partial \psi} + \dfrac{\left(A(n) + 2\varphi\beta_2 B(n)\right)}{(1-2\beta_2 B(n))}\left(\dfrac{\partial A(n)}{\partial \psi} + 2\varphi\beta_2 \dfrac{\partial B(n)}{\partial \psi}\right) \\
\qquad + \beta_2 \dfrac{\partial B(n)}{\partial \psi}\dfrac{\left(A(n) + 2\varphi\beta_2 B(n)\right)^2}{(1-2\beta_2 B(n))^2} \\
\dfrac{\partial D(n+1)}{\partial \psi} = \left(\alpha_1 + \alpha_2 \gamma^2\right)\dfrac{\partial D(n)}{\partial \psi} + \dfrac{\left(A(n) + 2\gamma\alpha_2 D(n)\right)}{(1-2\alpha_2 D(n))}\left(\dfrac{\partial A(n)}{\partial \psi} + 2\gamma\alpha_2 \dfrac{\partial D(n)}{\partial \psi}\right) \\
\qquad + \alpha_2 \dfrac{\partial D(n)}{\partial \psi}\dfrac{\left(A(n) + 2\gamma\alpha_2 D(n)\right)^2}{(1-2\alpha_2 D(n))^2} \\
\dfrac{\partial C(n+1)}{\partial \psi} = \dfrac{\partial C(n)}{\partial \psi} - \kappa\theta\dfrac{\partial A(n)}{\partial \psi} + \beta_0 \dfrac{\partial B(n)}{\partial \psi} + \beta_2 \dfrac{\partial B(n)}{\partial \psi}\dfrac{1}{(1-2\beta_2 B(n))} \\
\qquad + \alpha_0 \dfrac{\partial D(n)}{\partial \psi} + \alpha_2 \dfrac{\partial D(n)}{\partial \psi}\dfrac{1}{(1-2\alpha_2 D(n))}
\end{cases}
\tag{C.7}
$$

with the following initial values $A(0) = -\psi, B(0) = D(0) = C(0) = 0, \overline{R} = 1, \dfrac{\partial A(0)}{\partial \psi} = -1$,

and $\dfrac{\partial B(0)}{\partial \psi} = \dfrac{\partial D(0)}{\partial \psi} = \dfrac{\partial C(0)}{\partial \psi} = 0$.

On the other hand, the MGF of the short rate at time *t*+*n* under the forward measure Q^{t+n} is given by:

$$
\begin{aligned}
MGF_{rate}^{II}(\psi; t, n) &\equiv E_t^{Q^{t+n}}\left(\exp\{\psi r_{t+n}\}\right) \\
&= \frac{1}{P(t,t+n)} \times E_t^{Q}\left(\exp\left\{\psi r_{t+n} - \sum_{i=t}^{t+n-1} r_i\right\}\right) \\
&= \frac{\Pi(t, t+n; -\psi, 0,0,0,1)}{P(t,t+n)}
\end{aligned}
\tag{C.8}
$$

The probability $Q^{t+n}\left(r_{t+n} \geq K\right)$ is computed in a similar way as in Eq. (C.4).

Appendix D: Average rate options

The call option on the average rate is priced as:

$$Call_{avg}(t,n,m,K) \equiv E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\left(\tfrac{1}{n+m}\sum_{i=t-m}^{t+n-1} r_i - K\right)^+\right)$$

$$= \frac{1}{n+m}E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\sum_{i=t-m}^{t+n-1} r_i\right)\times Q^{Avg}(L) - \left(K - \frac{\sum_{i=t-m}^{t-1} r_i}{n+m}\right)\times P(t,t+n)\times Q^{t+n}(L) \quad \text{(D.1)}$$

where $L = \left\{\tfrac{1}{n+m}\sum_{i=t}^{t+n-1} r_i \geq K - \tfrac{1}{n+m}\sum_{i=t-m}^{t-1} r_i\right\}$, and Q^{Avg} has the following Radon-Nikodym derivative:

$$\frac{dQ^{Avg}}{dQ} \equiv \frac{\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\sum_{i=t}^{t+n-1} r_i}{E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\sum_{i=t}^{t+n-1} r_i\right)} \quad \text{(D.2)}$$

