

Liquidity Risk and the Dynamics of Arbitrage Capital

Péter Kondor

London School of Economics and CEPR

Dimitri Vayanos

London School of Economics, CEPR and NBER

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Abstract

We develop a dynamic model of liquidity provision, in which hedgers can trade multiple risky assets with arbitrageurs. We compute the equilibrium in closed form when arbitrageurs' utility over consumption is logarithmic or risk-neutral with a non-negativity constraint. Arbitrageurs increase their positions following high asset returns, and can choose to provide less insurance when hedgers are more risk-averse. The stationary distribution of arbitrageur wealth is bimodal when hedging needs are strong. Liquidity is increasing in arbitrageur wealth, while asset volatilities, correlations, and expected returns are hump-shaped. Assets that suffer the most when aggregate liquidity decreases offer the highest expected returns. This is because the arbitrageurs' portfolio is a pricing factor, and aggregate liquidity captures exactly that factor.

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1 Introduction

Liquidity in financial markets is often provided by specialized agents, such as market makers, trading desks in investment banks, and hedge funds. Adverse shocks to the capital of these agents cause liquidity to decline and risk premia to increase. Conversely, movements in the prices of assets held by liquidity providers feed back into these agents' capital.¹

In this paper we study the dynamics of liquidity providers' capital, the liquidity that these agents provide to other participants, and assets' risk premia. We strip out all frictions from our model except for the implicit constraint that liquidity providers cannot raise equity capital from other investors when returns are high. Our model yields closed-form solutions in two cases of interest, and these solutions reveal new insights on the dynamics of risk-sharing and asset prices. Moreover, while our model involves no explicit frictions, it can help provide a unified explanation for a number of puzzling empirical facts about liquidity, for which such frictions have been viewed as essential.

We assume a continuous-time infinite-horizon economy with two sets of competitive agents: hedgers, who receive a risky income flow and seek to reduce their risk by participating in financial markets, and arbitrageurs. Arbitrageurs take the other side of the trades that hedgers initiate, and in that sense provide liquidity to them. They also provide insurance because they absorb part of the hedgers' risk. Arbitrageurs in our model can be interpreted, for example, as speculators in futures markets or as sellers of catastrophe insurance. We assume that hedgers' demand for insurance is independent of their wealth. Because on the other hand, we endow arbitrageurs with constant relative risk aversion (CRRA) utility over consumption, the supply of insurance depends on their wealth, and wealth becomes the key state variable affecting risk-sharing and asset prices.

The market in our model consists of a riskless asset and multiple risky assets. We determine the prices of the risky assets endogenously in equilibrium, but fix the return on the riskless asset to an exogenous constant. Hence, price movements are driven purely by risk premia. Studying movements in risk premia in isolation from those in the riskless rate is a plausible simplification when movements concern one asset class, as in our applications, rather than the entire asset universe. Because we

¹A growing empirical literature documents the relationship between the capital of liquidity providers, the liquidity that these agents provide to other participants, and assets' risk premia. For example, Comerton-Forde, Hendershott, Jones, Moulton, and Seasholes (2010) find that bid-ask spreads quoted by specialists in the New York Stock Exchange widen when specialists experience losses. Aragon and Strahan (2012) find that following the collapse of Lehman Brothers in 2008, hedge funds doing business with Lehman experienced a higher probability of failure, and the liquidity of the stocks that they were trading declined. Jylha and Suominen (2011) find that outflows from hedge funds that perform the carry trade predict poor performance of that trade, with low interest-rate currencies appreciating and high-interest rate ones depreciating. Acharya, Lochstoer, and Ramadorai (2013) find that risk premia in commodity-futures markets are larger when broker-dealer balance sheets are shrinking.

fix the riskless rate rather than aggregate consumption, we cannot use standard methodologies of consumption-based asset-pricing models to compute risk-sharing and asset prices. Yet, our multi-asset setting delivers more tractability than what is typical in these models. We show all our results analytically, using closed-form solutions that we derive in two special cases of arbitrageur CRRA utility: risk-neutral with non-negative consumption, and logarithmic.

Our model yields new insights on the dynamics of risk-sharing and insurance provision. The risk aversion of arbitrageurs is driven not only by the static coefficient of risk aversion in their utility function, but also by a forward-looking component that reflects the equilibrium relationship between their wealth and asset prices. Arbitrageurs realize that in states where their portfolio performs poorly other arbitrageurs also perform poorly and insurance provision becomes more profitable. Limiting their investment in the risky assets enables them to have more wealth in those states and earn higher profits. This effect is the sole determinant of risk aversion in the case where arbitrageur utility is risk neutral and negative consumption is not allowed. We show that in that case risk aversion increases when hedgers become more risk averse or asset cashflows become more volatile. Our results have the surprising implication that more risk-averse hedgers may receive less insurance from arbitrageurs in equilibrium.

Our model yields additionally a new understanding of liquidity risk and its relationship with expected asset returns. A large empirical literature has documented that liquidity varies over time and in a correlated manner across assets. Moreover, aggregate liquidity appears to be a priced risk factor and carry a positive premium: assets that underperform the most during times of low aggregate liquidity earn higher expected returns than assets with otherwise identical characteristics.²

We define liquidity based on the impact that hedgers have on prices, following similar definitions in asset-pricing settings by Kyle and Xiong (2001), Xiong (2001), and Johnson (2008). We show that liquidity has a cross-sectional and a time-series dimension. In the cross-section, liquidity is lower for assets with more volatile cashflows. In the time-series, liquidity decreases following losses by arbitrageurs. Hence, liquidity varies over time in response to changes in arbitrageur wealth, and this variation is common across assets.

Expected returns in the cross-section of assets are proportional to the covariance with the

²Chordia, Roll, and Subrahmanyam (2000), Hasbrouck and Seppi (2001), and Huberman and Halka (2001) document the time-variation of liquidity and its correlation across assets. Amihud (2002) and Hameed, Kang, and Viswanathan (2010) link time-variation in aggregate liquidity to the returns of the aggregate stock market. Pastor and Stambaugh (2003) and Acharya and Pedersen (2005) find that aggregate liquidity is a priced risk factor in the stock market and carries a positive premium. Sadka (2010) and Franzoni, Nowak, and Phalippou (2012) find similar results for hedge-fund and private-equity returns, respectively. For more references, see Vayanos and Wang (2013) who survey the theoretical and empirical literature on market liquidity.

portfolio of arbitrageurs, which is the single pricing factor in our model. That factor may be hard to measure empirically: the portfolio of arbitrageurs is unobservable, and may differ from the market portfolio, consisting of the supply coming from asset issuers, because of the additional supply coming from hedgers. We show, however, that aggregate liquidity captures exactly that factor. Indeed, because arbitrageurs sell a fraction of their portfolio following losses, assets that covary the most with their portfolio suffer the most when liquidity decreases. Thus, an asset's covariance with aggregate liquidity is proportional to its covariance with the portfolio of arbitrageurs, which in turn is proportional to the asset's expected return. We show additionally that other liquidity-related covariances used in empirical work, such as between an asset's liquidity and aggregate liquidity or return, do not explain expected returns as well. This is because those covariances are proportional to the volatility of an asset's cashflows rather than to the asset's covariance with the arbitrageurs' portfolio.

Our model yields a number of additional results concerning the dynamics of asset prices and of arbitrageur wealth and positions. We show that arbitrageurs behave as momentum traders, increasing their positions following high asset returns. This is because when returns are high, arbitrageurs become wealthier, and their risk aversion decreases. We also show that the feedback effects between arbitrageur wealth and asset prices cause the volatility of asset returns to be hump-shaped in wealth. When wealth is small, shocks to wealth are small in absolute terms, and so is the price volatility that they generate. When instead wealth is large, arbitrageurs provide perfect liquidity to hedgers and prices are not sensitive to changes in wealth. This yields the hump-shaped pattern for volatility, from which those for correlations and expected returns follow. We show additionally that the term structure of risk premia depends on arbitrageur wealth, with its slope steepening (becoming more positive) when wealth increases. These results can help shed light on a number of empirical findings, discussed in later sections.

We finally show that the stationary distribution of arbitrageur wealth can be bimodal (as in Brunnermeier and Sannikov (2014)), with wealth less likely to take intermediate values than large or very small ones. The wealth of arbitrageurs reaches a non-degenerate stationary distribution because arbitrage activity is self-correcting. For example, when wealth is large, arbitrageurs hold large positions, offering more insurance to hedgers. As a consequence, equilibrium risk premia decrease, and this renders arbitrage less profitable, causing wealth to decrease. The stationary density becomes bimodal when hedging needs are strong. Intuitively, with strong hedging needs insurance provision is more profitable. Therefore, arbitrageur wealth grows fast, and large values of wealth can be more likely in steady state than intermediate values. At the same time, while

profitability (per unit of wealth) is highest when wealth is small, wealth grows away from small values slowly in absolute terms. Therefore, small values are more likely than intermediate values.

Our paper relates to a number of literatures. A first group of related papers are those studying the pricing of liquidity risk in the cross-section of assets. In Holmstrom and Tirole (2001), liquidity is defined in terms of firms' financial constraints. Firms avoid assets whose return is low when constraints are severe, and these assets offer high expected returns in equilibrium. Our result that arbitrageurs avoid assets whose return is low when liquidity provision becomes more profitable has a similar flavor. The covariance between asset returns and liquidity, however, is endogenous in our model because prices depend on arbitrageur wealth. In Amihud (2002) and Acharya and Pedersen (2005), illiquidity takes the form of exogenous time-varying transaction costs. An increase in the costs of trading an asset raises the expected return that investors require to hold it and lowers its price. A negative covariance between illiquidity and asset prices arises also in our model but because of an entirely different mechanism: low liquidity and low prices are endogenous symptoms of low arbitrageur wealth. The endogenous variation in liquidity is also what drives the cross-sectional relationship between expected returns and liquidity-related covariances.

A second group of related papers link arbitrage capital to liquidity and asset prices. Some of these papers emphasize margin constraints. In Gromb and Vayanos (2002), arbitrageurs intermediate trade between investors in segmented markets, and are subject to margin constraints. Because of the constraints, the liquidity that arbitrageurs provide to investors increases in their wealth. In Brunnermeier and Pedersen (2009), margin-constrained arbitrageurs intermediate trade in multiple assets across time periods. Assets with more volatile cashflows are more sensitive to changes in arbitrageur wealth. Garleanu and Pedersen (2011) introduce margin constraints in an infinite-horizon setting with multiple assets. They show that assets with higher margin requirements earn higher expected returns and are more sensitive to changes in the wealth of the margin-constrained agents. This result is suggestive of a priced liquidity factor. In our model cross-sectional differences in assets' covariance with aggregate liquidity arise because of differences in cashflow volatility and hedger supply rather than in margin constraints.

Other papers assume constraints on equity capital, which may be implicit (as in our paper) or explicit. In Xiong (2001) and Kyle and Xiong (2001), arbitrageurs with logarithmic utility over consumption can trade with long-term traders and noise traders over an infinite horizon. The liquidity that arbitrageurs can provide is increasing in their wealth, and asset volatilities are hump-shaped.³ In He and Krishnamurthy (2013), arbitrageurs can raise capital from other investors to

³Isaenko (2008) studies a related model in which long-term traders are utility maximizers, with constant absolute

invest in a risky asset over an infinite horizon, but this capital cannot exceed a fixed multiple of their internal capital. When arbitrageur wealth decreases, the constraint binds, and asset volatility and expected returns increase. In Brunnermeier and Sannikov (2014), arbitrageurs are more efficient holders of productive capital. The long-run stationary distribution of their wealth can have a decreasing or a bimodal density. These papers mostly focus on the case of one risky asset (two assets in Kyle and Xiong (2001)).

Finally, our paper is related to the literature on consumption-based asset pricing with heterogeneous agents, e.g., Dumas (1989), Wang (1996), Chan and Kogan (2002), Bhamra and Uppal (2009), Basak and Pavlova (2013), Chabakauri (2013), Ehling and Heyerdahl-Larsen (2013), Garleanu and Panageas (2014), and Longstaff and Wang (2014). In these papers, agents have CRRA utility and differ in their risk aversion. As the wealth of the less risk-averse agents increases, Sharpe ratios decrease, and this can cause volatilities and correlations to be hump-shaped. In contrast to these papers, we assume that only one set of agents has wealth-dependent risk aversion. This allows us to focus more sharply on the wealth effects of liquidity providers. We also fix the riskless rate, which in these papers is instead determined by aggregate consumption.

A methodological contribution relative to the above groups of papers is that we provide an analytically tractable model with multiple assets, dynamics, heterogeneous agents, and wealth effects. With a few exceptions, the dynamic models cited above compute the equilibrium by solving differential equations numerically. By contrast, we derive closed-form solutions and prove analytically each of our main results.⁴

We proceed as follows. In Section 2 we present the model. In Section 3 we derive risk-sharing, market prices of risk, and wealth dynamics. For expositional convenience we perform these derivations assuming that the risky assets are “short-lived” claims on the next instant’s cashflow. In Section 4 we show that the results with short-lived assets carry through identical when assets are “long-lived” cashflow streams. We also compute expected returns, volatilities, and correlations of long-lived assets. In Section 5 we explore the implications of our model for liquidity risk. In Section 6 we explore additional applications and extensions of our model. Section 7 concludes.

risk aversion utility, and there are transaction costs.

⁴Closed-form solutions are also derived in Danielsson, Shin, and Zigrand (2012) and Gromb and Vayanos (2015). In the former paper, risk-neutral arbitrageurs are subject to a VaR constraint and can trade with long-term traders, modeled as in Kyle and Xiong (2001). In the latter paper, arbitrageurs intermediate trade across segmented markets and are subject to margin constraints. Their activity is self-correcting, but involves no risk because the different legs of their trades cancel.

2 Model

Time t is continuous and goes from zero to infinity. Uncertainty is described by the N -dimensional Brownian motion B_t . There is a riskless asset whose instantaneous return is constant over time and equal to r . There are also N risky assets with cashflows

$$dD_t = \bar{D}dt + \sigma^\top dB_t, \tag{2.1}$$

and \bar{D} is a constant $N \times 1$ vector, σ is a constant and invertible $N \times N$ matrix, and \top denotes transpose. The cashflows (2.1) are *i.i.d.* The *i.i.d.* assumption is for simplicity, and we can introduce persistence without significant changes to our analysis. We denote by S_t the $N \times 1$ vector of risky-asset prices at time t , and by s the $N \times 1$ vector consisting of asset supplies measured in terms of number of shares. We set $\Sigma \equiv \sigma^\top \sigma$.

There are two sets of agents, hedgers and arbitrageurs. Each set forms a continuum with measure one. Hedgers choose asset positions at time t to maximize the mean-variance objective

$$E_t(dv_t) - \frac{\alpha}{2} \text{Var}_t(dv_t), \tag{2.2}$$

where dv_t is the change in wealth between t and $t + dt$, and α is a risk-aversion coefficient. To introduce hedging needs, we assume that hedgers receive a random endowment $u^\top dD_t$ at $t + dt$, where u is a constant $N \times 1$ vector. This endowment is added to dv_t . Since the hedgers' risk-aversion coefficient α and endowment variance $u^\top \Sigma u$ are constant over time, their demand for insurance, derived in the next section, is also constant. We intentionally simplify the model in this respect, so that we can focus on the supply of insurance, which is time-varying because of the wealth-dependent risk aversion of arbitrageurs.

One interpretation of the hedgers is as generations living over infinitesimal periods. The generation born at time t is endowed with initial wealth \bar{v} , and receives the additional endowment $u^\top dD_t$ at $t + dt$. It consumes all its wealth at $t + dt$ and dies. (In a discrete-time version of our model, each generation would be born in one period and die in the next.) If preferences over consumption are described by the VNM utility U , this yields the objective (2.2) with the risk-aversion coefficient $\alpha = -\frac{U''(\bar{v})}{U'(\bar{v})}$, which is constant over time.

In Section 6.3 we relax the assumption that hedgers maximize instantaneous mean-variance utility, and allow them instead to maximize expected utility of intertemporal consumption. We

assume that utility is time-additive with constant absolute risk aversion (CARA) $\frac{\alpha}{r}$:

$$-\mathbb{E}_t \left(\int_t^\infty e^{-\frac{\alpha}{r} \bar{c}_s} e^{-\bar{\rho}(s-t)} ds \right), \quad (2.3)$$

where \bar{c}_s is consumption at $s \geq t$ and $\bar{\rho}$ is a subjective discount rate. CARA utility preserves the property of mean-variance utility that the hedgers' demand for insurance is independent of their wealth. We assume that the coefficient of absolute risk aversion over consumption is $\frac{\alpha}{r}$ so that its counterpart coefficient over wealth is α (as shown in Section 6.3), identical to that under mean-variance utility. Under CARA utility, our main results remain the same but we lose the closed-form solutions.

Arbitrageurs maximize expected utility of intertemporal consumption. We assume that utility is time-additive with constant relative risk aversion (CRRA) $\gamma \geq 0$. When $\gamma \neq 1$, the arbitrageurs' objective at time t is

$$\mathbb{E}_t \left(\int_t^\infty \frac{c_s^{1-\gamma}}{1-\gamma} e^{-\rho(s-t)} ds \right), \quad (2.4)$$

where c_s is consumption at $s \geq t$ and ρ is a subjective discount rate. When $\gamma = 1$, the objective becomes

$$\mathbb{E}_t \left(\int_t^\infty \log(c_s) e^{-\rho(s-t)} ds \right). \quad (2.5)$$

Implicit in the definition of the arbitrageurs' objective for $\gamma > 0$, is that consumption is non-negative. The objective for $\gamma = 0$ can be defined for negative consumption, but we impose non-negativity as a constraint. Since negative consumption can be interpreted as a costly activity that arbitrageurs undertake to repay a loan, the non-negativity constraint can be interpreted as a collateral constraint: arbitrageurs cannot commit to engage in the costly activity, and can hence walk away from a loan not backed by collateral.

We solve for the equilibrium in steps. In Section 3 we derive risk-sharing, market prices of risk, and wealth dynamics. These derivations can be performed independently of those for price dynamics. Indeed, in Section 3 we replace the risky assets paying a infinite stream of cashflows by "short-lived" assets paying the next instant's cashflow. Short-lived assets are a useful expositional device: risk-sharing, market prices of risk, and wealth dynamics are identical as with long-lived assets, as we show in Section 4, but the derivations are simpler because there are no price dynamics. In Section 4 we compute the price dynamics of long-lived assets, and these assets' expected returns,

volatilities, and correlations. In both Sections 3 and 4 we assume that risky assets are in zero supply, i.e., $s = 0$. Even with zero supply, there is aggregate risk because of the hedgers' endowment, and risk premia are non-zero. We allow supply to be positive in Section 6, and show that our main results remain the same but we lose the closed-form solutions.

For zero supply our model could represent futures markets, with the assets being futures contracts and the arbitrageurs being the speculators. It could also represent the market for insurance against aggregate risks, e.g., weather or earthquakes, with the assets being insurance contracts and the arbitrageurs being the insurers. For positive supply, our model could represent stock or bond markets, with the arbitrageurs being hedge funds or other agents absorbing demand or supply imbalances.

3 Risk-Sharing

3.1 Equilibrium with Short-Lived Assets

At each time t , a new set of N short-lived risky assets can be traded. The assets available for trade at time t pay dD_t at $t + dt$. We denote by $\pi_t dt$ the $N \times 1$ vector of prices at which the assets trade at t , and by $dR_t \equiv dD_t - \pi_t dt$ the $N \times 1$ vector of returns that the assets earn between t and $t + dt$. (In a discrete-time version of our model, short-lived assets would be available for trade in one period and pay off in the next.) Eq. (2.1) implies that the instantaneous expected returns of the short-lived risky assets are

$$\frac{\mathbb{E}_t(dR_t)}{dt} = \bar{D} - \pi_t, \tag{3.1}$$

and the instantaneous covariance matrix of returns is

$$\frac{\text{Var}_t(dR_t)}{dt} = \frac{\mathbb{E}_t(dR_t dR_t^\top)}{dt} = \sigma^\top \sigma = \Sigma. \tag{3.2}$$

Note that dR_t is also a return in excess of the riskless asset since investing $\pi_t dt$ in the riskless asset yields return $r\pi_t(dt)^2$, which is negligible relative to dR_t .

We next compute the equilibrium with short-lived risky assets, and for simplicity refer to these assets as risky assets for the rest of this section. We first solve the hedgers' maximization problem. Consider a hedger who holds a position x_t in the risky assets at time t . The change in the hedger's

wealth between t and $t + dt$ is

$$dv_t = rv_t dt + x_t^\top (dD_t - \pi_t dt) + u^\top dD_t. \quad (3.3)$$

The first term in the right-hand side of (3.3) is the return from investing in the riskless asset, the second term is the return from investing in the risky assets, and the third term is the endowment. Substituting dD_t from (2.1) into (3.3), and the result into (2.2), we find the hedger's optimal asset demand.

Proposition 3.1 *The optimal policy of a hedger at time t is to hold a position*

$$x_t = \frac{\Sigma^{-1}(\bar{D} - \pi_t)}{\alpha} - u \quad (3.4)$$

in the risky assets.

The hedger's optimal demand for the risky assets consists of two components, which correspond to the two terms in the right-hand side of (3.4). The first term is the demand in the absence of the hedging motive. This demand consists of an investment in the tangent portfolio, scaled by the hedger's risk aversion coefficient α . The tangent portfolio is the inverse of the covariance matrix Σ of asset returns times the vector $\bar{D} - \pi_t$ of expected returns. The second term is the demand generated by the hedging motive. This demand consists of a short position in the portfolio u , which characterizes the sensitivity of hedgers' endowment to asset returns. Selling short an asset n for which u_n is positive hedges endowment risk.