The MGF of the average rate under Q^{Avg} is given by:

$$MGF_{avg}^I(\psi;t,n,m) \equiv E_t^{Q^{Avg}}\left(\exp\left\{\frac{\psi}{n+m}\sum_{i=t}^{t+n-1} r_i\right\}\right)$$

$$= \frac{1}{E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\sum_{i=t}^{t+n-1} r_i\right)}\times E_t^Q\left(\exp\left\{\left(\tfrac{\psi}{n+m}-1\right)\sum_{i=t}^{t+n-1} r_i\right\}\times\sum_{i=t}^{t+n-1} r_i\right) \quad \text{(D.3)}$$

$$= \frac{1}{\left.\frac{\partial\Pi(t,t+n;0,0,0,0,\overline{R})}{\partial\overline{R}}\right|_{\overline{R}=1}}\times\left.\frac{\partial\Pi(t,t+n;0,0,0,0,\overline{R})}{\partial\overline{R}}\right|_{\overline{R}=1-\frac{\psi}{n+m}}$$

Note that $\Pi(t,t+n;0,0,0,0,\overline{R})$ can be computed as:

$$\begin{cases}\Pi(t,t+n;0,0,0,0,\overline{R}) = \exp\left(-A(n)r_t + B(n)h_{t+1} + D(n)v_{t+1} + C(n)\right)\\ A(0)=0, B(0)=0, D(0)=0, C(0)=0\end{cases} \quad \text{(D.4)}$$

and its partial derivative can be computed recursively using Eq. (A.6) as:

$$\frac{\partial\Pi(t,t+n;0,0,0,0,\overline{R})}{\partial\overline{R}} = \left(-\frac{\partial A(n)}{\partial\overline{R}}r_t + \frac{\partial B(n)}{\partial\overline{R}}h_t + \frac{\partial D(n)}{\partial\overline{R}}v_t + \frac{\partial C(n)}{\partial\overline{R}}\right)\times\Pi(t,t+n;0,0,0,0,\overline{R}) \quad \text{(D.5)}$$

where we use the recurrence equations similar to Eq. (C.7), $\frac{\partial A(n)}{\partial \overline{R}} = \frac{1-(1-\kappa)^n}{\kappa}$ and $\frac{\partial B(0)}{\partial \overline{R}} = \frac{\partial D(0)}{\partial \overline{R}} = \frac{\partial C(0)}{\partial \overline{R}} = 0$.

On the other hand, the MGF of the average rate under the forward measure Q^{t+n} is given by:

$$\begin{aligned} MGF_{avg}^{II}(\psi; t, n, m) &\equiv E_t^{Q^{t+n}}\left(\exp\left\{ \frac{\psi}{n+m} \sum_{i=t}^{t+n-1} r_i \right\} \right) \\ &= \frac{1}{P(t,t+n)} \times E_t^Q \left(\exp\left\{ \left(\tfrac{\psi}{n+m} - 1 \right) \sum_{i=t}^{t+n-1} r_i \right\} \right) \\ &= \frac{\Pi(t, t+n; 0,0,0,0,1-\frac{\psi}{n+m})}{P(t,t+n)} \end{aligned} \tag{D.6}$$

Appendix E: Yield spread and yield basket options

The call option with maturity date $t+n$ and a strike K on the yield spread is given by:

$$\begin{aligned} & Call_{spread}(t, n, m_1, m_2, K) \\ & \equiv E_t^Q \left(\exp\left\{ -\sum_{i=t}^{t+n-1} r_i \right\} \times \left(Y(t+n, t+n+m_2) - Y(t+n, t+n+m_1) - K \right)^+ \right) \end{aligned} \tag{E.1}$$

The yield spread can be expressed as follows:

$$\begin{aligned} \Delta Y(t, n, m_1, m_2) &\equiv \frac{A(m_2) r_{t+n} - B(m_2) h_{t+n+1} - D(m_2) v_{t+n+1} - C(m_2)}{m_2} \\ &\quad - \frac{A(m_1) r_{t+n} - B(m_1) h_{t+n+1} - D(m_1) v_{t+n+1} - C(m_1)}{m_1} \\ &= \tilde{A} r_{t+n} - \tilde{B} h_{t+n+1} - \tilde{D} v_{t+n+1} - \tilde{C} \end{aligned} \tag{E.2}$$

where $\tilde{A} = \frac{\left(m_1 A(m_2) - m_2 A(m_1) \right)}{m_1 m_2}$ and with similar equations for $\tilde{B}, \tilde{D}$ and $\tilde{C}$. Therefore, we need to compute the following expectation:

$$E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\left(\Delta Y(t,n,m_1,m_2)-K\right)^+\right)$$
$$= E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\Delta Y(t,n,m_1,m_2)\right)\times Q^{\Delta Y}\left(\Delta Y(t,n,m_1,m_2)\geq K\right) \quad \text{(E.3)}$$
$$-P(t,t+n)\times K\times Q^{t+n}\left(\Delta Y(t,n,m_1,m_2)\geq K\right)$$

where $Q^{\Delta Y}$ has the following Radon-Nikodym derivative:

$$\frac{dQ^{\Delta Y}}{dQ} \equiv \frac{\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\Delta Y(t,n,m_1,m_2)}{E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\Delta Y(t,n,m_1,m_2)\right)} \quad \text{(E.4)}$$

The MGF of the yield spread under the forward measure $Q^{\Delta Y}$ is given by:

$$MGF_{spread}^{I}(\psi;t,n,m_1,m_2) \equiv E_t^{Q^{\Delta Y}}\left(\exp\{\psi\Delta Y(t,n,m_1,m_2)\}\right)$$
$$= \frac{E_t^Q\left(\exp\left\{\psi\Delta Y(t,n,m_1,m_2)-\sum_{i=t}^{t+n-1} r_i\right\}\times\Delta Y(t,n,m_1,m_2)\right)}{E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\Delta Y(t,n,m_1,m_2)\right)} \quad \text{(E.5)}$$
$$= \frac{1}{\left.\frac{\partial\Pi(t,t+n;-\psi\tilde{A},-\psi\tilde{B},-\psi\tilde{D},-\psi\tilde{C},1)}{\partial\psi}\right|_{\psi=0}}\times\frac{\partial\Pi(t,t+n;-\psi\tilde{A},-\psi\tilde{B},-\psi\tilde{D},-\psi\tilde{C},1)}{\partial\psi}$$

The MGF of the yield spread under the forward measure Q^{t+n} is given by:

$$MGF_{spread}^{II}(\psi;t,n,m_1,m_2) \equiv E_t^{Q^{t+n}}\left(\exp\{\psi\Delta Y(t,n,m_1,m_2)\}\right)$$
$$= \frac{1}{P(t,t+n)}\times E_t^Q\left(\exp\left\{\psi\Delta Y(t,n,m_1,m_2)-\sum_{i=t}^{t+n-1} r_i\right\}\right) \quad \text{(E.6)}$$
$$= \frac{\Pi(t,t+n;-\psi\tilde{A},-\psi\tilde{B},-\psi\tilde{D},-\psi\tilde{C},1)}{P(t,t+n)}$$

The call option with a maturity date $t+n$ and a strike K on the yield basket is given by:

$$Call_{basket}(t,n,\{m_1,...,m_l\},K) \equiv E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\left(\sum_{j=1}^{l}\omega_j Y(t+n,t+n+m_j)-K\right)^+\right)$$
$$= E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\sum_{j=1}^{l}\omega_j Y(t+n,t+n+m_j)\right)\times Q^{Basket}(L) - P(t,t+n)\times K\times Q^{t+n}(L)$$

(E.7)

where Q^{Basket} has the following Radon-Nikodym derivative:

$$\frac{dQ^{Basket}}{dQ} \equiv \frac{\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\sum_{j=1}^{l}\omega_j Y(t+n,t+n+m_j)}{E_t^Q\left(\exp\left\{-\sum_{i=t}^{t+n-1} r_i\right\}\times\sum_{j=1}^{l}\omega_j Y(t+n,t+n+m_j)\right)} \quad \text{(E.8)}$$

and $L = \left\{\sum_{j=1}^{l}\omega_j Y(t+n,t+n+m_j) \geq K\right\}$. The value of the yield basket can be expressed as follows:

$$\sum_{j=1}^{l}\omega_j Y(t+n,t+n+m_j) = \sum_{j=1}^{l}\tfrac{\omega_j}{m_j}\left\{A(m_j)r_{t+n} - B(m_j)h_{t+n+1} - D(m_j)v_{t+n+1} - C(m_j)\right\}$$
$$= \left(\sum_{j=1}^{l}\tfrac{\omega_j}{m_j}A(m_j)\right)r_{t+n} - \left(\sum_{j=1}^{l}\tfrac{\omega_j}{m_j}B(m_j)\right)h_{t+n+1} - \left(\sum_{j=1}^{l}\tfrac{\omega_j}{m_j}D(m_j)\right)v_{t+n+1} - \left(\sum_{j=1}^{l}\tfrac{\omega_j}{m_j}C(m_j)\right)$$
$$\equiv \Sigma_A r_{t+n} - \Sigma_B h_{t+n+1} - \Sigma_D v_{t+n+1} - \Sigma_C$$

(E.9)

where $\Sigma_A = \sum_{j=1}^{l}\frac{\omega_j}{m_j}A(m_j)$ and with similar equations for Σ_B, Σ_D and Σ_C. Following the derivation of the yield spread option pricing formula, we can easily show that the MGF of the yield basket under the forward measures Q^{Basket} and Q^{t+n} are obtained as:

$$MGF_{basket}^{I}(\psi;t,n,\{m_1,...,m_l\}) \equiv E_t^{Q^{Basket}}\left(\exp\left\{\psi\sum_{j=1}^{l}\omega_j Y(t+n,t+n+m_j)\right\}\right)$$
$$= \frac{\dfrac{\partial\Pi(t,t+n;-\psi\Sigma_A,-\psi\Sigma_B,-\psi\Sigma_D,-\psi\Sigma_C,1)}{\partial\psi}}{\left.\dfrac{\partial\Pi(t,t+n;-\psi\Sigma_A,-\psi\Sigma_B,-\psi\Sigma_D,-\psi\Sigma_C,1)}{\partial\psi}\right|_{\psi=0}} \quad \text{(E.10)}$$

and

$$MGF_{basket}^{II}(\psi;t,n,\{m_1,...,m_l\}) \equiv E_t^{Q^{t+n}}\left(\exp\left\{\psi\sum_{j=1}^{l}\omega_j Y(t+n,t+n+m_j)\right\}\right)$$
$$= \frac{\Pi(t,t+n;-\psi\Sigma_A,-\psi\Sigma_B,-\psi\Sigma_D,-\psi\Sigma_C,1)}{P(t,t+n)} \quad \text{(E.11)}$$

Appendix F: Generalization to a multi-factor GARCH

Consider the multi-factor GARCH model defined under the risk-neutral measure Q as follows:

$$\begin{cases} r_{t+1} = r_t + \kappa(\theta - r_t) + \sum_{j=1}^{m}\sqrt{h_{j,t+1}}z_{j,t+1}^* \\ h_{j,t+1} = \beta_{j,0} + \beta_{j,1}h_{j,t} + \beta_{j,2}\left(z_{j,t}^* - \varphi_j\sqrt{h_{j,t}}\right)^2 \quad ; \quad j \in \{1,2,...,m\} \end{cases} \quad \text{(F.1)}$$

where $\{z_{j,t}^* : 1 \le j \le m; 1 \le t\}$ is a double-indexed sequence of independent standard normal variables under Q. Since the derivation of discount bonds and derivatives on interest rates depends on the generalized zero-coupon, we will only focus on the generalization formula of $\Pi(t,t+n)$.