We next study the arbitrageurs' maximization problem. Consider an arbitrageur who has wealth w_t at time t and holds a position y_t in the risky assets. The arbitrageur's budget constraint is

$$dw_t = rw_t dt + y_t^\top (dD_t - \pi_t dt) - c_t dt. \quad (3.5)$$

The first term in the right-hand side of (3.5) is the return from investing in the riskless asset, the second term is the return from investing in the risky assets, and the third term is consumption. The arbitrageur's value function depends not only on his own wealth w_t , but also on the total wealth of all arbitrageurs since the latter affects asset prices π_t . In equilibrium own wealth and total wealth coincide because all arbitrageurs hold the same portfolio and are in measure one. For the purposes of optimization, however, we need to make the distinction. We reserve the notation w_t for total wealth and denote own wealth by \hat{w}_t . We likewise use (c_t, y_t) for total consumption

and position in the assets, and denote own consumption and position by (\hat{c}_t, \hat{y}_t) . We conjecture that the arbitrageur's value function is

$$V(\hat{w}_t, w_t) = q(w_t) \frac{\hat{w}_t^{1-\gamma}}{1-\gamma} \quad (3.6)$$

for $\gamma \neq 1$, and

$$V(\hat{w}_t, w_t) = \frac{1}{\rho} \log(\hat{w}_t) + q_1(w_t) \quad (3.7)$$

for $\gamma = 1$, where $q(w_t)$ and $q_1(w_t)$ are scalar functions of w_t . We set $q(w_t) = \frac{1}{\rho}$ for $\gamma = 1$. Substituting dD_t from (2.1) into (3.5), and the result into the arbitrageur's Bellman equation, we find the arbitrageur's optimal consumption and asset demand.

Proposition 3.2 *Given the value function (3.6) and (3.7), the optimal policy of an arbitrageur at time t is to consume*

$$\hat{c}_t = q(w_t)^{-\frac{1}{\gamma}} \hat{w}_t \quad (3.8)$$

and hold a position

$$\hat{y}_t = \frac{\hat{w}_t}{\gamma} \left(\Sigma^{-1}(\bar{D} - \pi_t) + \frac{q'(w_t)y_t}{q(w_t)} \right) \quad (3.9)$$

in the risky assets.

The arbitrageur's optimal consumption is proportional to his wealth \hat{w}_t , with the proportionality coefficient $q(w_t)^{-\frac{1}{\gamma}}$ being a function of total arbitrageur wealth w_t . The arbitrageur's optimal demand for the risky assets consists of two components, as for the hedgers. The first component is the demand in the absence of a hedging motive, and consists of an investment in the tangent portfolio, scaled by the arbitrageur's coefficient of absolute risk aversion $\frac{\gamma}{\hat{w}_t}$. The second component is the demand generated by intertemporal hedging (Merton (1973)). The arbitrageur hedges against changes in his investment opportunity set, and does so by holding a portfolio with weights proportional to the sensitivity of that set to asset returns. In our model the investment opportunity set is fully characterized by total arbitrageur wealth, and the sensitivity of that variable to asset returns is the average portfolio y_t of all arbitrageurs. Hence, the arbitrageur's hedging demand is a scaled version of y_t , as the second term in the right-hand side of (3.9) shows.

Since in equilibrium all arbitrageurs hold the same portfolio, both components of asset demand consist of an investment in the tangent portfolio. Setting $\hat{y}_t = y_t$ and $\hat{w}_t = w_t$ in (3.9), we find that the total asset demand of arbitrageurs is

$$y_t = \frac{\Sigma^{-1}(\bar{D} - \pi_t)}{A(w_t)}, \quad (3.10)$$

where

$$A(w_t) \equiv \frac{\gamma}{w_t} - \frac{q'(w_t)}{q(w_t)}. \quad (3.11)$$

Arbitrageurs' investment in the tangent portfolio is thus scaled by the coefficient $A(w_t)$, which measures effective risk aversion. Effective risk aversion is the sum of the static coefficient of absolute risk aversion $\frac{\gamma}{w_t}$, and of the term $-\frac{q'(w_t)}{q(w_t)}$ which corresponds to the intertemporal hedging demand.

Substituting the asset demand (3.4) of the hedgers and (3.10) of the arbitrageurs into the market-clearing equation

$$x_t + y_t = 0, \quad (3.12)$$

we find that asset prices π_t are

$$\pi_t = \bar{D} - \frac{\alpha A(w_t)}{\alpha + A(w_t)} \Sigma u. \quad (3.13)$$

Substituting (3.13) back into (3.10), we find that the arbitrageurs' position in the risky assets in equilibrium is

$$y_t = \frac{\alpha}{\alpha + A(w_t)} u. \quad (3.14)$$

Intuitively, hedgers want to sell the portfolio u to hedge their endowment. Arbitrageurs buy a fraction of that portfolio, and the rest remains with the hedgers. The fraction bought by arbitrageurs decreases in their effective risk aversion $A(w_t)$ and increases in the hedgers' risk aversion α , according to optimal risk-sharing. Expected asset returns are proportional to the covariance with the portfolio u , which is the single pricing factor in our model. The risk premium $\frac{\alpha A(w_t)}{\alpha + A(w_t)}$ of that factor increases in the arbitrageurs' effective risk aversion, and is hence time-varying. The arbitrageurs' Sharpe ratio, defined as the expected return of their portfolio divided by the portfolio's standard

deviation, also increases in their effective risk aversion. Using (3.13) and (3.14), we find that the Sharpe ratio is

$$SR_t \equiv \frac{y_t^\top (\bar{D} - \pi_t)}{\sqrt{y_t^\top \Sigma y_t}} = \frac{\alpha A(w_t)}{\alpha + A(w_t)} \sqrt{u^\top \Sigma u}. \quad (3.15)$$

Substituting the arbitrageurs' optimal policy from Proposition (3.2) into the Bellman equation, we can derive an ordinary differential equation (ODE) that the arbitrageurs' value function must satisfy.

Proposition 3.3 *If (3.6) is the value function for $\gamma \neq 1$, then $q(w_t)$ must solve the ODE*

$$\rho q = \gamma q^{1-\frac{1}{\gamma}} + \left(r - q^{-\frac{1}{\gamma}}\right) q' w + r q(1-\gamma) + \frac{1}{2} \left(q'' + \frac{2q'\gamma}{w} - \frac{2q'^2}{q} + \frac{q(1-\gamma)\gamma}{w^2} \right) \frac{\alpha^2}{\left(\alpha + \frac{\gamma}{w} - \frac{q'}{q}\right)^2} u^\top \Sigma u. \quad (3.16)$$

If (3.7) is the value function for $\gamma = 1$, then $q_1(w_t)$ must solve the ODE

$$\rho q_1 = \log(\rho) + \frac{r - \rho}{\rho} + (r - \rho) q_1' + \frac{1}{2} \left(q_1'' + \frac{2q_1'}{w} + \frac{1}{\rho w^2} \right) \frac{\alpha^2}{\left(\alpha + \frac{1}{w}\right)^2} u^\top \Sigma u. \quad (3.17)$$

3.2 Closed-Form Solutions

We next characterize the equilibrium more fully in two special cases: arbitrageurs have logarithmic preferences ($\gamma = 1$) and arbitrageurs are risk-neutral ($\gamma = 0$). A useful parameter in both cases is

$$z \equiv \frac{\alpha^2 u^\top \Sigma u}{2(\rho - r)}. \quad (3.18)$$

The parameter z is larger when hedgers are more risk averse (large α), or their endowment is riskier (large $u^\top \Sigma u$), or arbitrageurs are more patient (small ρ).

When $\gamma = 1$, (3.8) and $q(w_t) = \frac{1}{\rho}$ imply that arbitrageur consumption is equal to ρ times wealth. Eq. (3.11) implies that arbitrageur effective risk-aversion $A(w_t)$ is

$$A(w_t) = \frac{1}{w_t}. \quad (3.19)$$

Effective risk aversion is equal to the static coefficient of absolute risk aversion because the intertemporal hedging demand is zero.

When $\gamma = 0$, (3.8) implies that arbitrageur consumption is equal to zero in the region $q(w_t) > 1$ since $\frac{1}{\gamma} = \infty$. Moreover, $q(w_t) \geq 1$ since an arbitrageur can always consume his entire wealth \hat{w}_t instantly and achieve utility \hat{w}_t . Therefore, there are two regions, one in which $q(w_t) > 1$ and arbitrageurs do not consume, and in which $q(w_t) = 1$ and arbitrageurs consume instantly until their total wealth w_t reaches the other region. The two regions are separated by a threshold $\bar{w} > 0$: for $w_t < \bar{w}$ arbitrageurs do not consume, and for $w_t > \bar{w}$ they consume instantly until w_t decreases to \bar{w} . The marginal utility $q(w_t)$ of an arbitrageur's wealth is high when the total wealth w_t of all arbitrageurs is low because insurance provision is then more profitable. Arbitrageur effective risk-aversion $A(w_t)$ is the solution to a first-order ODE derived from (3.16). Proposition 3.20 solves this ODE in closed form in the limit when the riskless rate r goes to zero. For ease of exposition, we refer from now on to the $r \rightarrow 0$ limit in the risk-neutral case as the "limit risk-neutral case." In subsequent sections we occasionally also take the $r \rightarrow 0$ limit in the logarithmic case, and refer to it as the "limit logarithmic case."

Proposition 3.4 *In the limit risk-neutral case ($\gamma = 0, r \rightarrow 0$), arbitrageur effective risk aversion is given by*

$$A(w_t) = \frac{\alpha}{1+z} \left(\sqrt{z} \cot \left(\frac{\alpha w_t}{\sqrt{z}} \right) - 1 \right) \quad (3.20)$$

for $w_t < \bar{w}$, and $A(w_t) = 0$ for $w_t \geq \bar{w}$, where the threshold \bar{w} is given by

$$\cot \left(\frac{\alpha \bar{w}}{\sqrt{z}} \right) = \frac{1}{\sqrt{z}}. \quad (3.21)$$

The marginal utility of arbitrageur wealth is given by

$$q(w_t) = \exp \left\{ \frac{z}{1+z} \left[\log \sin \left(\frac{\alpha \bar{w}}{\sqrt{z}} \right) - \log \sin \left(\frac{\alpha w_t}{\sqrt{z}} \right) - \frac{\alpha}{1+z} (\bar{w} - w_t) \right] \right\} \quad (3.22)$$

for $w_t < \bar{w}$, and $q(w_t) = 1$ for $w_t \geq \bar{w}$.

Although arbitrageurs are risk-neutral, their effective risk aversion is positive in the region $w_t < \bar{w}$. This is because of the intertemporal hedging demand, which is the sole determinant of

effective risk aversion because the static coefficient of absolute risk aversion is zero. Intuitively, arbitrageurs realize that in states where their portfolio performs poorly, other arbitrageurs also perform poorly, and hence insurance provision becomes more profitable. To have more wealth to provide insurance in those states, arbitrageurs limit their investment in the risky assets, hence behaving as risk-averse.

Figure 1 plots arbitrageur effective risk aversion $A(w_t)$ as a function of wealth w_t . To choose values for α and $u^\top \Sigma u$, we set hedgers' initial wealth \bar{v} to one: this is without loss of generality because we can redefine the numeraire. Since $\bar{v} = 1$, the parameter $\alpha = -\frac{u''(\bar{v})}{u'(\bar{v})}$ coincides with the hedgers' relative risk aversion coefficient, and we set it to 2. Moreover, the parameter $\sqrt{u^\top \Sigma u}$ coincides with the annualized standard deviation of the hedgers' endowment as a function of their initial wealth, and we set it to 15%. We set the arbitrageurs' subjective discount rate ρ to 4%, and the riskless rate r to 2%.

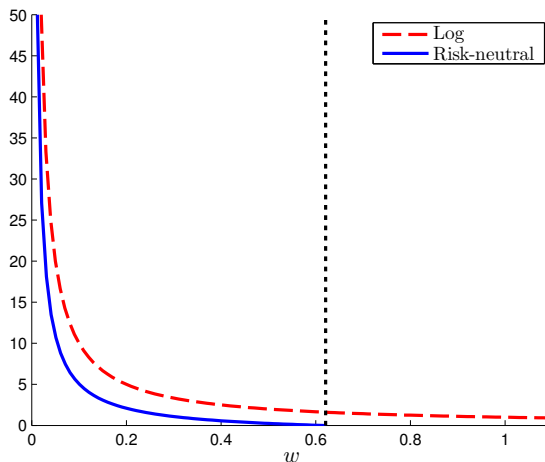


Figure 1: Arbitrageur effective risk aversion as a function of wealth in the logarithmic case (dashed line) and the risk-neutral case (solid line). The dotted vertical line hits the x -axis at \bar{w} and pertains to the risk-neutral case. Parameter values are $\alpha = 2$, $\sqrt{u^\top \Sigma u} = 15\%$, $\rho = 4\%$, and $r = 2\%$.

Figure 1 shows that in both the logarithmic and the risk-neutral cases, effective risk aversion $A(w_t)$ is decreasing and convex in arbitrageur wealth, and converges to infinity when wealth goes to zero. Moreover, effective risk aversion is smaller in the risk-neutral case than in the logarithmic case. These properties hold for all parameter values in the logarithmic case since $A(w_t) = \frac{1}{w_t}$. They also hold for all values of α , $u^\top \Sigma u$, and ρ in the limit risk-neutral case ($r \rightarrow 0$), as we show in the proof of Proposition 3.4.

We next examine how changes in arbitrageur wealth affect expected asset returns and the arbitrageurs' positions and Sharpe ratio. When arbitrageurs are wealthier, they have lower effective risk aversion, and absorb a larger fraction of the portfolio u that hedgers want to sell. Arbitrageur positions are thus larger in absolute value: more positive for positive elements of u , which correspond to assets that hedgers want to sell, and more negative for negative elements of u , which correspond to assets that hedgers want to buy. Since arbitrageurs are less risk averse, they require smaller compensation for providing insurance to hedgers. Expected asset returns, which measure that compensation, are thus smaller in absolute value: less positive for positive elements of u and less negative for negative elements of u . The same is true for the market prices of the Brownian risks, i.e., the expected returns per unit of risk exposure, and for the arbitrageurs' Sharpe ratio.

Proposition 3.5 *In both the logarithmic ($\gamma = 1$) and the limit risk-neutral ($\gamma = 0, r \rightarrow 0$) cases, an increase in arbitrageur wealth w_t :*

- (i) Raises the position of arbitrageurs in each asset in absolute value.*
- (ii) Lowers the expected return of each asset in absolute value.*
- (iii) Lowers the market price of each Brownian risk in absolute value.*
- (iv) Lowers the arbitrageurs' Sharpe ratio.*

The results of Proposition 3.5 are consistent with the empirical findings of Kang, Rouwenhorst, and Tang (2015). That paper finds that speculators in commodity futures markets act as momentum traders, buying when prices go up and selling when they go down. Moreover, following purchases by speculators expected returns are low, while they are high following speculator sales. These findings are consistent with Proposition 3.5, provided that arbitrageurs hold long positions in the risky assets, which is the case when u has positive elements. Indeed, when arbitrageurs are long, their wealth increases when assets earn high returns. Kang, Rouwenhorst, and Tang (2015) show that speculators are long on average for almost all of the commodities in their sample. Proposition 3.5 has the additional implication that when prices of some commodities go up speculators should buy *all* commodities and the expected returns on *all* commodities should decrease. This prediction is not tested in Kang, Rouwenhorst, and Tang (2015), but could perhaps be investigated as well.

We next derive the stationary distribution of arbitrageur wealth. Using that distribution, we can compute unconditional averages of endogenous variables, e.g., arbitrageurs' positions and Sharpe ratio.

Proposition 3.6 *If $z > 1$, then the stationary distribution of arbitrageur wealth has density*

$$d(w_t) = \frac{(\alpha w_t + 1)^2 w_t^{-\frac{1}{z}} \exp\left(-\frac{1}{2z}(\alpha^2 w_t^2 + 4\alpha w_t)\right)}{\int_0^\infty (\alpha w + 1)^2 w^{-\frac{1}{z}} \exp\left(-\frac{1}{2z}(\alpha^2 w^2 + 4\alpha w)\right) dw} \quad (3.23)$$

over the support $(0, \infty)$ in the logarithmic case ($\gamma = 1$), and density

$$d(w_t) = \frac{\left(\frac{\alpha + A(w_t)}{q(w_t)}\right)^2}{\int_0^{\bar{w}} \left(\frac{\alpha + A(w)}{q(w)}\right)^2 dw} \quad (3.24)$$

over the support $(0, \bar{w})$ in the limit risk-neutral case ($\gamma = 0, r \rightarrow 0$), where $A(w_t)$ and $q(w_t)$ are given by (3.20) and (3.22), respectively. If $0 < z < 1$, then wealth converges to zero in the long run, in both cases. If in the logarithmic case $z < 0$, then wealth converges to infinity in the long run.

The stationary distribution has a non-degenerate density if the parameter z defined by (3.18) is larger than one. This is the case when the hedgers' risk aversion α and endowment variance $u^\top \Sigma u$ are large, and the arbitrageurs' subjective discount rate ρ is small but exceeds the riskless rate r .

To provide an intuition for Proposition 3.6, we recall the standard Merton (1971) portfolio optimization problem in which an infinitely lived investor with CRRA coefficient γ can invest in a riskless asset with instantaneous return r and in N risky assets with instantaneous expected excess return vector μ and covariance matrix Σ . The investor's wealth converges to infinity in the long run when

$$r + \frac{1}{2} \mu^\top \Sigma^{-1} \mu > \rho, \quad (3.25)$$

i.e., when the riskless rate plus one-half of the squared Sharpe ratio achieved from investing in the risky assets exceeds the investor's subjective discount rate ρ . When instead (3.25) holds in the opposite direction, wealth converges to zero. Intuitively, wealth converges to infinity when the investor accumulates wealth at a rate that exceeds sufficiently the rate at which he consumes.

Our model differs from the Merton problem because the arbitrageurs' Sharpe ratio is endogenously determined in equilibrium and decreases in their wealth (Proposition 3.5). Using (3.15) to substitute for the arbitrageurs' Sharpe ratio, we can write (3.25) as

$$r + \frac{1}{2} \left(\frac{\alpha A(w_t)}{\alpha + A(w_t)} \right)^2 u^\top \Sigma u > \rho. \quad (3.26)$$

Transposing the result from the Merton problem thus suggests that there are three possibilities for the long-run dynamics. If (3.26) is satisfied for all values of w_t , then wealth converges to infinity. If (3.26) is violated for all values of w_t , then wealth converges to zero. If, finally, (3.26) is violated for large values but is satisfied for values close to zero, neither convergence occurs and wealth has a non-degenerate stationary density. Intuitively, a density can exist because the dynamics of arbitrageur wealth are self-correcting: when wealth becomes close to zero the Sharpe ratio increases and (3.26) becomes satisfied, and when wealth becomes large the Sharpe ratio decreases and (3.26) becomes violated.

When $\rho < r$, and so $z < 0$, (3.26) is satisfied for all values of w_t . Therefore, w_t converges to infinity. When $\rho > r$, and so $z > 0$, (3.26) is violated for values of w_t close to its upper bound (infinity in the logarithmic case and \bar{w} in the risk-neutral case) because $A(w_t)$ is close to zero for those values. Therefore, w_t either converges to zero or has a non-degenerate stationary density. Convergence to zero occurs if (3.26) is violated for w_t close to zero because it is then violated for all values of w_t . Since $A(w_t)$ is close to infinity for w_t close to zero, w_t converges to zero exactly when $z < 1$. Intuitively, wealth converges to zero when α and $u^\top \Sigma u$ are small because then arbitrageurs earn low expected returns for providing insurance to hedgers. When instead $z > 1$, w_t has a non-degenerate stationary density. Proposition 3.7 characterizes the shape of that density.

Proposition 3.7 *Suppose that $z > 1$. The density $d(w_t)$ of the stationary distribution:*

(i) *Is decreasing in w_t if $z < \frac{27}{8}$ in the logarithmic case ($\gamma = 1$) and if $z < 4$ in the limit risk-neutral case ($\gamma = 0, r \rightarrow 0$).*

(ii) *Is bimodal in w_t otherwise. That is, it is decreasing in w_t for $0 < w_t < \bar{w}_1$, increasing in w_t for $\bar{w}_1 < w_t < \bar{w}_2$, and again decreasing in w_t for $w_t > \bar{w}_2$. In the logarithmic case, the thresholds $\bar{w}_1 < \bar{w}_2$ are the two positive roots of*

$$(\alpha w)^3 + 3(\alpha w)^2 + (3 - 2z)\alpha w + 1 = 0. \quad (3.27)$$

In the limit risk-neutral case, they are given by

$$A(\bar{w}_1) \equiv \alpha \frac{z - 2 + \sqrt{z(z - 4)}}{2}, \quad (3.28)$$

$$A(\bar{w}_2) \equiv \alpha \frac{z - 2 - \sqrt{z(z - 4)}}{2}, \quad (3.29)$$

where $A(w_t)$ is given by (3.20), and they satisfy $0 < \bar{w}_1 < \bar{w}_2 < \bar{w}$.

(iii) Shifts to the right in the monotone likelihood ratio sense when α or $u^\top \Sigma u$ increase, in both the logarithmic and the limit risk-neutral cases.

The shape of the stationary density is fully determined by the parameter z . When z is not much larger than one, the density is decreasing, and so values close to zero are more likely than larger values. When instead z is sufficiently larger than one, the density becomes bimodal, with the two maxima being zero and an interior point \bar{w}_2 of the support. Values close to these maxima are more likely than intermediate values, meaning that the system spends more time at these values than in the middle. The intuition is that when the hedgers' risk aversion α and endowment variance $u^\top \Sigma u$ are large, arbitrageurs earn high expected returns for providing insurance, and their wealth grows fast. Therefore, large values of w_t are more likely in steady state than intermediate values. At the same time, while expected returns are highest when wealth is small, wealth grows away from small values slowly in absolute terms. Therefore, small values of w_t are more likely than intermediate values.