In the multi-factor setting, the generalized zero-coupon is defined as:

$$\Pi(t,t+n;\overline{A},(\overline{B}_j)_j,\overline{C},\overline{R}) \equiv E_t^Q\left(\exp\left\{-\overline{R}\sum_{i=t}^{t+n-1}r_i\right\}\times\exp\left\{-\overline{A}r_{t+n} + \sum_{j=1}^{m}\overline{B}_j h_{j,t+n+1} + \overline{C}\right\}\right) \quad \text{(F.2)}$$

We will show that $\Pi(t,t+n)$ is given by:

$$\Pi(t,t+n;\overline{A},(\overline{B}_j)_j,\overline{C},\overline{R}) = \exp\left(-A(n)r_t + \sum_{j=1}^{m}B_j(n)h_{j,t+1} + C(n)\right) \quad \text{(F.3)}$$

where the functions A, B_j and C are defined recursively with initial values of $\overline{A}$, $\overline{B}_j$ and $\overline{C}$, respectively. If we assume that Eq. (F.3) holds for a maturity n, it can be shown for a maturity n+1 that:

$$\Pi(t,t+n+1) \equiv E_t^Q\left(\exp\left\{-\overline{R}\sum_{i=t}^{t+n} r_i\right\}\times\exp\left\{-\overline{A}r_{t+n+1}+\sum_{j=1}^{m}\overline{B}_j h_{j,t+n+2}+\overline{C}\right\}\right)$$

$$= \exp\left\{-\overline{R}r_t\right\}\times E_t^Q\left(\Pi(t+1,t+n+1)\right) \quad \text{(F.4)}$$

$$= \exp\left\{-\overline{R}r_t\right\}\times E_t^Q\left(\exp\left\{-A(n)r_{t+1}+\sum_{j=1}^{m}B_j(n)h_{j,t+2}+C(n)\right\}\right)$$

The expectation above can be computed as follows:

$$E_t^Q\left(\exp\left\{-A(n)r_{t+1}+\sum_{j=1}^{m}B_j(n)h_{j,t+2}+C(n)\right\}\right)$$

$$= E_t^Q\left(\exp\left\{-A(n)\times\left(r_t+\kappa(\theta-r_t)+\sum_{j=1}^{m}\sqrt{h_{j,t+1}}z^*_{j,t+1}\right)+\sum_{j=1}^{m}B_j(n)h_{j,t+2}+C(n)\right\}\right)$$

$$= \exp\left\{-A(n)\times\left(r_t+\kappa(\theta-r_t)\right)+C(n)\right\}\times E_t^Q\left(\exp\left\{\sum_{j=1}^{m}\left(-A(n)\sqrt{h_{j,t+1}}z^*_{j,t+1}+B_j(n)h_{j,t+2}\right)\right\}\right)$$

$$= \exp\left\{-A(n)\times\left(r_t+\kappa(\theta-r_t)\right)+C(n)\right\}\times\prod_{j=1}^{m}E_t^Q\left(\exp\left\{-A(n)\sqrt{h_{j,t+1}}z^*_{j,t+1}+B_j(n)h_{j,t+2}\right\}\right)$$

(F.5)

Collecting and rearranging the terms, we obtain:

$$\begin{cases} A(n+1)=\overline{R}+(1-\kappa)A(n) \\ B_j(n+1)=\left(\beta_{j,1}+\beta_{j,2}\varphi_j{}^2\right)B_j(n)+\dfrac{1}{2(1-2\beta_{j,2}B_j(n))}\left(A(n)+2\varphi_j\beta_{j,2}B_j(n)\right)^2\ ;\ j\in\{1,2,...,m\} \\ C(n+1)=C(n)-\kappa\theta A(n)+\sum_{j=1}^{m}\left(\beta_{j,0}B_j(n)-\tfrac{1}{2}\ln\left(1-2\beta_{j,2}B_j(n)\right)\right) \end{cases}$$

(F.6)