Figure 2 plots the stationary density in the logarithmic and risk-neutral cases. The solid lines are drawn for the same parameter values as in Figure 1. The dashed lines are drawn for the same values except that hedger risk aversion α is raised from 2 to 4. The solid lines are decreasing in wealth, while the dashed lines are bimodal. These patterns are consistent with Proposition 3.7 since z is equal to 2.25 for the solid lines and to 9 for the dashed lines.

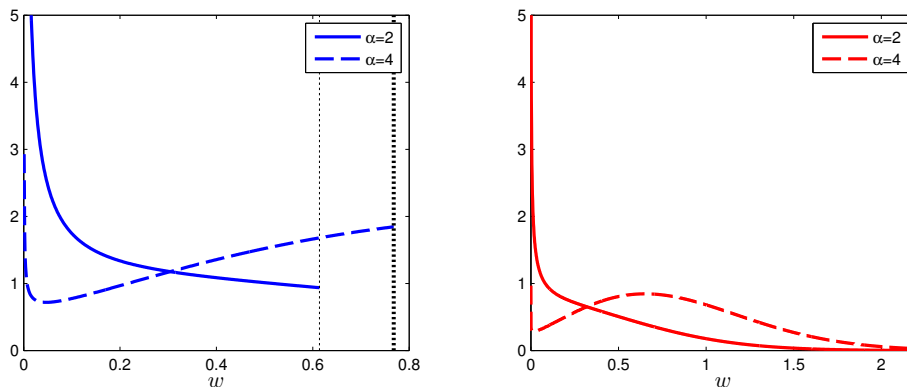


Figure 2: Stationary density of arbitrageur wealth in the logarithmic case (right panel) and the risk-neutral case (left panel). The solid lines are drawn for $\alpha = 2$, $\sqrt{u^\top \Sigma u} = 15\%$, $\rho = 4\%$, and $r = 2\%$. The dashed lines are drawn for the same values except that $\alpha = 4$.

We next perform comparative statics with respect to the hedgers' risk aversion α and endowment variance $u^\top \Sigma u$. We perform “conditional” comparative statics, where we compute how

changes in α and $u^\top \Sigma u$ affect endogenous variables, conditionally on a given level of arbitrageur wealth. We also perform “unconditional” comparative statics, where we compute how changes in α and $u^\top \Sigma u$ affect unconditional averages of the endogenous variables under the stationary distribution of wealth. The two types of comparative statics differ sharply.

Proposition 3.8 *Conditionally on a given level w_t of arbitrageur wealth, the following comparative statics hold:*

- (i) *An increase in the hedgers’ risk aversion α raises the arbitrageurs’ Sharpe ratio. In the logarithmic case ($\gamma = 1$), the position of arbitrageurs in each asset increases in absolute value. In the limit risk-neutral case ($\gamma = 0, r \rightarrow 0$), the position of arbitrageurs in each asset decreases in absolute value, except when w_t is below a threshold, which is negative if $z < 1$.*
- (ii) *An increase in the variance $u^\top \Sigma u$ of hedgers’ endowment raises the arbitrageurs’ Sharpe ratio. In the logarithmic case, arbitrageur positions do not change. In the limit risk-neutral case, the position of arbitrageurs in each asset decreases in absolute value.*

Result (i) of Proposition 3.8 concerns changes in hedger risk aversion. One would expect that when hedgers become more risk averse, they transfer more risk to arbitrageurs. This result holds in the logarithmic case, but the opposite result can hold in the risk-neutral case. This is because an increase in hedger risk aversion can generate an even larger increase in arbitrageur effective risk aversion through an increase in the intertemporal hedging demand. Recall that risk-neutral arbitrageurs behave as risk-averse because they seek to preserve wealth in states where other arbitrageurs realize losses and insurance provision becomes more profitable. When hedgers are more risk averse, this effect becomes stronger because insurance provision becomes more profitable for each level of arbitrageur wealth and more sensitive to changes in wealth. The effect is not present in the logarithmic case because effective risk aversion is equal to the static coefficient of absolute risk aversion, which depends only on wealth. In both the logarithmic and the risk-neutral cases, an increase in hedger risk aversion raises the Sharpe ratio of arbitrageurs because the expected return on their portfolio increases.

Result (ii) of Proposition 3.8 concerns changes in the variance of hedgers’ endowment. In the logarithmic case, such changes do not affect arbitrageur effective risk aversion and positions. In the risk-neutral case, however, there is an effect, which parallels that of hedger risk aversion. When the variance is high, e.g., because asset cashflows dD_t are more volatile, liquidity provision becomes more profitable. As a consequence, arbitrageurs have higher effective risk aversion and hold smaller

positions. In both the logarithmic and the risk-neutral cases, an increase in variance raises the arbitrageurs' Sharpe ratio.

We next turn to unconditional comparative statics. Figure 3 plots the unconditional Sharpe ratio of arbitrageurs as a function of α (left panel) and $u^\top \Sigma u$ (right panel). The results are in sharp contrast to the conditional comparative statics. While an increase in α and $u^\top \Sigma u$ raises the Sharpe ratio conditionally on a given level of wealth (Proposition 3.8), it can lower it when comparing unconditional averages. Intuitively, for larger values of α and $u^\top \Sigma u$, arbitrageur wealth grows faster, and its stationary density shifts to the right (Proposition 3.7). Therefore, while the conditional Sharpe ratio increases, the unconditional one can decrease because high values of wealth, which yield low Sharpe ratios, become more likely.

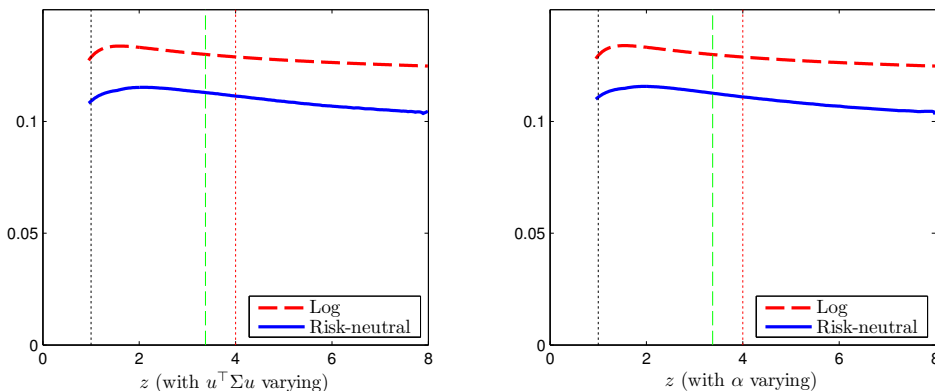


Figure 3: The unconditional Sharpe ratio of arbitrageurs as a function of α (left panel) and $u^\top \Sigma u$ (right panel), for the logarithmic case (dashed lines) and the risk-neutral case (solid lines). When α varies, the remaining parameters are set to $\sqrt{u^\top \Sigma u} = 15\%$, $\rho = 4\%$, and $r = 2\%$. When $u^\top \Sigma u$ varies, the remaining parameters are set to $\alpha = 2$, $\rho = 4\%$, and $r = 2\%$. The left-most vertical bar is the threshold $z = 1$ beyond which the stationary distribution has a non-degenerate density. The vertical bars to the right are the thresholds $z = \frac{27}{8}$ and $z = 4$ beyond which the density becomes bimodal in the logarithmic and in the limit risk-neutral cases, respectively.

4 Asset Prices

4.1 Equilibrium with Long-Lived Assets

We conjecture that in equilibrium the price vector S_t of the long-lived assets (introduced at the beginning of Section 2) follows the Ito process

$$dS_t = \mu_{S_t} dt + \sigma_{S_t}^\top dB_t, \quad (4.1)$$

where μ_{S_t} is a $N \times 1$ vector and σ_{S_t} is a $N \times N$ matrix. We denote by $dR_t \equiv dS_t + dD_t - rS_t dt$ the $N \times 1$ vector of returns that the long-lived assets earn between t and $t + dt$ in excess of the riskless asset. Eqs. (2.1) and (4.1) imply that the instantaneous expected returns of the long-lived assets are

$$\frac{\mathbb{E}_t(dR_t)}{dt} = \mu_{S_t} + \bar{D} - rS_t, \quad (4.2)$$

and the instantaneous covariance matrix of returns is

$$\frac{\text{Var}_t(dR_t)}{dt} = (\sigma_{S_t} + \sigma)^\top (\sigma_{S_t} + \sigma). \quad (4.3)$$

With long-lived assets, a hedger's budget constraint (3.3) becomes

$$dv_t = rv_t dt + X_t^\top (dS_t + dD_t - rS_t dt) + u^\top dD_t, \quad (4.4)$$

where X_t denotes the hedger's position in the long-lived assets. An arbitrageur's budget constraint (3.5) becomes similarly

$$dw_t = (rw_t - c_t)dt + Y_t^\top (dS_t + dD_t - rS_t dt), \quad (4.5)$$

where Y_t denotes the arbitrageur's position. Because the market is complete under long-lived assets, as it is under short-lived assets, the two asset structures generate the same allocation of risk.

Lemma 4.1 *An equilibrium (S_t, X_t, Y_t) with long-lived assets can be constructed from an equilibrium (π_t, x_t, y_t) with short-lived assets by:*

(i) *Choosing the price process S_t such that*

$$\left(\sigma^\top\right)^{-1} (\bar{D} - \pi_t) = \left((\sigma_{S_t} + \sigma)^\top\right)^{-1} (\mu_{S_t} + \bar{D} - rS_t). \quad (4.6)$$

(ii) *Choosing the asset positions X_t of hedgers and Y_t of arbitrageurs such that*

$$\sigma x_t = (\sigma_{S_t} + \sigma) X_t, \quad (4.7)$$

$$\sigma y_t = (\sigma_{S_t} + \sigma) Y_t. \quad (4.8)$$

In the equilibrium with long-lived assets the dynamics of arbitrageur wealth, the exposures of hedgers and arbitrageurs to the Brownian shocks, the market prices of the Brownian risks, and the arbitrageurs' Sharpe ratio are the same as in the equilibrium with short-lived assets.

Eqs. (4.7) and (4.8) construct positions of hedgers and arbitrageurs in the long-lived assets so that the exposures to the underlying Brownian shocks are the same as with short-lived assets. Eq. (4.6) constructs a price process such that the market prices of the Brownian risks are also the same. Given this price process, agents choose optimally the risk exposures in (4.7) and (4.8), and markets clear.

The price S_t of the long-lived assets is a function of arbitrageur wealth w_t only. Using Ito's lemma to compute the drift μ_{S_t} and diffusion σ_{S_t} of the price process as a function of the dynamics of w_t , and substituting into (4.6), we can determine $S(w_t)$ up to an ODE.

Proposition 4.1 *The price of the long-lived assets is given by*

$$S(w_t) = \frac{\bar{D} - \alpha \Sigma u}{r} + g(w_t) \Sigma u, \quad (4.9)$$

where the scalar function $g(w_t)$ satisfies the ODE

$$\left(r - q^{-\frac{1}{\gamma}}\right) w g' + \frac{\alpha^2}{2(\alpha + A)^2} u^\top \Sigma u g'' - r g = -\frac{\alpha^2}{\alpha + A}. \quad (4.10)$$

The price in (4.9) is the sum of two terms. The first term, $\frac{\bar{D} - \alpha \Sigma u}{r}$, is the price that would prevail in the absence of arbitrageurs. Indeed, if hedgers were the only traders in a market with short-lived assets, their demand (3.4) would equal the asset supply, which is zero. Solving for the market-clearing price yields $\pi_t = \bar{D} - \alpha \Sigma u$. Long-lived assets would trade at the present value of the infinite stream of these prices discounted at the riskless rate r , which is $\frac{\bar{D} - \alpha \Sigma u}{r}$. The second term, $g(w_t) \Sigma u$, measures the price impact of arbitrageurs. Since arbitrageurs buy a fraction of the portfolio u that hedgers want to sell, they cause assets covarying positively with that portfolio to become more expensive. Therefore, the function $g(w_t)$ should be positive, and equal to zero for $w_t = 0$. Moreover, since arbitrageurs have a larger impact the wealthier they are, $g(w_t)$ should be increasing in w_t , as we confirm in the special cases studied in Section 4.2.

Expected asset returns and the covariance matrix of returns are driven by the sensitivity of the price to changes in arbitrageur wealth w_t . Therefore, they are driven by the term $g(w_t) \Sigma u$ and do not depend on $\frac{\bar{D} - \alpha \Sigma u}{r}$. In the proof of Proposition 4.1 we show that expected returns are

$$\frac{E_t(dR_t)}{dt} = \frac{\alpha A(w_t)}{\alpha + A(w_t)} \left[f(w_t) u^\top \Sigma u + 1 \right] \Sigma u, \quad (4.11)$$

and the covariance matrix of returns is

$$\frac{\text{Var}_t(dR_t)}{dt} = f(w_t) \left[f(w_t) u^\top \Sigma u + 2 \right] \Sigma u u^\top \Sigma + \Sigma, \quad (4.12)$$

where

$$f(w_t) \equiv \frac{\alpha g'(w_t)}{\alpha + A(w_t)}. \quad (4.13)$$

The covariance matrix (4.12) is the sum of a “fundamental” component Σ , driven purely by shocks to assets’ underlying cashflows dD_t , and an “endogenous” component $f(w_t) \left[f(w_t) u^\top \Sigma u + 2 \right] \Sigma u u^\top \Sigma$, introduced because cashflow shocks affect arbitrageur wealth w_t which affects prices. Endogenous risk is zero in the case of short-lived assets because their payoff dD_t is not sensitive to changes in w_t . Changes in w_t , however, affect the payoff $dD_t + S_{t+dt}$ of long-lived assets because they affect the price S_{t+dt} . Therefore, endogenous risk arises only with long-lived assets, and we show that it drives the patterns of volatilities, correlations, and expected returns.

The effect of w_t on prices is proportional to the covariance Σu with the portfolio u . Therefore, the endogenous covariance between assets n and n' is proportional to the product between the elements n and n' of the vector Σu . Expected returns are proportional to Σu , as in the case of short-lived assets. The proportionality coefficient is different than in that case, however, because it is influenced by the endogenous covariance.

4.2 Closed-Form Solutions

We next characterize the equilibrium more fully in the logarithmic case ($\gamma = 1$) and the risk-neutral case ($\gamma = 0$). We compute the function $g'(w_t)$ that characterizes the sensitivity of the price to changes in arbitrageur wealth in closed form in the limit when the riskless rate r goes to zero. In that limit the price is not well defined because the constant term $\frac{\bar{D} - \alpha \Sigma u}{r}$ converges to infinity. The function $g(w_t)$ is well-defined, however, and so are expected asset returns and the covariance matrix of returns, which depend on the price only through $g'(w_t)$. Hence, as long as $g'(w_t)$ is continuous with respect to r , our results are informative about the properties of these quantities close to the limit, where the price is well defined.

Proposition 4.2 *The function $g'(w_t)$ is given by*

$$g'(w_t) = \frac{2w_t^{\frac{1}{z}} \exp\left(\frac{1}{2z}(\alpha^2 w_t^2 + 4\alpha w_t)\right)}{u^\top \Sigma u} \int_{w_t}^{\infty} \left(\alpha + \frac{1}{w}\right) w^{-\frac{1}{z}} \exp\left(-\frac{1}{2z}(\alpha^2 w^2 + 4\alpha w)\right) dw \quad (4.14)$$

for $w_t \in (0, \infty)$ in the limit logarithmic case ($\gamma = 1, r \rightarrow 0$), and by

$$g'(w_t) = \frac{2z}{(1+z)u^\top \Sigma u} \left[\log \sin\left(\frac{\alpha \bar{w}}{\sqrt{z}}\right) - \log \sin\left(\frac{\alpha w_t}{\sqrt{z}}\right) + \alpha(\bar{w} - w_t) \right] \quad (4.15)$$

for $w_t \in (0, \bar{w})$ in the limit risk-neutral case ($\gamma = 0, r \rightarrow 0$). In both cases $g'(w_t) > 0$.

We next examine how changes in arbitrageur wealth affect expected asset returns, volatilities, correlations, and arbitrageur positions.

Proposition 4.3 *An increase in arbitrageur wealth w_t has the following effects in both the limit logarithmic ($\gamma = 1, r \rightarrow 0$) and the limit risk-neutral ($\gamma = 0, r \rightarrow 0$) cases:*

- (i) *A hump-shaped effect on the expected return of each asset, in absolute value, except when $z < \frac{1}{2}$ in the logarithmic case, where the effect is decreasing. The hump peaks at a value \bar{w}_a that is common to all assets.*
- (ii) *A hump-shaped effect on the volatility of the return of each asset. The hump peaks at a value \bar{w}_b that is common to all assets and is larger than the corresponding value \bar{w}_a for the expected return.*
- (iii) *The same hump-shaped effect as in Part (ii) on the covariance between the returns of each asset pair (n, n') if $(\Sigma u)_n (\Sigma u)_{n'} > 0$, and the opposite, i.e., inverse hump-shaped effect, if $(\Sigma u)_n (\Sigma u)_{n'} < 0$.*
- (iv) *The same hump-shaped effect as in Part (ii) on the correlation between the returns of each asset pair (n, n') if*

$$\frac{(\Sigma u)_n (\Sigma u)_{n'} \Sigma_{nn} - (\Sigma u)_n^2 \Sigma_{nn'}}{f(w_t) [f(w_t) u^\top \Sigma u + 2] (\Sigma u)_n^2 + \Sigma_{nn}} + \frac{(\Sigma u)_n (\Sigma u)_{n'} \Sigma_{n'n'} - (\Sigma u)_{n'}^2 \Sigma_{nn'}}{f(w_t) [f(w_t) u^\top \Sigma u + 2] (\Sigma u)_{n'}^2 + \Sigma_{n'n'}} > 0, \quad (4.16)$$

and the opposite, i.e., inverse hump-shaped, effect if (4.16) holds in the opposite direction.

- (v) *An increasing effect on the position of arbitrageurs in each asset, in absolute value.*

Since the fundamental component Σ of the covariance matrix is independent of arbitrageur wealth, the hump-shaped patterns of volatilities, covariances, and correlations are driven by the endogenous component. The intuition for the hump shape in the case of volatilities can be seen by computing the diffusion of the price process. Ito's lemma implies that $\sigma_{St} = \sigma_{wt}S'(w_t)^\top$, i.e., price volatility (diffusion) is equal to the volatility of arbitrageur wealth times the sensitivity of the price to changes in wealth. The volatility of wealth is increasing in wealth, and converges to zero when wealth goes to zero. Intuitively, when arbitrageurs are poor, they hold small positions and take almost no risk. The sensitivity of price to changes in wealth is instead decreasing in wealth, and converges to zero when wealth becomes large (close to infinity in the logarithmic case and to \bar{w} in the risk-neutral case). Intuitively, when arbitrageurs are wealthy, they provide full insurance to hedgers, and changes to their wealth have no impact on the price. Therefore, price volatility converges to zero at both extremes of the wealth distribution, and this accounts for the hump-shaped pattern of return volatilities.⁵

The intuition for the hump shape in the case of covariances is similar to that for volatilities. Price movements caused by changes in arbitrageur wealth are small at the extremes of the wealth distribution and larger in the middle. This yields a hump-shaped pattern for the covariance between two assets n and n' , if the prices of these assets move in the same direction. Movements are in the same direction when the term (n, n') of the endogenous covariance matrix is positive. This term is equal to $(\Sigma u)_n(\Sigma u)_{n'}$, and is likely to be positive when the corresponding components of the vector u have the same sign, i.e., arbitrageurs either buy both assets from the hedgers or sell both assets to them. When, for example, both assets are bought by arbitrageurs, they both appreciate when arbitrageur wealth go up, yielding a positive covariance.

The effect on correlations is more complicated than that on covariances because it includes the effect on volatilities. Suppose that changes in arbitrageur wealth move the prices of assets n and n' in the same direction, and hence have a hump-shaped effect on their covariance. Because, however, the effect on volatilities, which are in the denominator, is also hump-shaped, the overall effect on the correlation can be inverse hump-shaped. Intuitively, arbitrageurs can cause assets to become less correlated because the increase in volatilities that they cause can swamp the increase in covariance.

⁵Price volatility converges to zero at the extremes of the wealth distribution because we are assuming for simplicity *i.i.d.* cashflows dD_t . Under *i.i.d.* cashflows, a cashflow shock does not have a direct effect on prices, i.e., does not affect prices holding arbitrageur wealth constant. Return volatility remains positive even at the extremes of the wealth distribution because a cashflow shock has a direct effect on returns. Under a persistent cashflow process, price volatility would not converge to zero at the extremes, but would remain hump-shaped.

The hump-shaped pattern of expected returns derives from that of volatilities. Expected returns per unit of risk exposure, i.e., the market prices of the Brownian risks, are the same as in the equilibrium with short-term assets, and are hence decreasing in wealth (Proposition 3.5). But because the volatility of long-term assets is hump-shaped in wealth, their expected returns are generally also hump-shaped.

Figures 4 and 5 illustrate the behavior of assets' Sharpe ratios, expected returns, volatilities, and correlations as a function of arbitrageur wealth in the logarithmic and risk-neutral cases, respectively. The figures are drawn for the same parameter values as in Figure 2.