Figure 1: Simulation of the Daily Short Rate in the Three-Factor GARCH Model

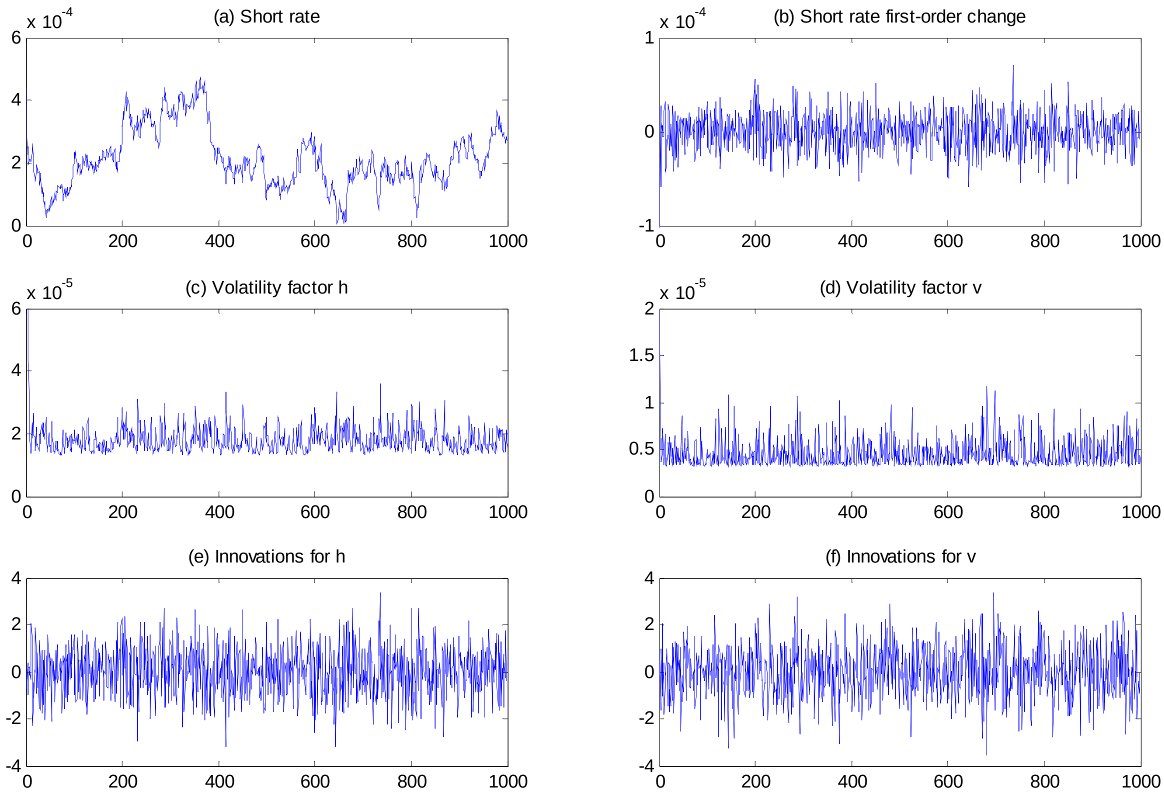

Figure 1 presents the simulation of the short rate in the three-factor GARCH model: (a) short rate, (b) short rate first-order change, (c) volatility factor h, (d) volatility factor v, (e) innovations corresponding to factor h, and (f) innovations corresponding to factor v. The parameters used are: r_0 = 0.02%, h_1 = 9e-7, v_1 = 1e-7, κ = 0.01, θ = 2e-4, λ = -3.6, μ = -0.4, β_0 = 9e-11, β_1 = 4.5, β_2 = 9e-11, $\tilde{\varphi}$ = 13.6, α_0 = 1e-11, α_1 = 0.05, α_2 = 1e-11, and $\tilde{\gamma}$ = 14.4.

Table 1: Summary Statistics of the Daily Treasury Rates (10/1/2001 – 9/12/2008)

	1-month	3-month	6-month	1-year	2-year	3-year	5-year	7-year	10-year
Min	0.26	0.61	0.82	0.88	1.1	1.34	2.08	2.63	3.13
First Quartile	1.23	1.3325	1.56	1.69	2.07	2.46	3.2	3.67	4.07
Median	1.895	1.91	2.1	2.36	3.03	3.33	3.85	4.08	4.37
Third Quartile	3.95	4.01	4.31	4.33	4.36	4.35	4.48	4.59	4.71
Max	5.27	5.19	5.33	5.3	5.29	5.26	5.23	5.29	5.44
Mean	2.553	2.635	2.784	2.909	3.166	3.396	3.822	4.115	4.385
Std. Dev.	1.481	1.498	1.515	1.398	1.182	1.016	0.737	0.584	0.451
Skewness	0.560	0.504	0.429	0.363	0.155	0.038	-0.084	-0.094	0.001
Excess Kurtosis	-1.192	-1.297	-1.378	-1.362	-1.326	-1.257	-1.079	-0.850	-0.528