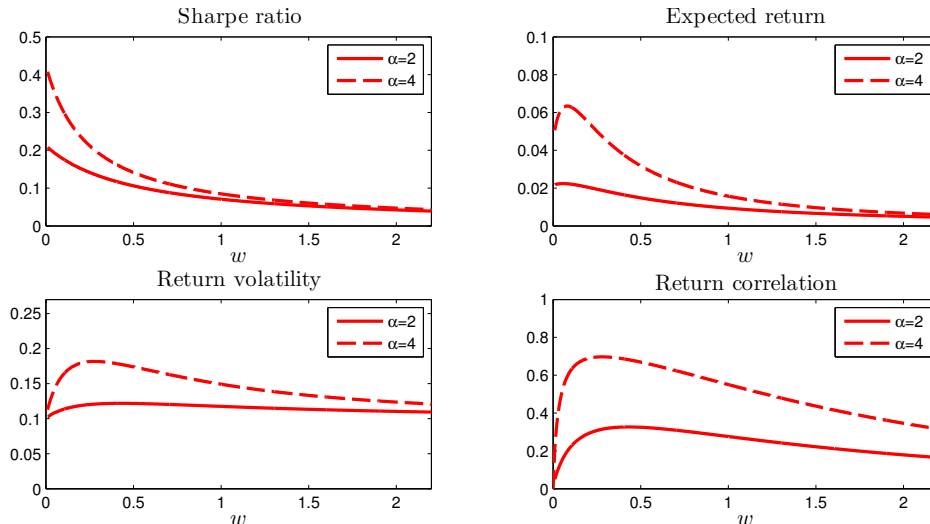


Figure 4: Assets' Sharpe ratios, expected returns, volatilities, and correlations as a function of arbitrageur wealth in the logarithmic case. The solid lines are drawn for $\alpha = 2$, $\sqrt{u^\top \Sigma u} = 15\%$, $\rho = 4\%$, $r = 2\%$, $N = 2$ symmetric assets with independent cashflows, and $\Sigma_{11} = \Sigma_{22} = 10\%$. The dashed lines are drawn for the same values except that $\alpha = 4$.

Using Figures 4 and 5, we can compare the logarithmic and risk-neutral cases. The assets' Sharpe ratios are higher in the logarithmic case, as one would expect since risk aversion is higher. Expected returns, however, can be higher in the risk-neutral case (as the figures show more clearly for $\alpha = 4$). This surprising result derives from volatilities, whose endogenous component can be higher in the risk-neutral case.

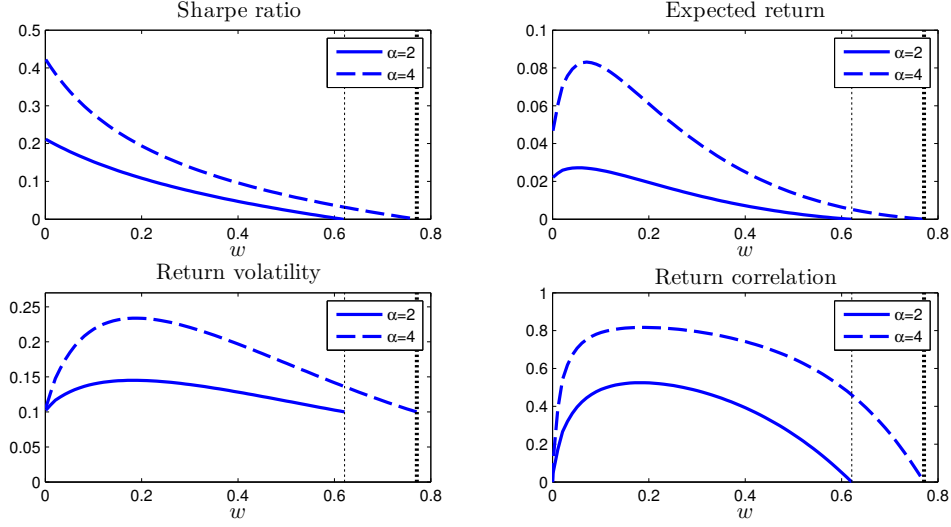


Figure 5: Assets' Sharpe ratios, expected returns, volatilities, and correlations as a function of arbitrageur wealth in the risk-neutral case. The solid lines are drawn for $\alpha = 2$, $\sqrt{u^\top \Sigma u} = 15\%$, $\rho = 4\%$, $r = 2\%$, $N = 2$ symmetric assets with independent cashflows, and $\Sigma_{11} = \Sigma_{22} = 10\%$. The dashed lines are drawn for the same values except that $\alpha = 4$.

5 Liquidity Risk

In this section we explore the implications of our model for liquidity risk. We assume long-lived assets, as in the previous section. We define liquidity based on the impact that hedgers have on prices. Consider an increase in the parameter u_n that characterizes hedgers' willingness to sell asset n . This triggers a decrease $\frac{\partial X_{nt}}{\partial u_n}$ in the quantity of the asset held by the hedgers, and a decrease $\frac{\partial S_{nt}}{\partial u_n}$ in the asset price. Asset n has low liquidity if the price change per unit of quantity traded is large. That is, the illiquidity of asset n is defined by

$$\lambda_{nt} \equiv \frac{r \frac{\partial S_{nt}}{\partial u_n}}{\frac{\partial X_{nt}}{\partial u_n}}, \tag{5.1}$$

where we multiply by r to ensure a well-behaved limit for our closed-form solutions. Defining illiquidity as price impact follows Kyle (1985). Kyle and Xiong (2001), Xiong (2001), and Johnson (2008) perform similar constructions to ours in asset-pricing settings by defining illiquidity as the derivative of price with respect to supply.⁶

⁶A drawback of the measure (5.1) in the context of our model is that u_n is constant over time, and hence λ_{nt} cannot be computed by an empiricist. One interpretation of (5.1) is that there are small shocks to u_n , which an empiricist can observe and use to compute λ_{nt} . In the Conclusion we sketch how our analysis can be extended to stochastic u .

Proposition 5.1 *Illiquidity λ_{nt} is equal to*

$$\left(1 + \frac{A(w_t)}{\alpha} + g'(w_t)u^\top \Sigma u\right) (\alpha - rg(w_t)) \Sigma_{nn}. \quad (5.2)$$

An increase in arbitrageur wealth w_t lowers illiquidity in both the limit logarithmic ($\gamma = 1, r \rightarrow 0$) and the limit risk-neutral ($\gamma = 0, r \rightarrow 0$) cases.

Proposition 5.1 identifies a time-series and a cross-sectional dimension of illiquidity. In the time-series, illiquidity varies in response to changes in arbitrageur wealth, and is a decreasing function of wealth. This variation is common across assets and corresponds to the two terms in parentheses in (5.2). In the cross-section, illiquidity is higher for assets with more volatile cashflows. The dependence of illiquidity on the asset index n is through the asset's cashflow variance Σ_{nn} , the last term in (5.2). The time-series and cross-sectional dimensions of illiquidity interact: assets with more volatile cashflows have higher illiquidity for any given level of wealth, and the time-variation of their illiquidity is more pronounced.

Using Proposition 5.1, we can compute the covariance between asset returns and aggregate illiquidity. Since illiquidity varies over time because of arbitrageur wealth, and with an inverse relationship, the covariance of the return vector with illiquidity is equal to the covariance with wealth times a negative coefficient. Proposition 4.1 implies, in turn, that the covariance of the return vector with wealth is proportional to Σu . This is the covariance between asset cashflows and the cashflows of the portfolio u , which characterizes hedgers' supply. The intuition for the proportionality is that when arbitrageurs realize losses, they sell a fraction of u , and this lowers asset prices according to the covariance with u . Therefore, the covariance between asset returns and aggregate illiquidity $\Lambda_t \equiv \frac{\sum_{n=1}^N \lambda_{nt}}{N}$ is

$$\frac{\text{Cov}_t(d\Lambda_t, dR_t)}{dt} = C^\Lambda(w_t)\Sigma u, \quad (5.3)$$

where $C^\Lambda(w_t)$ is a negative coefficient. Assets that suffer the most when aggregate illiquidity increases and arbitrageurs sell a fraction of the portfolio u , are those corresponding to large components $(\Sigma u)_n$ of Σu . They have volatile cashflows (high Σ_{nn}), or are in high supply by hedgers (high u_n), or correlate highly with assets with those characteristics.

Using Proposition 5.1, we can compute two additional liquidity-related covariances: the covariance between an asset's illiquidity and aggregate illiquidity, and the covariance between an

asset's illiquidity and aggregate return. We take the aggregate return to be that of the portfolio u , which characterizes hedgers' supply. Acharya and Pedersen (2005) show theoretically, within a model with exogenous transaction costs, that both covariances are linked to expected returns in the cross-section. In our model, the time-variation of an asset's illiquidity is proportional to the asset's cashflow variance Σ_{nn} . Therefore, the covariances between the asset's illiquidity on one hand, and aggregate illiquidity or return on the other, are proportional to Σ_{nn} .

Corollary 5.1 *In the cross-section of assets:*

- (i) *The covariance between asset n 's return dR_{nt} and aggregate illiquidity Λ_t is proportional to the covariance $(\Sigma u)_n$ between the asset's cashflows and the cashflows of the hedger-supplied portfolio u .*
- (ii) *The covariance between asset n 's illiquidity λ_{nt} and aggregate illiquidity Λ_t is proportional to the variance Σ_{nn} of the asset's cashflows.*
- (iii) *The covariance between asset n 's illiquidity λ_{nt} and aggregate return $u^\top dR_t$ is proportional to the variance Σ_{nn} of the asset's cashflows.*

The proportionality coefficients are negative, positive, and negative, respectively, in both the limit logarithmic ($\gamma = 1, r \rightarrow 0$) and the limit risk-neutral ($\gamma = 0, r \rightarrow 0$) cases.

We next determine the link between liquidity-related covariances and expected returns. Recall from (4.11) that the expected return of asset n is proportional to $(\Sigma u)_n$. This is exactly proportional to the covariance between the asset's return and aggregate illiquidity. Thus, aggregate illiquidity is a priced risk factor that explains expected returns perfectly. Intuitively, assets are priced by the portfolio of arbitrageurs, who are the marginal agents. Moreover, the covariance between asset returns and aggregate illiquidity identifies that portfolio perfectly. This is because (i) changes in aggregate illiquidity are driven by arbitrageur wealth, and (ii) the portfolio of trades that arbitrageurs perform when their wealth changes is proportional to their existing portfolio and impacts returns proportionately to the covariance with that portfolio.

The covariances between an asset's illiquidity on one hand, and aggregate illiquidity or returns on the other, are less informative about expected returns. Indeed, these covariances are proportional to cashflow variance Σ_{nn} , which is only a component of $(\Sigma u)_n$. Thus, these covariances proxy for the true pricing factor but imperfectly so.

Corollary 5.2 *In the cross-section of assets, expected returns are proportional to the covariance between returns and aggregate illiquidity. The proportionality coefficient is negative, in both the limit logarithmic ($\gamma = 1, r \rightarrow 0$) and the limit risk-neutral ($\gamma = 0, r \rightarrow 0$) cases.*

The premium associated to the illiquidity risk factor is the expected return per unit of covariance with the factor. We denote it by $\Pi^\Lambda(w_t)$:

$$\frac{E_t(dR_t)}{dt} \equiv \Pi^\Lambda(w_t) \frac{\text{Cov}_t(d\Lambda_t, dR_t)}{dt}. \quad (5.4)$$

Eqs. (4.2) and (5.3) imply that $\Pi^\Lambda(w_t)$ is related to the common component $C^\Lambda(w_t)$ of assets' covariance with aggregate illiquidity through

$$\Pi^\Lambda(w_t) = \frac{\alpha A(w_t) [f(w_t)u^\top \Sigma u + 1]}{[\alpha + A(w_t)] C^\Lambda(w_t)}. \quad (5.5)$$

The quantities $\Pi^\Lambda(w_t)$ and $C^\Lambda(w_t)$ vary over time in response to changes in arbitrageur wealth. When wealth is low, illiquidity is high and highly sensitive to changes in wealth. Because of this effect, assets' covariance with illiquidity is large and decreases when wealth increases. Conversely, because the premium of the illiquidity risk factor is the expected return per unit of covariance, it is low when wealth is low and increases when wealth increases. For large values of wealth, the premium can decrease again because the decrease in expected returns can dominate the decrease in covariance. Proposition 5.2 derives these results in the limit when r goes to zero, and Figure 6 illustrates them in a numerical example.

Proposition 5.2 *In both the limit logarithmic ($\gamma = 1, r \rightarrow 0$) and the limit risk-neutral ($\gamma = 0, r \rightarrow 0$) cases:*

- (i) *The common component $C^\Lambda(w_t) < 0$ of assets' covariance with aggregate illiquidity converges to minus infinity when arbitrageur wealth w_t goes to zero. It remains negative when wealth reaches \bar{w} in the limit risk-neutral case, and converges to zero when wealth goes to infinity in the limit logarithmic case.*
- (ii) *The premium $\Pi^\Lambda(w_t) < 0$ of the illiquidity risk factor converges to zero when arbitrageur wealth w_t goes to zero. In the limit risk-neutral case, it is inverse-hump shaped in wealth and reaches zero when wealth reaches \bar{w} . In the limit logarithmic case, it converges to minus infinity when wealth goes to infinity.*

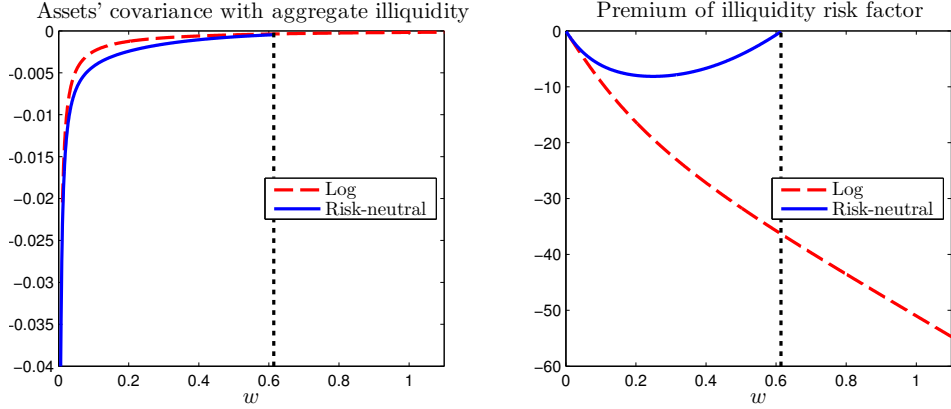


Figure 6: Assets' covariance with aggregate illiquidity (left panel) and premium of illiquidity risk factor (right panel) as a function of arbitrageur wealth, in the logarithmic case (dashed lines) and the risk-neutral case (solid lines). The premium of the illiquidity risk factor is the expected return per unit of covariance. Parameter values are $\alpha = 2$, $\sqrt{u^\top \Sigma u} = 15\%$, $\rho = 4\%$, $r = 2\%$, $N = 2$ symmetric assets with independent cashflows, and $\Sigma_{11} = \Sigma_{22} = 10\%$.

6 Other Applications and Extensions

6.1 Term Structure of Risk Premia

Section 3 derives the cost of obtaining insurance against risks that transpire in the immediate future. This cost is characterized by the expected returns $\bar{D} - \pi_t$ of the short-lived assets trading at time t : when expected returns are high in absolute value, insurance costs more. We can also compute the cost of obtaining insurance against risks that transpire in the more distant future. For this, we compute the price $\pi_{t,t'}$ at which agents would agree at time t to trade at a future time $t' > t$ the short-lived assets paying at time $t' + dt$. The expected returns $\bar{D} - \pi_{t,t'}$ associated to these forward contracts characterize the time- t cost of obtaining insurance against risks that transpire at time $t' + dt$. We refer to the function $t' \rightarrow |\bar{D}_n - \pi_{n,t,t'}|$ as the time- t term structure of the risk premia associated to long-lived asset n . We define the slope of the term structure as the difference between its values at the long end ($t' = \infty$) and the short end ($t' = t$).

An agreement at time t to buy at time t' the short-lived assets paying off at $t' + dt$ yields net payoff $\pi_{t'} - \pi_{t,t'}$. Since this payoff has zero expectation under the risk-neutral measure,

$$\pi_{t,t'} = E_t^*(\pi_{t'}) = \bar{D} - E_t^* \left(\frac{\alpha A(w_{t'})}{\alpha + A(w_{t'})} \right) \Sigma u, \quad (6.1)$$

where the second equality follows from (3.13) and the superscript $*$ denotes the risk-neutral measure.

Proposition 6.1 *In the logarithmic ($\gamma = 1$) case with $z > 0$, and in the limit risk-neutral ($\gamma = 0, r \rightarrow 0$) case, the term structure of the risk premia associated to long-lived asset n :*

(i) *Is upward-sloping and converges to $|\alpha(\Sigma u)_n|$ when t' goes to ∞ .*

(ii) *Becomes steeper (higher slope) when arbitrageur wealth w_t increases.*

The result that the term structure of risk premia is upward-sloping for all values of arbitrageur wealth may seem surprising. Indeed, when wealth is low, the cost of insurance is expected to decline over time because wealth is expected to rise. One would expect, in the spirit of the expectations hypothesis, that the decline in expected cost yields a downward-sloping term structure. Countering this effect, however, is that the term structure is given by the expected cost under the risk-neutral measure, which is higher than under the physical measure. This is because the cost goes up in states where wealth goes down, and these states are given larger weight by the risk-neutral measure than by the physical measure. The adjustment to the risk-neutral measure lowers arbitrageurs' expected return on wealth to the riskless rate, and this causes expected wealth to decline over time in both the logarithmic and the limit risk-neutral cases. Expected wealth converges to zero in the long run under the risk-neutral measure, causing the long end of the term structure to be equal to the maximum cost of insurance. When wealth w_t increases, the term structure steepens because the short end shifts down.

Binsbergen, Hueskes, Koijen, and Vrugt (2013) find empirically that the term structure of risk premia in the aggregate stock market was upward-sloping in the years preceding the recent financial crisis but became downward-sloping during the crisis. Our model can generate the decline in slope under the assumption that the crisis was accompanied by a reduction in wealth. It cannot generate, however, a negative slope. An extension of the model where there is entry by new arbitrageurs could perhaps generate a negative slope because wealth would increase more rapidly following crises.

6.2 Positive Supply

Our analysis so far assumes that the risky assets are in zero supply and that hedgers maximize instantaneous mean-variance utility. In this section we relax the first assumption, allowing supply to be positive. In Section 6.3 we relax both assumptions, allowing both for positive supply and for infinitely lived hedgers with CARA utility over intertemporal consumption.

Proposition 6.2 *Suppose that long-lived assets are in positive supply s . Then, their price is given by*

$$S(w_t) = \frac{\bar{D} - \alpha \Sigma(s + u)}{r} + g(w_t) \Sigma(s + u), \quad (6.2)$$

where the scalar function $g(w_t)$ solves the ODE (A.120).

The equation for the price is the same as (4.9), derived in the case of zero supply, except that the function $g(w_t)$ is given by a different ODE (not solvable in closed form), and that u is replaced by $s + u$. Thus, the effect of the supply s coming from asset issuers on cross-sectional asset pricing is the same as of the supply u coming from hedgers, and it is only the aggregate supply $s + u$ that matters. The same conclusion holds for expected returns: in the proof of Proposition 6.2 we show that expected returns are

$$\frac{E_t(dR_t)}{dt} = \frac{\alpha A(w_t) [\alpha + A(w_t) + \alpha g'(w_t)(s + u)^\top \Sigma u]}{[\alpha + A(w_t) - \alpha g'(w_t)(s + u)^\top \Sigma s]^2} \Sigma(s + u). \quad (6.3)$$

Expected returns in the cross-section are thus proportional to the covariance with the aggregate supply $s + u$. Since the effect of changes in arbitrageur wealth on asset prices is also proportional to that covariance, aggregate illiquidity is a priced risk factor and explains expected returns perfectly, as in the case of zero supply.

6.3 Infinitely Lived CARA Hedgers

Proposition 6.3 *Suppose that long-lived assets are in positive supply s and hedgers are infinitely lived with the CARA utility (2.3) over intertemporal consumption. Then, the hedgers' value function is given by*

$$V(v_t, w_t) = -e^{-[\alpha v_t + F(w_t)]} \quad (6.4)$$

and the price of the long-lived assets is given by (6.2), where the scalar functions $(F(w_t), g(w_t))$ solve the ODEs (A.138) and (A.141).

When hedgers are infinitely lived with CARA utility over intertemporal consumption, their demand includes an intertemporal hedging component. The intertemporal hedging demand depends only on arbitrageur wealth because wealth is the only variable that affects the investment

opportunity set. In the proof of Proposition 6.3 we show that because of the intertemporal hedging demand, the hedgers' effective risk aversion becomes $\alpha - F'(w_t)$. The term α is the coefficient of absolute risk aversion over changes in wealth, and the term $F'(w_t)$ (for the function $F(w_t)$ defined in Proposition 6.3) is the contribution of the intertemporal hedging demand.

Assuming infinitely-lived CARA hedgers does not affect the characterization of price and expected returns. The equation for the price is (6.2), derived for mean-variance hedgers, except that the function $g(w_t)$ is given by a different ODE. The equation for expected returns is

$$\frac{E_t(dR_t)}{dt} = \frac{\alpha A(w_t) [\alpha + A(w_t) - F'(w_t) + \alpha g'(w_t)(s + u)^\top \Sigma u]}{[\alpha + A(w_t) - F'(w_t) - \alpha g'(w_t)(s + u)^\top \Sigma s]^2} \Sigma (s + u). \quad (6.5)$$

and has the same form as (6.3), except that it also includes the term $F'(w_t)$ corresponding to the intertemporal hedging demand. The price and expected returns in the cross section depend only on the covariance with aggregate supply. Hence, aggregate illiquidity is a priced risk factor and explains expected returns perfectly.

Figure 7 illustrates the behavior of assets' Sharpe ratios, expected returns, volatilities, correlations, illiquidity, and the premium of the illiquidity risk factor, as a function of arbitrageur wealth in three cases: zero supply and mean-variance hedgers (baseline), positive supply and mean-variance hedgers (Section 6.2), and positive supply and CARA hedgers (Section 6.3). For simplicity we consider only the case of logarithmic arbitrageurs. The parameter values for Figure 7 are the same as for Figure 4, and when the supply s is positive it is set equal to $\frac{u}{2}$. Figure 7 shows that introducing positive supply and infinitely lived CARA hedgers does not change the way equilibrium quantities depend on arbitrageur wealth, although the effects can differ in magnitude.

7 Concluding Remarks

We develop a dynamic model of liquidity provision, in which hedgers can trade multiple risky assets with arbitrageurs. Arbitrageurs have CRRA utility over consumption, and their wealth is a state variable affecting risk sharing and asset prices in equilibrium. We compute equilibrium prices and quantities in closed form when arbitrageurs' utility function is logarithmic or risk-neutral with a non-negativity constraint. Our multi-asset setting delivers more tractability than what is typical in consumption-based asset pricing models, while also incorporating wealth effects.

Our model yields a new understanding of liquidity risk and its relationship with expected

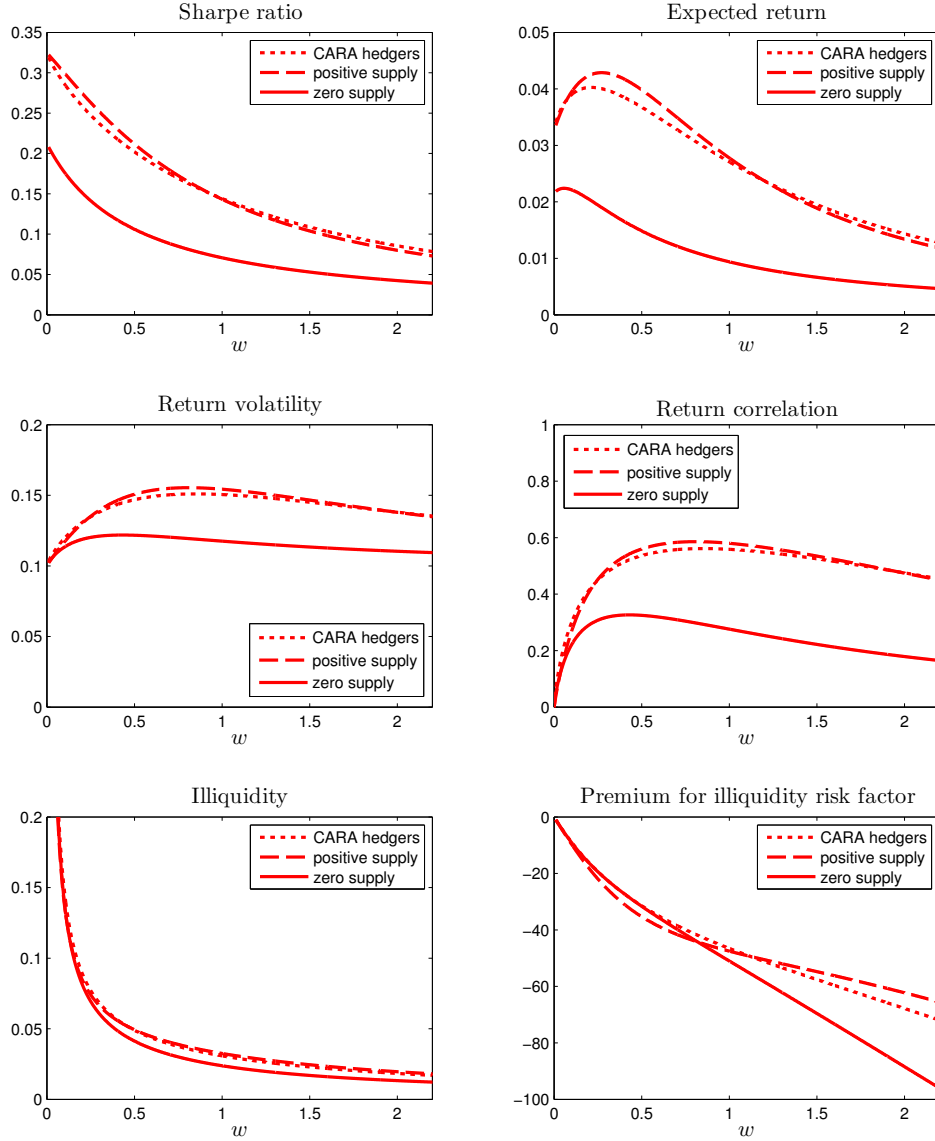


Figure 7: Assets' Sharpe ratios, expected returns, volatilities, correlations, illiquidity, and the premium of the illiquidity risk factor, as a function of arbitrageur wealth in the logarithmic case. The solid lines are drawn for the case of zero supply ($s = 0$) and mean-variance hedgers. The dashed lines are drawn for $s = \frac{u}{2}$ and mean-variance hedgers. The dotted lines are drawn for $s = \frac{u}{2}$ and infinitely lived CARA hedgers. The remaining parameter values are $\alpha = 2$, $\sqrt{u^\top \Sigma u} = 15\%$, $\rho = 4\%$, $r = 2\%$, $N = 2$ symmetric assets with independent cashflows, and $\Sigma_{11} = \Sigma_{22} = 10\%$.

asset returns. We show that aggregate liquidity is a priced risk factor in the cross-section of assets because it captures information about the arbitrageurs' portfolio. Because arbitrageurs sell a fraction of their portfolio following losses, assets that covary the most with their portfolio are those that suffer the most when liquidity decreases. Our model yields a number of additional results concerning the dynamics of risk sharing, asset prices, and arbitrageur wealth. For example,

we show that arbitrageurs may choose to provide less insurance when hedgers are more risk averse; arbitrageurs behave as momentum traders, increasing their positions following high asset returns; the stationary distribution of arbitrageur wealth can be bimodal, with wealth less likely to take intermediate values than large or very small ones; expected returns, volatilities and correlations are hump-shaped functions of wealth; and the term structure of risk premia depends on wealth, with its slope steepening (becoming more positive) when wealth increases.

Throughout our analysis we assume that the supply u coming from hedgers is constant over time. Our model can be extended to stochastic supply without loss of tractability: this can be accomplished by restricting the variance $u^\top \Sigma u$ of hedgers' endowment to remain constant over time. Such a restriction is possible because u is a $N \times 1$ vector and $u^\top \Sigma u$ is a one-dimensional statistic. The extension to stochastic supply gives our measure of illiquidity (5.1) stronger empirical content because that measure is based on changes in u . In future research, we plan to use that extension to compute measures of illiquidity commonly used in empirical work, such as those proposed by Amihud (2002) and Pastor and Stambaugh (2003), and compare them to our measure. We can also examine the behavior of the measures within our model, e.g., whether they capture mainly shocks to hedger demand or to arbitrageur wealth, compute their correlation over different frequencies, etc.

Extending our model to stochastic u can not only sharpen our measure of illiquidity but also strengthen our interpretation of arbitrageurs as specialized liquidity providers. Indeed, a common view of liquidity providers (e.g., Grossman and Miller (1988)) is that they absorb temporary imbalances between demand and supply. Such imbalances can correspond to shocks to u . In the presence of these shocks, liquidity provision becomes distinct from sharing the total risk in the economy, which also includes the supply s coming from issuers. The supply s may be absorbed by many agents in the economy, but arbitrageurs may be uniquely able to absorb shocks to u . To model this idea, we can introduce an additional class of investors who can trade only the portfolio s . Such investors can be interpreted as "indexers," and the constraint that they can trade only s may reflect an institutional friction or a lack of information to identify shocks to u . Under this extension, the result of Section 6 that s and u are symmetric in terms of their effects on prices and expected returns breaks down: shocks to u have a large effect on prices that depends strongly on arbitrageur wealth, while shocks to s can have a smaller effect that is more stable over time. Moreover, illiquidity concerns mainly the effect of shocks to u , and it is the effect of these shocks that is tied to arbitrageur wealth.

APPENDIX—Proofs

Proof of Proposition 3.1: Substituting dD_t from (2.1), we can write (3.3) as

$$dv_t = rv_t dt + x_t^\top (\bar{D} - \pi_t) dt + u^\top \bar{D} dt + (x_t + u)^\top \sigma^\top dB_t. \quad (\text{A.1})$$

Substituting dv_t from (A.1) into (2.2), we can write the hedger's maximization problem as

$$\max_{x_t} \left\{ x_t^\top (\bar{D} - \pi_t) - \frac{\alpha}{2} (x_t + u)^\top \Sigma (x_t + u) \right\}. \quad (\text{A.2})$$

The first-order condition is

$$\bar{D} - \pi_t - \alpha \Sigma (x_t + u). \quad (\text{A.3})$$

Solving for x_t , we find (3.4). ■

Proof of Proposition 3.2: The Bellman equation is

$$\rho V = \max_{\hat{c}_t, \hat{y}_t} \left\{ u(\hat{c}_t) + V_{\hat{w}} \mu_{\hat{w}t} + \frac{1}{2} V_{\hat{w}\hat{w}} \sigma_{\hat{w}t}^\top \sigma_{\hat{w}t} + V_w \mu_{wt} + \frac{1}{2} V_{ww} \sigma_{wt}^\top \sigma_{wt} + V_{\hat{w}w} \sigma_{\hat{w}t}^\top \sigma_{wt} \right\}, \quad (\text{A.4})$$

where $u(\hat{c}_t) = \frac{\hat{c}_t^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$ and $u(\hat{c}_t) = \log(\hat{c}_t)$ for $\gamma = 1$, $(\mu_{\hat{w}t}, \sigma_{\hat{w}t})$ are the drift and diffusion of the arbitrageur's own wealth \hat{w}_t , and (μ_{wt}, σ_{wt}) are the drift and diffusion of the arbitrageurs' total wealth. Substituting dD_t from (2.1), we can write (3.5) as

$$dw_t = (rw_t - c_t) dt + y_t^\top (\bar{D} - \pi_t) dt + y_t^\top \sigma^\top dB_t. \quad (\text{A.5})$$

Eq. (A.5) written for own wealth implies that

$$\mu_{\hat{w}t} = r\hat{w}_t - \hat{c}_t + \hat{y}_t^\top (\bar{D} - \pi_t), \quad (\text{A.6})$$

$$\sigma_{\hat{w}t} = \sigma \hat{y}_t, \quad (\text{A.7})$$

and written for total wealth implies that

$$\mu_{wt} = rw_t - c_t + y_t^\top (\bar{D} - \pi_t), \quad (\text{A.8})$$

$$\sigma_{wt} = \sigma y_t. \quad (\text{A.9})$$

When $\gamma \neq 1$, we substitute (3.6) and (A.6)-(A.9) into (A.4) to write it as

$$\begin{aligned} \rho q(w_t) \frac{\hat{w}_t^{1-\gamma}}{1-\gamma} = \max_{\hat{c}_t, \hat{y}_t} & \left\{ \frac{\hat{c}_t^{1-\gamma}}{1-\gamma} + q(w_t) \hat{w}_t^{-\gamma} \left(r \hat{w}_t - \hat{c}_t + \hat{y}_t^\top (\bar{D} - \pi_t) \right) - \frac{1}{2} q(w_t) \gamma \hat{w}_t^{-\gamma-1} \hat{y}_t^\top \Sigma \hat{y}_t \right. \\ & \left. + q'(w_t) \frac{\hat{w}_t^{1-\gamma}}{1-\gamma} \left(r w_t - c_t + y_t^\top (\bar{D} - \pi_t) \right) + \frac{1}{2} q''(w_t) \frac{\hat{w}_t^{1-\gamma}}{1-\gamma} y_t^\top \Sigma y_t + q'(w_t) \hat{w}_t^{-\gamma} \hat{y}_t^\top \Sigma y_t \right\}. \end{aligned} \quad (\text{A.10})$$

The first-order conditions with respect to \hat{c}_t and \hat{y}_t are (3.8) and (3.9), respectively.

When $\gamma = 1$, we substitute (3.7), (A.6), (A.7), (A.8), and (A.9) into (A.4) to write it as

$$\begin{aligned} \rho \left(\frac{1}{\rho} \log(\hat{w}_t) + q_1(w_t) \right) = \max_{\hat{c}_t, \hat{y}_t} & \left\{ \log(\hat{c}_t) + \frac{1}{\rho \hat{w}_t} \left(r \hat{w}_t - \hat{c}_t + \hat{y}_t^\top (\bar{D} - \pi_t) \right) - \frac{1}{2 \rho \hat{w}_t^2} \hat{y}_t^\top \Sigma \hat{y}_t \right. \\ & \left. + q_1'(w_t) \left(r w_t - c_t + y_t^\top (\bar{D} - \pi_t) \right) + \frac{1}{2} q_1''(w_t) y_t^\top \Sigma y_t \right\}. \end{aligned} \quad (\text{A.11})$$

The first-order conditions with respect to \hat{c}_t and \hat{y}_t are (3.8) and (3.9) for $q(w_t) = \frac{1}{\rho}$. ■

Proof of Proposition 3.3: Since in equilibrium $\hat{c}_t = c_t$ and $\hat{w}_t = w_t$, (3.8) implies that

$$c_t = q(w_t)^{-\frac{1}{\gamma}} w_t. \quad (\text{A.12})$$

Substituting (3.13) and (3.14) into (3.9) and using the definition of $A(w_t)$ from (3.11), we find

$$\hat{y}_t = \frac{\alpha \hat{w}_t}{(\alpha + A(w_t)) w_t} u. \quad (\text{A.13})$$

When $\gamma \neq 1$, we substitute (3.8), (3.13), (3.14), (A.12), and (A.13) into (A.10). The terms in \hat{w}_t cancel, and the resulting equation is

$$\begin{aligned} \rho q(w_t) = & q(w_t)^{1-\frac{1}{\gamma}} + \left(q'(w_t) + \frac{q(w_t)(1-\gamma)}{w_t} \right) \left(r w_t - q(w_t)^{-\frac{1}{\gamma}} w_t + \frac{\alpha^2 A(w_t)}{(\alpha + A(w_t))^2} u^\top \Sigma u \right) \\ & + \frac{1}{2} \left(q''(w_t) - \frac{q(w_t)\gamma(1-\gamma)}{w_t^2} + \frac{2q'(w_t)(1-\gamma)}{w_t} \right) \frac{\alpha^2}{(\alpha + A(w_t))^2} u^\top \Sigma u. \end{aligned} \quad (\text{A.14})$$

Using the definition of $A(w_t)$ and rearranging, we find (3.16).

When $\gamma = 1$, we substitute (3.8), (3.13), (3.14), (A.12), and (A.13) into (A.11), setting $q(w_t) = \frac{1}{\rho}$. (Note that this value of $q(w_t)$ solves (3.16) for $\gamma = 1$.) The terms in \hat{w}_t cancel, and the resulting

equation is

$$\begin{aligned} \rho q_1(w_t) &= \log(\rho) + \left(q_1'(w_t) + \frac{1}{\rho w_t} \right) \left(r w_t - \rho w_t + \frac{\alpha^2 A(w_t)}{(\alpha + A(w_t))^2} u^\top \Sigma u \right) \\ &\quad + \frac{1}{2} \left(q_1''(w_t) - \frac{1}{\rho w_t^2} \right) \frac{\alpha^2}{(\alpha + A(w_t))^2} u^\top \Sigma u. \end{aligned} \quad (\text{A.15})$$

Using the definition of $A(w_t)$ and rearranging, we find (3.17). ■

Lemma A.1 recalls some useful properties of the cotangent function.

Lemma A.1 *The function $x \cot(x)$ is decreasing for $x \in [0, \frac{\pi}{2}]$. Its asymptotic behavior for x close to zero is*

$$x \cot(x) = 1 - \frac{x^2}{3} + o(x^2). \quad (\text{A.16})$$

Proof: Differentiating $x \cot(x)$ with respect to x , we find

$$\begin{aligned} \frac{d}{dx} [x \cot(x)] &= \cot(x) - x [1 + \cot^2(x)] \\ &= \cot(x) \left[1 - \frac{x}{\sin(x) \cos(x)} \right] \\ &= \cot(x) \left[1 - \frac{2x}{\sin(2x)} \right]. \end{aligned} \quad (\text{A.17})$$

The function $x - \sin(x)$ is equal to zero for $x = 0$, and its derivative $1 - \cos(x)$ is positive for $x \in (0, \pi)$. Therefore, $x > \sin(x)$ for $x \in (0, \pi)$. Since, in addition, $\sin(x) > 0$ for $x \in (0, \pi)$ and $\cot(x) > 0$ for $x \in (0, \frac{\pi}{2})$, (A.17) is negative for $x \in (0, \frac{\pi}{2})$ and so $x \cot(x)$ is decreasing. Using the asymptotic behavior of $\sin(x)$ and $\cos(x)$ for x close to zero, we find

$$\cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1 - \frac{x^2}{2} + o(x^2)}{x - \frac{x^3}{6} + o(x^3)} = \frac{1}{x} \left(1 - \frac{x^2}{3} + o(x^2) \right),$$

which implies (A.16). ■

Proof of Proposition 3.4: For $\gamma = 0$ and $w_t < \bar{w}$, (3.16) becomes

$$(\rho - r)q = r q' w + \frac{1}{2} \left(q'' - \frac{2q'^2}{q} \right) \frac{\alpha^2}{\left(\alpha - \frac{q'}{q} \right)^2} u^\top \Sigma u. \quad (\text{A.18})$$

Dividing both sides by $q(w_t)$, and noting that $A(w_t) = -\frac{q'(w_t)}{q(w_t)}$ for $\gamma = 0$, we can write (A.18) as

$$\rho - r = -rAw - \frac{1}{2} (A' + A^2) \frac{\alpha^2}{(\alpha + A)^2} u^\top \Sigma u. \quad (\text{A.19})$$

Eq. (A.19) is a first-order ODE in the function $A(w_t)$. It must be solved with the boundary condition $\lim_{w_t \rightarrow 0} A(w_t) = \infty$. This is because when w_t goes to zero, arbitrageurs' position y_t in the risky assets should go to zero so that w_t remains non-negative, and y_t is given by (3.14).

In the limit when r goes to zero, (A.19) becomes

$$\begin{aligned} \rho &= -\frac{1}{2} (A' + A^2) \frac{\alpha^2}{(\alpha + A)^2} u^\top \Sigma u \\ \Leftrightarrow 1 &= - (A' + A^2) \frac{z}{(\alpha + A)^2} \end{aligned} \quad (\text{A.20})$$

$$\begin{aligned} \Leftrightarrow -\frac{zA'}{zA^2 + (\alpha + A)^2} &= 1 \\ \Leftrightarrow -\frac{\frac{zA'}{1+z}}{\left(A + \frac{\alpha}{1+z}\right)^2 + \frac{z\alpha^2}{(1+z)^2}} &= 1, \end{aligned} \quad (\text{A.21})$$

where (A.20) follows from the definition of z . Setting

$$\hat{A}(w_t) \equiv \frac{1+z}{\alpha\sqrt{z}} \left(A(w_t) + \frac{\alpha}{1+z} \right),$$

we can write (A.21) as

$$-\frac{\hat{A}'}{\hat{A}^2 + 1} = \frac{\alpha}{\sqrt{z}}. \quad (\text{A.22})$$

Eq. (A.22) integrates to

$$\operatorname{arccot}(\hat{A}(w_t)) - \operatorname{arccot}(\hat{A}(0)) = \frac{\alpha w_t}{\sqrt{z}}. \quad (\text{A.23})$$

The boundary condition $\lim_{w_t \rightarrow 0} A(w_t) = \infty$ implies $\lim_{w_t \rightarrow 0} \hat{A}(w_t) = \infty$ and hence

$$\operatorname{arccot}(\hat{A}(0)) = 0.$$

Substituting into (A.23), we find

$$\hat{A}(w_t) = \cot\left(\frac{\alpha w_t}{\sqrt{z}}\right). \quad (\text{A.24})$$

Eq. (A.24) and the definition of $\hat{A}(w_t)$ imply that $A(w_t)$ is given by (3.20) for $w_t < \bar{w}$. Since $q(w_t) = 1$ for $w_t \geq \bar{w}$, $A(w_t) = -\frac{q'(w_t)}{q(w_t)}$ implies that $A(w_t) = 0$ for $w_t \geq \bar{w}$. Smooth-pasting implies that $A(w_t)$ given by (3.20) must be equal to zero for $w_t = \bar{w}$. This yields (3.21).

To determine $q(w_t)$, we solve

$$\frac{q'}{q} = -A \quad (\text{A.25})$$

with the boundary condition $q(\bar{w}) = 1$. Eq. (A.25) integrates to

$$\begin{aligned} \log q(w_t) - \log q(\bar{w}) &= \int_{w_t}^{\bar{w}} A(w) dw \\ \Rightarrow \log q(w_t) &= \int_{w_t}^{\bar{w}} A(w) dw, \end{aligned} \quad (\text{A.26})$$

where the second step follows from the boundary condition. Substituting $A(w_t)$ from (3.20) into (A.26) and integrating, we find (3.22).