Table 2: Summary Statistics of the Daily Treasury Rate First Order Changes (10/1/2001 – 9/12/2008)

	1-month	3-month	6-month	1-year	2-year	3-year	5-year	7-year	10-year
Min	-1.05	-0.64	-0.45	-0.4	-0.28	-0.27	-0.24	-0.21	-0.21
First Quartile	-0.02	-0.01	-0.02	-0.02	-0.04	-0.04	-0.04	-0.04	-0.04
Median	0	0	0	0	0	0	0	0	0
Third Quartile	0.02	0.02	0.02	0.02	0.03	0.04	0.04	0.03	0.03
Max	0.8	0.61	0.4	0.35	0.33	0.3	0.28	0.28	0.25
Mean	-0.001	-0.001	0.000	0.000	0.000	0.000	-0.001	-0.001	0.000
Std. Dev.	0.087	0.058	0.044	0.049	0.065	0.068	0.067	0.065	0.059
Skewness	-0.588	-0.646	-0.389	-0.207	0.175	0.205	0.247	0.266	0.303
Excess Kurtosis	29.265	33.458	19.490	7.255	2.015	1.822	1.536	1.353	1.257

Table 3: Covariance Matrix of the Daily Variances of the Treasury Rates

	1-month	3-month	6-month	1-year	2-year	3-year	5-year	7-year	10-year
1-month	9.58E-04	2.04E-04	2.29E-05	1.46E-05	8.99E-06	8.74E-06	3.98E-06	2.62E-06	1.50E-06
3-month	2.04E-04	1.43E-04	2.43E-05	1.61E-05	6.99E-06	7.20E-06	3.75E-06	2.37E-06	1.21E-06
6-month	2.29E-05	2.43E-05	5.60E-06	4.12E-06	2.25E-06	2.07E-06	1.06E-06	6.46E-07	3.18E-07
1-year	1.46E-05	1.61E-05	4.12E-06	3.86E-06	2.81E-06	2.34E-06	1.20E-06	7.42E-07	3.93E-07
2-year	8.99E-06	6.99E-06	2.25E-06	2.81E-06	3.91E-06	3.06E-06	1.58E-06	9.72E-07	5.30E-07
3-year	8.74E-06	7.20E-06	2.07E-06	2.34E-06	3.06E-06	2.61E-06	1.39E-06	8.67E-07	4.83E-07
5-year	3.98E-06	3.75E-06	1.06E-06	1.20E-06	1.58E-06	1.39E-06	8.34E-07	5.32E-07	3.12E-07
7-year	2.62E-06	2.37E-06	6.46E-07	7.42E-07	9.72E-07	8.67E-07	5.32E-07	3.58E-07	2.15E-07
10-year	1.50E-06	1.21E-06	3.18E-07	3.93E-07	5.30E-07	4.83E-07	3.12E-07	2.15E-07	1.41E-07

Table 4: Principal Component Analysis of the Daily Variances of the Treasury Rates

% Variance	90.10%	9.02%	0.72%	0.11%	0.03%	0.02%	0.01%	0.00%	0.00%
1-month	0.6576	-0.5743	0.2848	-0.2604	0.2654	-0.0992	0.0922	0.0097	0.0018
3-month	-0.732	-0.3473	0.2964	-0.2721	0.3658	-0.143	0.1645	0.0196	0.0031
6-month	0.1765	0.7216	0.2172	-0.2533	0.4736	-0.2023	0.2654	0.0315	0.0048
1-year	0.0188	-0.1612	-0.7605	0.098	0.239	-0.2718	0.5012	0.0579	0.0103
2-year	0.0073	-0.01	0.4218	0.3579	-0.4628	-0.2609	0.6389	0.0571	0.0105
3-year	-0.0096	0.0359	-0.1171	-0.6365	-0.3391	0.5381	0.3965	0.1328	0.018
5-year	0.0123	-0.0379	0.1176	0.5005	0.4272	0.6844	0.2125	0.1927	0.0279
7-year	-0.0002	0.0023	-0.0027	-0.0235	-0.0513	-0.1687	-0.1789	0.9397	0.2307
10-year	-0.0001	0.0003	0.0004	0.0007	0.0047	0.0177	0.0148	-0.2325	0.9723