We finally show that $A(w_t)$ is decreasing and convex, converges to ∞ when w_t goes to zero, and is smaller than $\frac{1}{w_t}$. Since the right-hand side of (3.21) is positive, $\frac{\alpha \bar{w}}{\sqrt{z}} < \frac{\pi}{2}$. Since $\cot(x)$ is decreasing for $x \in (0, \frac{\pi}{2})$, (3.20) implies that $A(w_t)$ is decreasing for $w_t \in (0, \bar{w}]$. Differentiating (3.20) with respect to w_t yields

$$A'(w_t) = -\frac{\alpha^2}{1+z} \left(1 + \cot^2\left(\frac{\alpha w_t}{\sqrt{z}}\right) \right). \quad (\text{A.27})$$

Since $\cot(x)$ is positive and decreasing for $x \in (0, \frac{\pi}{2})$, $A'(w_t)$ is increasing for $w_t \in (0, \bar{w}]$. Therefore, $A(w_t)$ is convex. Since $\cot(x)$ converges to ∞ when x goes to zero, (3.20) implies that $A(w_t)$ converges to ∞ when w_t goes to zero. Since the function $x \cot(x)$ is decreasing, it is smaller than one, its limit when x goes to zero (Lemma A.1). Therefore, (3.20) implies that

$$w_t A(w_t) = \frac{\alpha w_t}{1+z} \left(\sqrt{z} \cot\left(\frac{\alpha w_t}{\sqrt{z}}\right) - 1 \right) < \frac{\alpha \sqrt{z} w_t}{1+z} \cot\left(\frac{\alpha w_t}{\sqrt{z}}\right) < \frac{z}{1+z} < 1,$$

and hence $A(w_t) < \frac{1}{w_t}$. ■

Proof of Proposition 3.5: In both the logarithmic and the limit risk-neutral cases, $A(w_t)$ is decreasing in w_t . Part (i) follows from this property, (3.1) and (3.13). Part (ii) follows from the same property and (3.14). Part (iii) follows from the same property and because (3.13) implies that the market prices of risk are given by

$$\eta_t \equiv \left(\sigma^\top\right)^{-1} (\bar{D} - \pi_t) = \frac{\alpha A(w_t)}{\alpha + A(w_t)} \sigma u. \quad (\text{A.28})$$

Part (iv) follows from the same property and (3.15). ■

Proof of Proposition 3.6: Substituting (3.13), (3.14), and (A.12) into (A.5) we can write the dynamics of arbitrageur wealth w_t as

$$dw_t = \mu_{wt} dt + \sigma_{wt}^\top dB_t, \quad (\text{A.29})$$

where

$$\mu_{wt} = \left(r - q(w_t)^{-\frac{1}{\gamma}}\right) w_t + \frac{\alpha^2 A(w_t)}{(\alpha + A(w_t))^2} u^\top \Sigma u, \quad (\text{A.30})$$

$$\sigma_{wt} = \frac{\alpha}{\alpha + A(w_t)} \sigma u. \quad (\text{A.31})$$

We first determine the stationary distribution in the limit risk-neutral case. Wealth evolves in $(0, \bar{w})$, with an upper reflecting barrier at \bar{w} . Since the consumption rate $q(w_t)^{-\frac{1}{\gamma}}$ is equal to zero in $(0, \bar{w})$ and r is equal to zero in the limit risk-neutral case, we can write the drift (A.30) as

$$\mu_{wt} = \frac{\alpha^2 A(w_t)}{(\alpha + A(w_t))^2} u^\top \Sigma u. \quad (\text{A.32})$$

If the stationary distribution has density $d(w_t)$, then $d(w_t)$ satisfies the ODE

$$-(\mu_w d)' + \frac{1}{2} (\sigma_w^T \sigma_w d)'' = 0 \quad (\text{A.33})$$

over $(0, \bar{w})$, and the boundary condition

$$-\mu_w d + \frac{1}{2} (\sigma_w^T \sigma_w d)' = 0 \quad (\text{A.34})$$

at the reflecting barrier \bar{w} . Integrating (A.33) using (A.34) yields the ODE

$$-\mu_w d + \frac{1}{2}(\sigma_w^T \sigma_w d)' = 0. \quad (\text{A.35})$$

Setting $D(w_t) \equiv \sigma_{w_t}^T \sigma_{w_t} d(w_t)$, we can write (A.35) as

$$\frac{D'}{D} = \frac{2\mu_w}{\sigma_w^T \sigma_w}. \quad (\text{A.36})$$

Eq. (A.36) integrates to

$$D(w_t) = D(\bar{w}) \exp\left(-\int_{w_t}^{\bar{w}} \frac{2\mu_w}{\sigma_w^T \sigma_w} dw\right),$$

yielding

$$d(w_t) = D(\bar{w}) \frac{\exp\left(-\int_{w_t}^{\bar{w}} \frac{2\mu_w}{\sigma_w^T \sigma_w} dw\right)}{\sigma_{w_t}^T \sigma_{w_t}}. \quad (\text{A.37})$$

We can determine the multiplicative constant $D(\bar{w})$ by the requirement that $d(w_t)$ must integrate to one, i.e.,

$$\int_0^{\bar{w}} d(w_t) dw_t = D(\bar{w}) \int_0^{\bar{w}} \frac{\exp\left(-\int_{w_t}^{\bar{w}} \frac{2\mu_w}{\sigma_w^T \sigma_w} dw\right)}{\sigma_{w_t}^T \sigma_{w_t}} dw_t = 1. \quad (\text{A.38})$$

Eq. (A.38) determines a positive $D(\bar{w})$, and hence a positive $d(w_t)$, if the integral multiplying $D(\bar{w})$ is finite. If the integral is infinite, then (A.38) implies that $D(\bar{w}) = 0$, and the stationary distribution does not have a density but is concentrated at zero. The integral multiplying $D(\bar{w})$ is infinite when the integrand converges to infinity at a fast enough rate when w_t goes to zero.

Substituting μ_{w_t} and σ_{w_t} from (A.32) and (A.31), respectively, into (A.37), we find

$$\begin{aligned} d(w_t) &= \frac{D(\bar{w})}{\alpha^2 u^\top \Sigma u} (\alpha + A(w_t))^2 \exp\left(-2 \int_{w_t}^{\bar{w}} A(w) dw\right) \\ &= \frac{D(\bar{w})}{\alpha^2 u^\top \Sigma u} \left(\frac{\alpha + A(w_t)}{q(w_t)}\right)^2, \end{aligned} \quad (\text{A.39})$$

where the second step follows from (A.26). Eqs. (A.38) and (A.39) imply (3.24). Eqs. (3.20) and (3.22), and Lemma A.1, imply that when w_t is close to zero,

$$\left(\frac{\alpha + A(w_t)}{q(w_t)}\right)^2 \approx \Gamma\left(\frac{\frac{1}{w_t}}{\exp\left(-\frac{z}{1+z} \log(w_t)\right)}\right)^2 = \Gamma\left(\frac{w_t^{\frac{z}{1+z}}}{w_t}\right)^2 = \Gamma w_t^{-\frac{2}{1+z}},$$

where Γ is a positive constant. Therefore, the integral multiplying $D(\bar{w})$ in (A.38) is finite when

$$\frac{2}{1+z} < 1 \Leftrightarrow z > 1.$$

We next determine the stationary distribution in the logarithmic case. Wealth evolves in $(0, \infty)$. Noting that $c_t = \rho w_t$ and $A(w_t) = \frac{1}{w_t}$, we can write the drift (A.30) and the diffusion (A.31) as

$$\mu_{wt} \equiv (r - \rho)w_t + \frac{\alpha^2 w_t}{(\alpha w_t + 1)^2} u^\top \Sigma u, \quad (\text{A.40})$$

$$\sigma_{wt} \equiv \frac{\alpha w_t}{\alpha w_t + 1} \sigma u, \quad (\text{A.41})$$

respectively. In the logarithmic case there is no reflecting barrier, but (A.35) still holds. Intuitively, (A.35) holds for any reflecting barrier, and the effect of a reflecting barrier on the stationary distribution converges to zero when the barrier goes to infinity. To compute the density $d(w_t)$ of the stationary distribution, we thus need to integrate (A.36). Integrating between an arbitrary value \bar{w}_0 and w_t , we find

$$d(w_t) = D(\bar{w}_0) \frac{\exp\left(-\int_{w_t}^{\bar{w}_0} \frac{2\mu_w}{\sigma_w^\top \sigma_w} dw\right)}{\sigma_{wt}^\top \sigma_{wt}}. \quad (\text{A.42})$$

We can determine the multiplicative constant $D(\bar{w}_0)$ by the requirement that $d(w_t)$ must integrate to one, i.e.,

$$\int_0^\infty d(w_t) dw_t = D(\bar{w}_0) \int_0^\infty \frac{\exp\left(-\int_{w_t}^{\bar{w}_0} \frac{2\mu_w}{\sigma_w^\top \sigma_w} dw\right)}{\sigma_{wt}^\top \sigma_{wt}} dw_t = 1. \quad (\text{A.43})$$

Substituting μ_{wt} and σ_{wt} from (A.40) and (A.41) into (A.42), we find

$$\begin{aligned} d(w_t) &= D(\bar{w}_0) \frac{\exp\left(-\int_{w_t}^{\bar{w}_0} \left(-\frac{w(\alpha + \frac{1}{w})^2}{z} + \frac{2}{w}\right) dw\right)}{\frac{\alpha^2 u^\top \Sigma u}{\left(\alpha + \frac{1}{w_t}\right)^2}} \\ &= C(\bar{w}_0) \frac{\exp\left(-\frac{1}{z} \left(\frac{1}{2} \alpha^2 w_t^2 + 2\alpha w_t + \log(w_t)\right) + 2 \log(w_t)\right)}{\frac{\alpha^2 u^\top \Sigma u w_t^2}{(\alpha w_t + 1)^2}} \\ &= \frac{C(\bar{w}_0)}{\alpha^2 u^\top \Sigma u} (\alpha w_t + 1)^2 w_t^{-\frac{1}{z}} \exp\left(-\frac{1}{2z} (\alpha^2 w_t^2 + 4\alpha w_t)\right), \end{aligned} \quad (\text{A.44})$$

where

$$C(\bar{w}_0) \equiv D(\bar{w}_0) \exp \left(\frac{1}{z} \left(\frac{1}{2} \alpha^2 \bar{w}_0^2 + 2\alpha \bar{w}_0 + \log(\bar{w}_0) \right) - 2 \log(\bar{w}_0) \right).$$

Eqs. (A.43) and (A.44) imply (3.23). If $z < 0$, then the integral multiplying $D(\bar{w}_0)$ is infinite because of the behavior of the integrand when z goes to ∞ . If $z > 0$, then the integral can be infinite because of the behavior of the integrand when w_t is close to zero. Since

$$(\alpha w_t + 1)^2 w_t^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 w_t^2 + 4\alpha w_t) \right) \approx w_t^{-\frac{1}{z}}$$

when w_t is close to zero, the integral multiplying $D(\bar{w})$ in (A.43) is finite when $z > 1$. ■

Proof of Proposition 3.7: Eq. (3.23) implies that in the logarithmic case, the derivative of $d(w_t)$ with respect to w_t has the same sign as the derivative of

$$(\alpha w_t + 1)^2 w_t^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 w_t^2 + 4\alpha w_t) \right).$$

The latter derivative is

$$\frac{1}{z} (\alpha w_t + 1) w_t^{-\frac{1}{z}-1} \exp \left(-\frac{1}{2z} (\alpha^2 w_t^2 + 4\alpha w_t) \right) [2\alpha z w_t - (\alpha w_t + 1) - (\alpha w_t + 1) \alpha w_t (\alpha w_t + 2)]$$

and has the same sign as

$$- [(\alpha w_t)^3 + 3(\alpha w_t)^2 + (3 - 2z)\alpha w_t + 1].$$

The function

$$F(x) \equiv x^3 + 3x^2 + (3 - 2z)x + 1$$

is equal to 1 for $x = 0$, and its derivative with respect to x is

$$F'(x) = 3x^2 + 6x + (3 - 2z).$$

If $z < \frac{3}{2}$, then $F'(x) > 0$ for all $x > 0$, and hence $F(x) > 0$ for all $x > 0$. If $z > \frac{3}{2}$, then $F'(x)$ has the positive root

$$x'_1 \equiv -1 + \sqrt{\frac{2z}{3}},$$

and is negative for $0 < x < x'_1$ and positive for $x > x'_1$. Therefore, if $F(x'_1) > 0$ then $F(x) > 0$ for all $x > 0$, and if $F(x'_1) < 0$ then $F(x)$ has two positive roots $x_1 < x'_1 < x_2$ and is positive outside the roots and negative inside. Since

$$F(x'_1) = \left(-1 + \sqrt{\frac{2z}{3}}\right)^3 + 3\left(-1 + \sqrt{\frac{2z}{3}}\right)^2 + (3-2z)\left(-1 + \sqrt{\frac{2z}{3}}\right) + 1 = \frac{2z}{3}\left(3 - 2\sqrt{\frac{2z}{3}}\right),$$

$F(x'_1)$ is positive if

$$3 - 2\sqrt{\frac{2z}{3}} > 0 \equiv z < \frac{27}{8},$$

and is negative if $z > \frac{27}{8}$. Therefore, if $z < \frac{27}{8}$ then the derivative of $d(w_t)$ is negative, and if $z > \frac{27}{8}$ then the derivative of $d(w_t)$ is negative for $w_t \in (0, \bar{w}_1) \cup (\bar{w}_2, \infty)$ and positive for $w_t \in (\bar{w}_1, \bar{w}_2)$, where $\bar{w}_i \equiv \frac{x_i}{\alpha}$ for $i = 1, 2$. This proves Part (i).

Eq. (3.24) implies that in the limit risk-neutral case, the derivative of $d(w_t)$ with respect to w_t has the same sign as the derivative of $\frac{\alpha + A(w_t)}{q(w_t)}$. The latter derivative is

$$\begin{aligned} \frac{d}{dw_t} \left(\frac{\alpha + A(w_t)}{q(w_t)} \right) &= \frac{A'(w_t)q(w_t) - (\alpha + A(w_t))q'(w_t)}{q(w_t)^2} \\ &= \frac{A'(w_t) + A(w_t)(\alpha + A(w_t))}{q(w_t)} \\ &= \frac{-\frac{(\alpha + A(w_t))^2}{z} + \alpha A(w_t)}{q(w_t)}, \end{aligned}$$

where the second step follows from (A.25) and the third from (A.20). Therefore, the derivative of $d(w_t)$ with respect to w_t has the same sign as

$$- [\alpha^2 + A(w_t)^2 - (z-2)\alpha A(w_t)]. \quad (\text{A.45})$$

The term in square brackets is a quadratic function of $A(w_t)$ and is always positive if

$$(z-2)^2 - 4 < 0 \Leftrightarrow z < 4.$$

Therefore, if $z < 4$, then the derivative of $d(w_t)$ is negative. If $z > 4$, then the quadratic function has two positive roots, given by

$$\frac{z-2 \pm \sqrt{z(z-4)}}{2},$$

and is positive outside the roots and negative inside. We define the thresholds \bar{w}_1 and \bar{w}_2 such that $A(\bar{w}_1)$ is equal to the smaller root and $A(\bar{w}_2)$ is equal to the larger root. Since $A(w_t)$ decreases from infinity to zero when w increases from zero to \bar{w} , the thresholds \bar{w}_1 and \bar{w}_2 are uniquely defined and satisfy $0 < \bar{w}_1 < \bar{w}_2 < \bar{w}$. Moreover, the derivative of $d(w_t)$ is negative for $w_t \in (0, \bar{w}_1) \cup (\bar{w}_2, \bar{w})$ and positive for $w_t \in (\bar{w}_1, \bar{w}_2)$. This proves Part (ii).

The density $d(w_t)$ shifts to the right in the monotone likelihood ratio sense when a parameter θ increases if

$$\frac{\partial^2 \log(d(w_t, \theta))}{\partial \theta \partial w_t} > 0. \quad (\text{A.46})$$

Using (3.23), we find that in the logarithmic case,

$$\frac{\partial \log(d(w_t))}{\partial w_t} = \frac{2\alpha}{\alpha w_t + 1} - \frac{1}{z w_t} - \frac{1}{z} (\alpha^2 w_t + 2\alpha). \quad (\text{A.47})$$

An increase in α (which also affects z) raises the right-hand side of (A.47). Therefore, $d(w_t)$ satisfies (A.46) with respect to α . An increase in z also raises the right-hand side of (A.47). Therefore, $d(w_t)$ satisfies (A.46) with respect to $u^\top \Sigma u$. Using (3.24), we find that in the limit risk-neutral case,

$$\begin{aligned} \frac{\partial \log(d(w_t))}{\partial w_t} &= \frac{2A'(w_t)}{\alpha + A(w_t)} - \frac{2q'(w_t)}{q(w_t)} \\ &= \frac{2A'(w_t)}{\alpha + A(w_t)} + 2A(w_t), \end{aligned} \quad (\text{A.48})$$

where the second step follows from (A.25). Eqs. (3.20) and (A.27) imply that

$$\frac{A'(w_t)}{\alpha + A(w_t)} = - \frac{1 + \cot^2\left(\frac{\alpha w_t}{\sqrt{z}}\right)}{\sqrt{z} \left(\cot\left(\frac{\alpha w_t}{\sqrt{z}}\right) + \sqrt{z} \right)}. \quad (\text{A.49})$$

An increase in α (which also affects z) raises the right-hand side of (A.49). Since it also raises $A(w_t)$ (Part (i) of Lemma 3.4), (A.48) implies that $d(w_t)$ satisfies (A.46) with respect to α . Differentiating (A.49) with respect to \sqrt{z} , we find

$$\frac{\partial}{\partial \sqrt{z}} \left(\frac{A'(w_t)}{\alpha + A(w_t)} \right) = \frac{\left(1 + \cot^2\left(\frac{\alpha w_t}{\sqrt{z}}\right)\right) \left[\left(2\sqrt{z} + \cot\left(\frac{\alpha w_t}{\sqrt{z}}\right)\right) \left(1 - \frac{\alpha w_t}{\sqrt{z}} \cot\left(\frac{\alpha w_t}{\sqrt{z}}\right)\right) + \frac{\alpha w_t}{\sqrt{z}} \right]}{z \left(\cot\left(\frac{\alpha w_t}{\sqrt{z}}\right) + \sqrt{z} \right)^2}.$$

Since the function $x \cot(x)$ is smaller than one (Lemma A.1), the term in square brackets is positive. Therefore, an increase in z raises the right-hand side of (A.49). Since it also raises $A(w_t)$ (Part (iii) of Lemma 3.4), (A.48) implies that $d(w_t)$ satisfies (A.46) with respect to $u^\top \Sigma u$. This proves Part (iii). \blacksquare

Lemma A.2 shows some useful properties of $A(w_t)$.

Lemma A.2 *Suppose that $\gamma = r = 0$.*

(i) *An increase in α raises $A(w_t)$.*

(ii) *An increase in α raises $\frac{A(w_t)}{\alpha}$ except when w_t is below a threshold, which is negative if $z < 1$.*

(iii) *An increase in $\frac{u^\top \Sigma u}{\rho}$ raises $A(w_t)$.*

Proof: We first prove Part (i). Differentiating (3.20) with respect to α , and noting that α also affects z , we find

$$\begin{aligned} \frac{\partial A(w_t)}{\partial \alpha} &= \frac{2\sqrt{z} \cot\left(\frac{\alpha w_t}{\sqrt{z}}\right) - 1}{1+z} - \frac{2z \left(\sqrt{z} \cot\left(\frac{\alpha w_t}{\sqrt{z}}\right) - 1\right)}{(1+z)^2} \\ &= \frac{2\sqrt{z} \cot\left(\frac{\alpha w_t}{\sqrt{z}}\right) + z - 1}{(1+z)^2}. \end{aligned} \tag{A.50}$$

(All partial derivatives with respect to α in this and subsequent proofs take into account the dependence of z on α , instead of treating z as a constant.) Since $\cot(x)$ is decreasing for $x \in (0, \frac{\pi}{2})$, the numerator in (A.50) is larger than

$$2\sqrt{z} \cot\left(\frac{\alpha \bar{w}}{\sqrt{z}}\right) + z - 1 = 1 + z > 0,$$

where the first step follows from (3.21). Therefore, an increase in α raises $A(w_t)$.

We next prove Part (ii). Using (3.20), we find

$$\frac{\partial \frac{A(w_t)}{\alpha}}{\partial \alpha} = \frac{\sqrt{z}(1-z) \cot\left(\frac{\alpha w_t}{\sqrt{z}}\right) + 2z}{\alpha(1+z)^2}. \tag{A.51}$$

The numerator in (A.51) is positive for $z < 1$. For $z > 1$, the numerator is increasing in w_t , converges to $-\infty$ when w_t goes to zero, and is equal to $1+z > 0$ for $w_t = \bar{w}$ because of (3.21).

Therefore, for $z > 1$, the numerator is negative for w_t below a threshold $\bar{w}_2 < \bar{w}$ and is positive for $w_t > \bar{w}_2$. The effect of α on $\frac{A(w_t)}{\alpha}$ is thus as in the lemma.

We finally prove Part (iii). Differentiating (3.20) with respect to $\frac{u^\top \Sigma u}{\rho}$ is equivalent to differentiating with respect to z holding α constant. We compute the derivative with respect to \sqrt{z} , which has the same sign as that with respect to z . We find

$$\frac{\partial A(w_t)}{\partial \sqrt{z}} = \alpha \frac{(1-z) \cot\left(\frac{\alpha w_t}{\sqrt{z}}\right) + 2\sqrt{z} + \frac{\alpha(1+z)w_t}{\sqrt{z}} \left(1 + \cot^2\left(\frac{\alpha w_t}{\sqrt{z}}\right)\right)}{(1+z)^2}. \quad (\text{A.52})$$

We set $\hat{w} \equiv \frac{\alpha w_t}{\sqrt{z}}$ and write the numerator in (A.52) as

$$N(\hat{w}, z) \equiv (1-z) \cot(\hat{w}) + 2\sqrt{z} + (1+z)\hat{w} (1 + \cot^2(\hat{w})).$$

The function $N(\hat{w}, z)$ is positive for $z \leq 1$. Therefore, it is positive for all $z > 0$ if its derivative with respect to z

$$\frac{\partial N(\hat{w}, z)}{\partial z} = -\cot(\hat{w}) + \frac{1}{\sqrt{z}} + \hat{w} (1 + \cot^2(\hat{w}))$$

is positive. Eq. (A.16) implies that for \hat{w} close to zero,

$$\frac{\partial N(\hat{w}, z)}{\partial z} = -\frac{1}{\hat{w}} \left(1 - \frac{\hat{w}^2}{3}\right) + \frac{1}{\sqrt{z}} + \hat{w} \left(1 + \frac{1}{\hat{w}^2} \left(1 - \frac{\hat{w}^2}{3}\right)^2\right) + o(\hat{w}) = \frac{1}{\sqrt{z}} + o(1) > 0.$$

Therefore, $\frac{\partial N(\hat{w}, z)}{\partial z}$ is positive if its derivative with respect to \hat{w}

$$\frac{\partial^2 N(\hat{w}, z)}{\partial \hat{w} \partial z} = 2(1 + \cot^2(\hat{w})) - 2\hat{w} \cot(\hat{w}) (1 + \cot^2(\hat{w})) = 2(1 - \hat{w} \cot(\hat{w})) (1 + \cot^2(\hat{w}))$$

is positive. Since the function $x \cot(x)$ is decreasing, it is smaller than one, its limit when x goes to zero (Lemma A.1). Therefore, $\frac{\partial^2 N(\hat{w}, z)}{\partial \hat{w} \partial z}$ is positive, and so is $N(\hat{w}, z)$, implying that an increase in z raises $A(w_t)$.