Table 5: Call Option on Discount Bond in the Three-Factor GARCH Model

3-month Call on a 3-month Discount Bond

Quadrature Order	Price
10	0.9767
11	1.0370
12	1.0883
13	1.1312
14	1.1665
15	1.1950
16	1.2176
17	1.2352
18	1.2487
19	1.2588
20	1.2663
21	1.2717
22	1.2756
23	1.2783
24	1.2802
25	1.2815
26	1.2823
27	1.2829
28	1.2832
29	1.2835
30	1.2837
Monte Carlo Price	1.2831
Standard Deviation	7.46E-04

3-month Call on a 6-month Discount Bond

Quadrature Order	Price
10	1.6043
11	1.6593
12	1.6988
13	1.7263
14	1.7447
15	1.7566
16	1.7640
17	1.7685
18	1.7710
19	1.7725
20	1.7732
21	1.7736
22	1.7738
23	1.7739
24	1.7740
25	1.7740
26	1.7740
27	1.7740
28	1.7740
29	1.7741
30	1.7742
Monte Carlo Price	1.7736
Standard Deviation	1.19E-03

3-month Call on a 1-year Discount Bond

Quadrature Order	Price
10	1.8922
11	1.9334
12	1.9595
13	1.9752
14	1.9843
15	1.9892
16	1.9918
17	1.9930
18	1.9936
19	1.9938
20	1.9939
21	1.9940
22	1.9940
23	1.9940
24	1.9940
25	1.9940
26	1.9940
27	1.9940
28	1.9940
29	1.9941
30	1.9942
Monte Carlo Price	1.9936
Standard Deviation	1.34E-03

6-month Call on a 1-year Discount Bond

Quadrature Order	Price
10	0.5593
11	0.6036
12	0.6440
13	0.6807
14	0.7136
15	0.7428
16	0.7686
17	0.7910
18	0.8103
19	0.8268
20	0.8408
21	0.8525
22	0.8622
23	0.8701
24	0.8766
25	0.8818
26	0.8860
27	0.8893
28	0.8919
29	0.8939
30	0.8955
Monte Carlo Price	0.8998
Standard Deviation	7.29E-04

Table 5 presents the pricing results for at-the-money forward call options on discount bonds in the three-factor GARCH model. The parameters used are: r_0 = 0.02%, h_1 = 9e-7, v_1 = 1e-7, κ = 0.01, θ = 2e-4, λ = -3.6, μ = -0.4, β_0 = 9e-11, β_1 = 4.5, β_2 = 9e-11, $\tilde{\varphi}$ = 13.6, α_0 = 1e-11, α_1 = 0.05, α_2 = 1e-11, and $\tilde{\gamma}$ = 14.4. The strike prices are \$98.5229, \$96.9252, \$93.6139 and \$93.3474, respectively.

Table 6: Call Option on Coupon Bond in the Three-Factor GARCH Model

3-month Call on a 1-year Coupon Bond		3-month Call on a 1-year Step-Up Bond	
Quadrature Order	Price	Quadrature Order	Price
10	1.9541	10	2.0607
11	2.0127	11	2.1289
12	2.0536	12	2.1778
13	2.0810	13	2.2118
14	2.0987	14	2.2345
15	2.1097	15	2.2491
16	2.1162	16	2.2582
17	2.1199	17	2.2636
18	2.1219	18	2.2668
19	2.1230	19	2.2685
20	2.1235	20	2.2694
21	2.1238	21	2.2699
22	2.1239	22	2.2701
23	2.1240	23	2.2703
24	2.1240	24	2.2703
25	2.1240	25	2.2703
26	2.1240	26	2.2703
27	2.1240	27	2.2704
28	2.1240	28	2.2703
29	2.1241	29	2.2704
30	2.1243	30	2.2706

Table 6 presents the valuation results for at-the-money forward call options on coupon bonds in the three-factor GARCH model. The parameters are the same as those in Table 5. The first bond has a quarterly coupon of $4, a strike price of $108.99 and a stochastic duration of 205 days. The second bond has a quarterly step-up coupon of $4, $6, $8 and $10, respectively, a strike price of $120.35 and a stochastic duration of 181 days.