Proof of Proposition 3.8: The results for the logarithmic case follow from (3.14), (3.15), and $A(w_t) = \frac{1}{w_t}$. We next prove the results for the limit risk-neutral case. The result for the Sharpe ratio in Part (i) follows from (3.15) and because an increase in α raises $\frac{\alpha A(w_t)}{\alpha + A(w_t)}$. The latter property

follows from

$$\frac{\partial}{\partial \alpha} \left(\frac{\alpha A(w_t)}{\alpha + A(w_t)} \right) = \frac{A(w_t)^2 + \alpha^2 \frac{\partial A(w_t)}{\partial \alpha}}{(\alpha + A(w_t))^2}$$

and Part (i) of Lemma A.2. The result for arbitrageur positions in Part (i) follows from (3.14) and Part (ii) of Lemma A.2. The result for the Sharpe ratio in Part (ii) follows from (3.15) and Part (iii) of Lemma A.2. The result for arbitrageur positions in Part (ii) follows from (3.14) and Part (iii) of Lemma A.2. \blacksquare

Proof of Lemma 4.1: Using (2.1) and (4.1), we can write (4.4) and (4.5) as

$$dv_t = rv_t dt + X_t^\top (\mu_{S_t} + \bar{D} - rS_t) dt + u^\top \bar{D} dt + \left(X_t^\top (\sigma_{S_t} + \sigma)^\top + u^\top \sigma^\top \right) dB_t, \quad (\text{A.53})$$

$$dw_t = (rw_t - c_t) dt + Y_t^\top (\mu_{S_t} + \bar{D} - rS_t) dt + Y_t^\top (\sigma_{S_t} + \sigma)^\top dB_t, \quad (\text{A.54})$$

respectively. If S_t , X_t , and Y_t satisfy (4.6), (4.7), and (4.8), then (A.53) is identical to (A.1), and (A.54) to (A.5). Therefore, if x_t and y_t maximize the objective of hedgers and of arbitrageurs, respectively, given π_t , then the same is true for X_t and Y_t , given S_t . Moreover, if x_t and y_t satisfy the market-clearing equation (3.12), then X_t and Y_t satisfy the market-clearing equation

$$X_t + Y_t = 0 \quad (\text{A.55})$$

because of (4.7) and (4.8). Since (A.53) is identical to (A.1), and (A.54) to (A.5), the dynamics of arbitrageur wealth and the exposures of hedgers and arbitrageurs to the Brownian shocks are the same in the equilibrium (S_t, X_t, Y_t) as in (π_t, x_t, y_t) . The market prices η_t of the Brownian risks in the two equilibria are $(\sigma^\top)^{-1}(\bar{D} - \pi_t)$ and $((\sigma_{S_t} + \sigma)^\top)^{-1}(\mu_{S_t} + \bar{D} - rS_t)$, and are the same because of (4.6). The arbitrageurs' Sharpe ratios in the two equilibria are $\frac{y_t^\top (\bar{D} - \pi_t)}{\sqrt{y_t^\top \sigma^\top \sigma y_t}}$ and $\frac{Y_t^\top (\mu_{S_t} + \bar{D} - rS_t)}{\sqrt{Y_t^\top (\sigma_{S_t} + \sigma)^\top (\sigma_{S_t} + \sigma) Y_t}}$, and are the same because of (4.6) and (4.8). \blacksquare

Proof of Proposition 4.1: Setting $S_t = S(w_t)$ and combining Ito's lemma with (4.1), we find

$$\begin{aligned} \mu_{S_t} &= \mu_{w_t} S'(w_t) + \frac{1}{2} \sigma_{w_t}^\top \sigma_{w_t} S''(w_t) \\ &= \left(r - q(w_t)^{-\frac{1}{\gamma}} \right) w_t S'(w_t) + \frac{\alpha^2}{(\alpha + A(w_t))^2} u^\top \Sigma u \left(A(w_t) S'(w_t) + \frac{1}{2} S''(w_t) \right), \end{aligned} \quad (\text{A.56})$$

where the second step follows from (A.30), and

$$\begin{aligned}\sigma_{S_t} &= \sigma_{w_t} S'(w_t)^\top \\ &= \frac{\alpha}{\alpha + A(w_t)} \sigma u S'(w_t)^\top,\end{aligned}\tag{A.57}$$

where the second step follows from (A.31).

Multiplying (4.6) from the left by $(\sigma_{S_t} + \sigma)^\top$, and using (3.13), we find

$$\mu_{S_t} + \bar{D} - r S_t = \frac{\alpha A(w_t)}{\alpha + A(w_t)} (\sigma_{S_t} + \sigma)^\top \sigma u.\tag{A.58}$$

Substituting $(\mu_{S_t}, \sigma_{S_t})$ from (A.56) and (A.57) into (A.58), we find the ODE

$$\left(r - q^{-\frac{1}{\gamma}}\right) w S' + \frac{\alpha^2}{2(\alpha + A)^2} u^\top \Sigma u S'' + \bar{D} - r S = \frac{\alpha A}{\alpha + A} \Sigma u.\tag{A.59}$$

A solution $S(w_t)$ to (A.59) must be of the form (4.9). Substituting (4.9) into (A.59), we find that $g(w_t)$ solves the ODE (4.10).

Substituting μ_{S_t} from (A.56) into (4.2), and using (4.9) and (4.10), we can write expected returns as (4.11). Substituting σ_{S_t} from (A.57) into (4.3), and using (4.9), we can write the covariance matrix of returns as (4.12). ■

Proof of Proposition 4.2: We first compute $g(w_t)$ in the limit logarithmic case. Noting that $q(w_t) = \frac{1}{\rho}$ and $A(w_t) = \frac{1}{w_t}$, and taking the limit when r goes to zero, we can write (4.10) as

$$\begin{aligned}-\rho w g' + \frac{\alpha^2 w^2}{2(\alpha w + 1)^2} u^\top \Sigma u g'' &= -\frac{\alpha^2 w}{\alpha w + 1} \\ \Leftrightarrow -\frac{(\alpha w + 1)^2}{z w} g' + g'' &= -\frac{2(\alpha w + 1)}{u^\top \Sigma u w}.\end{aligned}\tag{A.60}$$

Multiplying both sides of (A.60) by the integrating factor

$$\exp\left(-\int \frac{(\alpha w + 1)^2}{z w} dw\right) = w^{-\frac{1}{z}} \exp\left(-\frac{1}{2z} (\alpha^2 w^2 + 4\alpha w)\right),$$

we find

$$\left[g' w^{-\frac{1}{z}} \exp\left(-\frac{1}{2z} (\alpha^2 w^2 + 4\alpha w)\right)\right]' = -\frac{2(\alpha w + 1)}{u^\top \Sigma u w} w^{-\frac{1}{z}} \exp\left(-\frac{1}{2z} (\alpha^2 w^2 + 4\alpha w)\right).\tag{A.61}$$

Integrating (A.61) once with the boundary condition that $g'(w_t) = 0$ remains bounded when w_t goes to ∞ , we find

$$g'(w_t)w_t^{-\frac{1}{z}} \exp\left(-\frac{1}{2z}(\alpha^2 w_t^2 + 4\alpha w_t)\right) = \frac{2}{u^\top \Sigma u} \int_{w_t}^{\infty} \left(\alpha + \frac{1}{w}\right) w^{-\frac{1}{z}} \exp\left(-\frac{1}{2z}(\alpha^2 w^2 + 4\alpha w)\right) dw,$$

which yields (4.14).

We next compute $g'(w_t)$ in the limit risk-neutral case. Noting that $q(w_t)^{-\frac{1}{\gamma}}$ is equal to zero in $(0, \bar{w})$, and taking the limit when r goes to zero, we can write (4.10) as

$$\begin{aligned} \frac{\alpha^2}{2(\alpha + A)^2} u^\top \Sigma u g'' &= -\frac{\alpha^2}{\alpha + A} \\ \Leftrightarrow g'' &= -\frac{2(\alpha + A)}{u^\top \Sigma u}. \end{aligned} \tag{A.62}$$

Integrating (A.62) once with the boundary condition $g'(\bar{w}) = 0$, we find

$$g'(w_t) = \frac{2}{u^\top \Sigma u} \int_{w_t}^{\bar{w}} (\alpha + A(w)) dw. \tag{A.63}$$

The boundary condition is implied by smooth-pasting and because $g(w_t)$ is independent of w_t for $w_t \geq \bar{w}$. Using (3.20) to compute the integral in (A.63), we find (4.15). \blacksquare

Proof of Proposition 4.3: We first show the results in the limit risk-neutral case. Eq. (4.11) implies that Part (i) holds if the function

$$K(w_t) \equiv \frac{\alpha A(w_t)}{\alpha + A(w_t)} \left[f(w_t) u^\top \Sigma u + 1 \right] \tag{A.64}$$

is increasing in w_t for $w_t < \bar{w}_a$ and decreasing for $w_t > \bar{w}_a$. The derivative of $K(w_t)$ with respect to w_t is

$$K'(w_t) = \frac{\alpha^2}{(\alpha + A(w_t))^2} \left\{ A'(w_t) \left[\frac{(\alpha - A(w_t))g'(w_t)}{\alpha + A(w_t)} u^\top \Sigma u + 1 \right] + A(w_t)g''(w_t)u^\top \Sigma u \right\} \tag{A.65}$$

$$= \frac{\alpha^2}{(\alpha + A(w_t))^2} \left\{ A'(w_t) \left[\frac{(\alpha - A(w_t))g'(w_t)}{\alpha + A(w_t)} u^\top \Sigma u + 1 \right] - 2A(w_t)(\alpha + A(w_t)) \right\}, \tag{A.66}$$

where (A.65) follows from (4.13), and (A.66) because (A.63) implies that

$$g''(w_t) = -\frac{2(\alpha + A(w_t))}{u^\top \Sigma u}. \tag{A.67}$$

Since $A'(w_t) < 0$, the term $A(w_t) - \alpha$ is positive when w_t is below the threshold \bar{w}_c defined by $A(\bar{w}_c) = \alpha$ and is negative when $w_t > \bar{w}_c$. Therefore, (A.66) implies that $K'(w_t) < 0$ for $w_t \geq \bar{w}_c$. For $w_t < \bar{w}_c$, $K'(w_t)$ has the same sign as

$$\begin{aligned} K_1(w_t) &\equiv g'(w_t)u^\top \Sigma u - \frac{\alpha + A(w_t)}{A(w_t) - \alpha} + \frac{2A(w_t)(\alpha + A(w_t))^2}{A'(w_t)(A(w_t) - \alpha)} \\ &= g'(w_t)u^\top \Sigma u - \frac{\alpha + A(w_t)}{A(w_t) - \alpha} - \frac{2A(w_t)(\alpha + A(w_t))^2}{\left(A(w_t)^2 + \frac{(\alpha + A(w_t))^2}{z}\right)(A(w_t) - \alpha)}, \end{aligned} \quad (\text{A.68})$$

where the second step follows from (A.20). The function $K_1(w_t)$ converges to ∞ when w_t goes to zero because $g'(w_t)$ and $A(w_t)$ converge to ∞ , and converges to $-\infty$ when w_t goes to \bar{w}_c from below. If, therefore, $K_1(w_t)$ is decreasing in w_t , it is positive when w_t is below a threshold $\bar{w}_a \in (0, \bar{w}_c)$ and is negative when $w_t > \bar{w}_a$. The first term in (A.68) is decreasing in w_t because (A.67) implies that $g'(w_t)$ is decreasing. The second term is increasing in w_t because $A(w_t)$ is decreasing in w_t and the function

$$x \rightarrow \frac{\alpha + x}{x - \alpha}$$

is decreasing in x for $x \in (\alpha, \infty)$. Likewise, the third term is increasing in w_t if the function

$$x \rightarrow \frac{2x(\alpha + x)^2}{\left(x^2 + \frac{(\alpha + x)^2}{z}\right)(x - \alpha)}$$

is decreasing in x for $x \in (\alpha, \infty)$. The derivative of the latter function with respect to x has the same sign as

$$\begin{aligned} &[(\alpha + x)^2 + 2x(\alpha + x)] \left(x^2 + \frac{(\alpha + x)^2}{z}\right) (x - \alpha) \\ &- x(\alpha + x)^2 \left[2 \left(x + \frac{\alpha + x}{z}\right) (x - \alpha) + \left(x^2 + \frac{(\alpha + x)^2}{z}\right)\right] \\ &= \alpha(\alpha + x) \left[x^2(\alpha - 3x) - \frac{(\alpha + x)^3}{z}\right], \end{aligned}$$

which is negative for $x \in (\alpha, \infty)$. Therefore, $K_1(w_t)$ is decreasing in w_t , and so $K(w_t)$ is increasing in w_t for $w_t < \bar{w}_a$ and decreasing for $w_t > \bar{w}_a$.

Eq. (4.12) implies that Parts (ii) and (iii) hold if $f(w_t)$ is increasing in w_t for $w_t < \bar{w}_b$ and decreasing for $w_t > \bar{w}_b$. If, in particular, such a threshold \bar{w}_b exists, it is larger than the threshold \bar{w}_a

in Part (i) because $K(w_t)$ is the product of $\frac{\alpha A(w_t)}{\alpha + A(w_t)}$, which is decreasing in w_t , times $f(w_t)u^\top \Sigma u + 1$.

The derivative of $f(w_t)$ with respect to w_t is

$$f'(w_t) = \frac{\alpha}{(\alpha + A(w_t))^2} [-A'(w_t)g'(w_t) + g''(w_t)(\alpha + A(w_t))] \quad (\text{A.69})$$

$$= -\frac{\alpha}{(\alpha + A(w_t))^2} \left[A'(w_t)g'(w_t) + \frac{2(\alpha + A(w_t))^2}{u^\top \Sigma u} \right], \quad (\text{A.70})$$

where (A.69) follows from the definition of $f(w_t)$, and (A.70) from (A.67). Since $A'(w_t) < 0$, (A.70) implies that $f'(w_t)$ has the same sign as

$$\begin{aligned} H_1(w_t) &\equiv g'(w_t)u^\top \Sigma u + \frac{2(\alpha + A(w_t))^2}{A'(w_t)} \\ &= g'(w_t)u^\top \Sigma u - \frac{2(\alpha + A(w_t))^2}{A(w_t)^2 + \frac{(\alpha + A(w_t))^2}{z}} \\ &= g'(w_t)u^\top \Sigma u - \frac{2z}{z \left[\frac{A(w_t)}{\alpha + A(w_t)} \right]^2 + 1}, \end{aligned} \quad (\text{A.71})$$

where the second step follows from (A.20). The function $H_1(w_t)$ converges to ∞ when w_t goes to zero because $g'(w_t)$ and $A(w_t)$ converge to ∞ , and converges to $-2z < 0$ when w_t goes to \bar{w} because $g'(w_t)$ and $A(w_t)$ converge to zero. If, therefore, $H_1(w_t)$ is decreasing in w_t , it is positive when w_t is below a threshold \bar{w}_b and is negative when $w_t > \bar{w}_b$. The first term in (A.71) is decreasing in w_t because $g'(w_t)$ is decreasing. The second term is increasing in w_t because $A(w_t)$ is decreasing. Therefore, $H_1(w_t)$ is decreasing in w_t , and so $f(w_t)$ is increasing in w_t for $w_t < \bar{w}_b$ and decreasing for $w_t > \bar{w}_b$.

To show Part (iv), we use (4.12) to write the correlation as

$$\text{Corr}_t(dR_{nt}, dR_{n't}) = \frac{f(w_t) [f(w_t)u^\top \Sigma u + 2] (\Sigma u)_n (\Sigma u)_{n'} + \Sigma_{nn'}}{\sqrt{\{f(w_t) [f(w_t)u^\top \Sigma u + 2] (\Sigma u)_n^2 + \Sigma_{nn}\} \{f(w_t) [f(w_t)u^\top \Sigma u + 2] (\Sigma u)_{n'}^2 + \Sigma_{n'n'}\}}}. \quad (\text{A.72})$$

Differentiating (A.72) with respect to $f(w_t)$, we find that $\text{Corr}_t(dR_{nt}, dR_{n't})$ is increasing in $f(w_t)$ if (4.16) holds and is decreasing in $f(w_t)$ if (4.16) holds in the opposite direction. Part (iv) then follows from the behavior of $f(w_t)$ shown in the proof of Parts (ii) and (iii).

To show Part (v), we use (4.9) and (A.57) to write (4.8) as

$$\begin{aligned}
\sigma y_t &= \sigma \left(I + f(w_t)uu^\top \Sigma \right) Y_t \\
&\Leftrightarrow y_t = \left(I + f(w_t)uu^\top \Sigma \right) Y_t \\
&\Leftrightarrow Y_t + f(w_t)u^\top \Sigma Y_t u = \frac{\alpha}{\alpha + A(w_t)}u,
\end{aligned} \tag{A.73}$$

where I is the $N \times N$ identity matrix, and the third step follows from (3.14). Eq. (A.73) implies that Y_t is collinear with u . Setting $Y_t = \nu u$ in (A.73), we find

$$\frac{\alpha}{\alpha + A(w_t)} = \nu + f(w_t)\nu u^\top \Sigma u \Rightarrow \nu = \frac{\alpha}{(\alpha + A(w_t))(1 + f(w_t)u^\top \Sigma u)},$$

and so

$$Y_t = \frac{\alpha}{(\alpha + A(w_t))(1 + f(w_t)u^\top \Sigma u)}u = \frac{\alpha}{\alpha + A(w_t) + \alpha g'(w_t)u^\top \Sigma u}u. \tag{A.74}$$

Part (v) follows from (A.74) and because $A(w_t)$ and $g'(w_t)$ are decreasing in w_t .

We next show the results in the limit logarithmic case. We start by determining the asymptotic behavior of $g'(w_t)$ for w_t close to zero and w_t close to ∞ . For w close to zero, the integrand in (4.14) is

$$w^{-1-\frac{1}{z}} + o\left(w^{-1-\frac{1}{z}}\right).$$

Hence, for w_t close to zero, the integral in (4.14) is

$$zw_t^{-\frac{1}{z}} + o\left(w_t^{-\frac{1}{z}}\right),$$

and (4.14) implies that

$$\lim_{w_t \rightarrow 0} g'(w_t) = \frac{2z}{u^\top \Sigma u}. \tag{A.75}$$

To determine the asymptotic behavior for w_t close to ∞ , we set $w = w_t + x$ and write the integral

in (4.14) as

$$\begin{aligned}
& \int_0^\infty \left(\alpha + \frac{1}{w_t + x} \right) (w_t + x)^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 (w_t + x)^2 + 4\alpha (w_t + x)) \right) dx \\
&= w_t^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 w_t^2 + 4\alpha w_t) \right) \\
&\times \int_0^\infty \left(\alpha + \frac{1}{w_t + x} \right) \left(1 + \frac{x}{w_t} \right)^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (2\alpha^2 w_t x + \alpha^2 x^2 + 4\alpha x) \right) dx. \tag{A.76}
\end{aligned}$$

We can further write the integral in (A.76) as

$$\int_0^\infty Q \left(x, \frac{1}{w_t} \right) \exp(-Rw_t x) dx, \tag{A.77}$$

where

$$\begin{aligned}
Q(x, y) &\equiv \left(\alpha + \frac{y}{1 + xy} \right) (1 + xy)^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 x^2 + 4\alpha x) \right), \\
R &\equiv \frac{\alpha^2}{z}.
\end{aligned}$$

Because of the term $\exp(-Rw_t x)$, the behavior of the integral (A.77) for large w_t is determined by the behavior of the function $Q(x, y)$ for (x, y) close to zero. We set

$$Q(x, y) = Q(0, 0) + \frac{\partial Q}{\partial x}(0, 0)x + \frac{\partial Q}{\partial y}(0, 0)y + \hat{Q}(x, y), \tag{A.78}$$

where $\hat{Q}(x, y)$ involves terms of order two and higher in (x, y) . Substituting (A.78) into (A.77), and integrating, we find

$$\begin{aligned}
& \int_0^\infty Q \left(x, \frac{1}{w_t} \right) \exp(-Rw_t x) dx \\
&= Q(0, 0) \frac{1}{Rw_t} + \frac{\partial Q}{\partial x}(0, 0) \frac{1}{R^2 w_t^2} + \frac{\partial Q}{\partial y}(0, 0) \frac{1}{Rw_t^2} + \int_0^\infty \hat{Q} \left(x, \frac{1}{w_t} \right) \exp(-Rw_t x) dx. \tag{A.79}
\end{aligned}$$

Since

$$\begin{aligned}
Q(0, 0) &= \alpha, \\
\frac{\partial Q}{\partial x}(0, 0) &= -\frac{2\alpha^2}{z}, \\
\frac{\partial Q}{\partial y}(0, 0) &= 1,
\end{aligned}$$

and the integral in \hat{Q} yields terms of order smaller than $\frac{1}{w_t^2}$ for large w_t , (A.79) implies that

$$\int_0^\infty Q\left(x, \frac{1}{w_t}\right) \exp(-Rw_t x) dx = \frac{z}{\alpha w_t} - \frac{z}{\alpha^2 w_t^2} + o\left(\frac{1}{w_t^2}\right). \quad (\text{A.80})$$

Substituting back into (A.76) and then back into (4.14), we find that for w_t close to ∞ ,

$$g'(w_t) = \frac{2z}{u^\top \Sigma u \alpha w_t} - \frac{2z}{u^\top \Sigma u \alpha^2 w_t^2} + o\left(\frac{1}{w_t^2}\right). \quad (\text{A.81})$$

We next show that $g'(w_t)$ is decreasing in w_t . Assume, by contradiction, that there exists \underline{w} such that $g''(\underline{w}) \geq 0$. Since $g'(w_t)$ is positive and converges to zero when w_t converges to ∞ , there exists $\bar{w} > \underline{w}$ such that $g''(\bar{w}) < 0$. Therefore, the function $g''(w_t)$ must cross the x -axis from above in $[\underline{w}, \bar{w}]$, i.e., there must exist $\hat{w} \in [\underline{w}, \bar{w}]$ such that $g''(\hat{w}) = 0$ and $g'''(\hat{w}) \leq 0$. Since $g'(w_t)$ satisfies the ODE (A.60), it also satisfies

$$-\frac{\alpha}{z} g' - \frac{\alpha w + 1}{z} g'' + \frac{d}{dw} \left(\frac{w}{\alpha w + 1} \right) g'' + \frac{w}{\alpha w + 1} g''' = 0, \quad (\text{A.82})$$

which follows from (A.60) by multiplying both sides by $\frac{w}{\alpha w + 1}$ and differentiating with respect to w . Eq. (A.82) cannot hold at \hat{w} because $g'(\hat{w}) > 0$, $g''(\hat{w}) = 0$, and $g'''(\hat{w}) \leq 0$, a contradiction. Therefore, $g''(w_t) < 0$ for all w_t .

Part (v) follows from the arguments in the limit risk-neutral case and because the functions $A(w_t) = \frac{1}{w_t}$ and $g'(w_t)$ are positive and decreasing in w_t . Part (i) also follows from the arguments in that case if the function $K(w_t)$ defined by (A.64) is decreasing in w_t in the case $z < \frac{1}{2}$, and is increasing in w_t for $w_t < \bar{w}_a$ and decreasing for $w_t > \bar{w}_a$ in the case $z > \frac{1}{2}$. Using $A(w_t) = \frac{1}{w_t}$, we can write the derivative of $K(w_t)$ with respect to w_t , given by (A.65), as

$$K'(w_t) = \frac{\alpha^2}{(\alpha w_t + 1)^2} \left[\frac{(1 - \alpha w_t) g'(w_t)}{\alpha w_t + 1} u^\top \Sigma u - 1 + w_t g''(w_t) u^\top \Sigma u \right] \quad (\text{A.83})$$

$$= \frac{\alpha^2}{(\alpha w_t + 1)^3} \left[\left(1 - \alpha w_t + \frac{(\alpha w_t + 1)^3}{z} \right) g'(w_t) u^\top \Sigma u - (2\alpha w_t + 3)(\alpha w_t + 1) \right], \quad (\text{A.84})$$

where the second step follows by substituting $g''(w_t)$ from (A.60). Eqs. (A.83) and $g'(w_t) < 0$ imply

that $K'(w_t) < 0$ for $w_t \geq \frac{1}{\alpha}$. For $w_t < \frac{1}{\alpha}$, (4.14) and (A.84) imply that $K'(w_t)$ has the same sign as

$$K_2(w_t) \equiv 2 \int_{w_t}^{\infty} \left(\alpha + \frac{1}{w} \right) w^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 w^2 + 4\alpha w) \right) dw \\ - \frac{(2\alpha w_t + 3)(\alpha w_t + 1) w_t^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 w_t^2 + 4\alpha w_t) \right)}{1 - \alpha w_t + \frac{(\alpha w_t + 1)^3}{z}}.$$

The derivative of $K_2(w_t)$ with respect to w_t is

$$K_2'(w_t) = -2 \left(\alpha + \frac{1}{w_t} \right) w_t^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 w_t^2 + 4\alpha w_t) \right) - \frac{w_t^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 w_t^2 + 4\alpha w_t) \right)}{\left(1 - \alpha w_t + \frac{(\alpha w_t + 1)^3}{z} \right)^2} \\ \times \left[\left(\alpha(4\alpha w_t + 5) - \frac{(2\alpha w_t + 3)(\alpha w_t + 1)^3}{z w_t} \right) \left(1 - \alpha w_t + \frac{(\alpha w_t + 1)^3}{z} \right) \right. \\ \left. - \alpha \left(-1 + \frac{3(\alpha w_t + 1)^2}{z} \right) (2\alpha w_t + 3)(\alpha w_t + 1) \right],$$

and has the same sign as

$$K_3(w_t) \equiv -\frac{2(\alpha^2 w_t^2 + 3\alpha w_t + 1)}{(\alpha w_t + 1)^3} + \frac{4\alpha^2 w_t^2 + 3\alpha w_t - 1}{z} + \frac{(\alpha w_t + 1)^3}{z^2}.$$

The function $K_3(w_t)$ is equal to

$$K_3(0) = -2 - \frac{1}{z} + \frac{1}{z^2} = \frac{(1 - 2z)(1 + z)}{z^2}$$

for $w_t = 0$, and is increasing in w_t because the function $\frac{\alpha^2 w_t^2 + 3\alpha w_t + 1}{(\alpha w_t + 1)^3}$ is decreasing in w_t . The

function $K_2(w_t)$ is equal to

$$K_2(w_t) = 2z w_t^{-\frac{1}{z}} - \frac{3z}{1+z} w_t^{-\frac{1}{z}} + o \left(w_t^{-\frac{1}{z}} \right) = \frac{(2z-1)z}{1+z} w_t^{-\frac{1}{z}} + o \left(w_t^{-\frac{1}{z}} \right)$$

for w_t close to zero. Moreover, $K_2 \left(\frac{1}{\alpha} \right) < 0$ because the functions $K'(w_t)$ and $K_2(w_t)$ have the same sign for $w_t \leq \frac{1}{\alpha}$ and (A.83) implies that $K' \left(\frac{1}{\alpha} \right) < 0$.

- When $z < \frac{1}{2}$, $K_3(0) > 0$ and $K_3(w_t)$ increasing in w_t imply that $K_3(w_t) > 0$. Therefore, $K_2(w_t)$ is increasing in w_t . Since $K_2 \left(\frac{1}{\alpha} \right) < 0$, $K_2(w_t)$ is negative for $w_t < \frac{1}{\alpha}$. Therefore, $K(w_t)$ is decreasing in w_t for all $w_t \in (0, \infty)$.

- When $z > \frac{1}{2}$, $K_3(0) < 0$ and $K_3(w_t)$ increasing in w_t imply that $K_3(w_t) < 0$ for $w_t \in (0, \frac{1}{\alpha})$ except possibly in an interval ending at $\frac{1}{\alpha}$. Therefore, $K_2(w_t)$ is decreasing in w_t for $w_t \in (0, \frac{1}{\alpha})$ except possibly in an interval ending at $\frac{1}{\alpha}$ where it is increasing. Since $K_2(w_t)$ is positive for w_t close to zero and $K_2(\frac{1}{\alpha}) < 0$, $K_2(w_t)$ is positive when w_t is below a threshold $\bar{w}_a \in (0, \frac{1}{\alpha})$ and negative when $w_t \in (\bar{w}_a, \frac{1}{\alpha})$. Therefore, $K(w_t)$ is increasing in w_t for $w_t \in (0, \bar{w}_a)$ and decreasing for $w_t \in (\bar{w}_a, \infty)$.

Parts (ii), (iii), and (iv) follow from the arguments in the limit risk-neutral case if the function $f(w_t)$ is increasing in w_t for $w_t < \bar{w}_b$ and decreasing for $w_t > \bar{w}_b$. Using $A(w_t) = \frac{1}{w_t}$ and substituting $g''(w_t)$ from (A.60), we can write the derivative of $f(w_t)$ with respect to w_t , given by (A.69), as

$$f'(w_t) = \frac{\alpha}{(\alpha w_t + 1)^2} \left[\left(\frac{(\alpha w_t + 1)^3}{z} + 1 \right) g'(w_t) - \frac{2(\alpha w_t + 1)^2}{u^\top \Sigma u} \right]. \quad (\text{A.85})$$

Eqs. (4.14) and (A.85) imply that $f'(w_t)$ has the same sign as

$$H_2(w_t) \equiv \int_{w_t}^{\infty} \left(\alpha + \frac{1}{w} \right) w^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 w^2 + 4\alpha w) \right) dw - \frac{(\alpha w_t + 1)^2 w_t^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 w_t^2 + 4\alpha w_t) \right)}{\frac{(\alpha w_t + 1)^3}{z} + 1}.$$

The derivative of $H_2(w_t)$ with respect to w_t is

$$\begin{aligned} H_2'(w_t) = & - \left(\alpha + \frac{1}{w_t} \right) w_t^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 w_t^2 + 4\alpha w_t) \right) - \frac{w_t^{-\frac{1}{z}} \exp \left(-\frac{1}{2z} (\alpha^2 w_t^2 + 4\alpha w_t) \right)}{\left(\frac{(\alpha w_t + 1)^3}{z} + 1 \right)^2} \\ & \times \left[\left(2\alpha(\alpha w_t + 1) - \frac{(\alpha w_t + 1)^4}{z w_t} \right) \left(\frac{(\alpha w_t + 1)^3}{z} + 1 \right) - \frac{3\alpha(\alpha w_t + 1)^4}{z} \right], \end{aligned}$$

and has the same sign as

$$H_3(w_t) \equiv -\frac{2\alpha w_t + 1}{(\alpha w_t + 1)^3} + \frac{\alpha w_t - 1}{z}.$$

The function $H_3(w_t)$ is negative for $w_t = 0$ and converges to ∞ when w_t goes to ∞ . Moreover, it is increasing in w_t because the function $\frac{2\alpha w_t + 1}{(\alpha w_t + 1)^3}$ is decreasing in w_t . Therefore, $H_3(w_t) < 0$ when w_t is below a threshold \bar{w}_d and $H_3(w_t) > 0$ when $w_t > \bar{w}_d$. For w_t close to zero, the function $H_2(w_t)$ is equal to

$$H_2(w_t) = z w_t^{-\frac{1}{z}} - \frac{z}{1+z} w_t^{-\frac{1}{z}} + o \left(w_t^{-\frac{1}{z}} \right) = \frac{z^2}{1+z} w_t^{-\frac{1}{z}} + o \left(w_t^{-\frac{1}{z}} \right),$$

and hence is positive. Moreover, $H_2(w_t)$ converges to zero when w_t goes to ∞ . Since $H_2(w_t)$ is decreasing in w_t for $w_t < \bar{w}_d$ and increasing for $w_t > \bar{w}_d$, it is positive when w_t is below a threshold $\bar{w}_b < \bar{w}_d$ and negative when $w_t > \bar{w}_b$. Therefore, $f(w_t)$ is increasing in w_t for $w_t < \bar{w}_b$ and decreasing for $w_t > \bar{w}_b$. \blacksquare

Proof of Proposition 5.1: Using (4.9), (A.55), and (A.74) to compute the partial derivatives in (5.1), we find (5.2). In the limit when r goes to zero, λ_{nt} converges to

$$\left(1 + \frac{A(w_t)}{\alpha} + g'(w_t)u^\top \Sigma u\right) \alpha \Sigma_{nn}.$$

This expression is decreasing in w_t because $A(w_t)$ is decreasing (shown in the proof of Proposition 3.4 for the limit risk-neutral case) and $g'(w_t)$ is decreasing (shown in the proof of Proposition 4.3).

Proof of Corollary 5.1: We set

$$\lambda_{nt} = \left(1 + \frac{A(w_t)}{\alpha} + g'(w_t)u^\top \Sigma u\right) (\alpha - rg(w_t)) \Sigma_{nn} \equiv L(w_t) \Sigma_{nn}. \quad (\text{A.86})$$

Using (A.86) and Ito's lemma, we find

$$\text{Cov}_t(d\Lambda_t, dR_t) = L'(w_t) \frac{\sum_{n'=1}^N \Sigma_{n'n'}}{N} \text{Cov}_t(dw_t, dR_t), \quad (\text{A.87})$$

$$\text{Cov}_t(d\Lambda_t, d\lambda_{nt}) = (L'(w_t))^2 \frac{\Sigma_{nn} \sum_{n'=1}^N \Sigma_{n'n'}}{N} \text{Var}_t(dw_t), \quad (\text{A.88})$$

$$\text{Cov}_t(d(u^\top dR_t), d\lambda_{nt}) = L'(w_t) \Sigma_{nn} u^\top \text{Cov}_t(dw_t, dR_t). \quad (\text{A.89})$$

The diffusion matrix of the return vector dR_t is

$$\begin{aligned} (\sigma_{S_t} + \sigma)^\top &= \left(\frac{\alpha}{\alpha + A(w_t)} \sigma u S'(w_t)^\top + \sigma \right)^\top \\ &= \left(\frac{\alpha g'(w_t)}{\alpha + A(w_t)} \sigma u u^\top \Sigma + \sigma \right)^\top, \end{aligned} \quad (\text{A.90})$$

where the first step follows from (A.57) and the second from (4.9). The covariance between wealth and the return vector dR_t is

$$\begin{aligned} \text{Cov}_t(dw_t, dR_t) &= (\sigma_{S_t} + \sigma)^\top \sigma_{wt} \\ &= \left(\frac{\alpha g'(w_t)}{\alpha + A(w_t)} \sigma u u^\top \Sigma + \sigma \right)^\top \frac{\alpha}{\alpha + A(w_t)} \sigma u \\ &= \frac{\alpha}{\alpha + A(w_t)} \left[f(w_t) u^\top \Sigma u + 1 \right] \Sigma u, \end{aligned} \quad (\text{A.91})$$

where the second step follows from (A.31) and (A.90). Part (i) of the corollary follows by substituting (A.91) into (A.87). The proportionality coefficient is

$$C^\Lambda(w_t) = L'(w_t) \frac{\alpha \sum_{n'=1}^N \Sigma_{n'n'}}{N(\alpha + A(w_t))} \left[f(w_t) u^\top \Sigma u + 1 \right], \quad (\text{A.92})$$

and is negative in the limit when r goes to zero because $L(w_t)$ is decreasing in w_t . Part (ii) of the corollary follows from (A.88). The proportionality coefficient is positive for any r . Part (iii) of the corollary follows by substituting (A.91) into (A.89). The proportionality coefficient is negative in the limit when r goes to zero because $L(w_t)$ is decreasing in w_t . ■

Proof of Corollary 5.2: The proportionality result follows from (4.11), (A.87), and (A.91). These equations imply that the proportionality coefficient is

$$\Pi^\Lambda(w_t) = \frac{A(w_t)}{L'(w_t) \frac{\sum_{n'=1}^N \Sigma_{n'n'}}{N}}. \quad (\text{A.93})$$

This coefficient is negative in the limit when r goes to zero because $L(w_t)$ is decreasing in w_t . ■

Proof of Proposition 5.2: In the limit when r goes to zero, (A.86) implies that $L(w_t)$ converges to

$$\left(1 + \frac{A(w_t)}{\alpha} + g'(w_t) u^\top \Sigma u \right) \alpha.$$

Substituting into (A.92) and (A.93), we find

$$C^\Lambda(w_t) = \left(\frac{A'(w_t)}{\alpha} + g''(w_t) u^\top \Sigma u \right) \frac{\alpha^2 \sum_{n'=1}^N \Sigma_{n'n'}}{N(\alpha + A(w_t))} \left[f(w_t) u^\top \Sigma u + 1 \right], \quad (\text{A.94})$$

$$\Pi^\Lambda(w_t) = \frac{A(w_t)}{\left(\frac{A'(w_t)}{\alpha} + g''(w_t) u^\top \Sigma u \right) \frac{\alpha \sum_{n'=1}^N \Sigma_{n'n'}}{N}}. \quad (\text{A.95})$$

We first show the properties of $C^\Lambda(w_t)$ and $\Pi^\Lambda(w_t)$ in the limit risk-neutral case. Using (4.13), (A.20), (A.62), and (A.63), we can write (A.94) and (A.95) as

$$C^\Lambda(w_t) = - \left[\frac{A(w_t)^2 + \frac{(\alpha + A(w_t))^2}{z}}{\alpha} + 2(\alpha + A(w_t)) \right] \frac{\alpha^2 \sum_{n'=1}^N \Sigma_{n'n'}}{N(\alpha + A(w_t))} \left[\frac{2\alpha \int_{w_t}^{\bar{w}} (\alpha + A(w)) dw}{\alpha + A(w_t)} + 1 \right], \quad (\text{A.96})$$

$$\Pi^\Lambda(w_t) = - \frac{A(w_t)}{\left[\frac{A(w_t)^2 + \frac{(\alpha + A(w_t))^2}{z}}{\alpha} + 2(\alpha + A(w_t)) \right] \frac{\alpha \sum_{n'=1}^N \Sigma_{n'n'}}{N}}. \quad (\text{A.97})$$

When w_t goes to zero, $A(w_t)$ converges to ∞ . Therefore, (A.96) implies that $C^\Lambda(w_t)$ converges to $-\infty$, and (A.97) implies that $\Pi^\Lambda(w_t)$ converges to zero. For $w_t = \bar{w}$, $A(\bar{w}) = 0$. Therefore, (A.96) implies that

$$C^\Lambda(\bar{w}) = - \left(\frac{1}{z} + 2 \right) \frac{\alpha^2 \sum_{n'=1}^N \Sigma_{n'n'}}{N} < 0$$

and (A.97) implies that $\Pi^\Lambda(\bar{w}) = 0$. To show the inverse hump shape of $\Pi^\Lambda(w_t)$, we write (A.97) as

$$\Pi^\Lambda(w_t) = - \frac{1}{\left[\frac{A(w_t)}{\alpha} \left(1 + \frac{1}{z} \right) + 2 \left(1 + \frac{1}{z} \right) + \frac{\alpha}{A(w_t)} \left(2 + \frac{1}{z} \right) \right] \frac{\alpha \sum_{n'=1}^N \Sigma_{n'n'}}{N}}. \quad (\text{A.98})$$

The term in square brackets in the denominator of (A.98) is an inverse hump-shaped function of $A(w_t)$. Since $A(w_t)$ is decreasing in w_t , $\Pi^\Lambda(w_t)$ is an inverse hump-shaped function of w_t .

We next show the properties of $C^\Lambda(w_t)$ and $\Pi^\Lambda(w_t)$ in the limit logarithmic case. Using $A(w_t) = \frac{1}{w_t}$, (4.13), and (A.60), we can write (A.94) and (A.95) as

$$C^\Lambda(w_t) = \left[-\frac{1}{\alpha w_t^2} + \frac{\alpha w_t + 1}{w_t} \left(\frac{(\alpha w_t + 1) u^\top \Sigma u}{z} g'(w_t) - 2 \right) \right] \frac{\alpha^2 \sum_{n'=1}^N \Sigma_{n'n'}}{N \left(\alpha + \frac{1}{w_t} \right)} \left[\frac{\alpha g'(w_t)}{\alpha + \frac{1}{w_t}} u^\top \Sigma u + 1 \right], \quad (\text{A.99})$$

$$\Pi^\Lambda(w_t) = \frac{\frac{1}{w_t}}{\left[-\frac{1}{\alpha w_t^2} + \frac{\alpha w_t + 1}{w_t} \left(\frac{(\alpha w_t + 1) u^\top \Sigma u}{z} g'(w_t) - 2 \right) \right] \frac{\alpha \sum_{n'=1}^N \Sigma_{n'n'}}{N}}. \quad (\text{A.100})$$

When w_t goes to zero, $g'(w_t)$ converges to the positive limit (A.75). Therefore, (A.99) implies that $C^\Lambda(w_t)$ converges to $-\infty$, and (A.100) implies that $\Pi^\Lambda(w_t)$ converges to zero. When w_t goes to ∞ , (A.81) implies that $g'(w_t)$ is of order $\frac{1}{w_t}$ and

$$\frac{(\alpha w_t + 1) u^\top \Sigma u}{z} g'(w_t) - 2 = -\frac{2}{\alpha^2 w_t^2} + o\left(\frac{1}{w_t^2}\right).$$

Therefore, (A.99) implies that $C^\Lambda(w_t)$ converges to zero, and (A.100) implies that $\Pi^\Lambda(w_t)$ converges to $-\infty$. ■

Proof of Proposition 6.1: The dynamics of arbitrageur wealth w_t under the risk-neutral measure are

$$dw_t = \left(\mu_{wt} - \sigma_{wt}^\top \eta_t \right) dt + \sigma_{wt}^\top dB_t^* \equiv \mu_{wt}^* dt + \sigma_{wt}^\top dB_t^*, \quad (\text{A.101})$$

where (μ_{wt}, σ_{wt}) are the drift and diffusion under the physical measure, η_t is the vector of market prices of risk, and B_t^* is a N -dimensional Brownian motion under the risk-neutral measure. Using (A.28), (A.30), and (A.31) to substitute for $(\eta_t, \mu_{wt}, \sigma_{wt})$, we find

$$\mu_{wt}^* = \left(r - q(w_t)^{-\frac{1}{\gamma}} \right) w_t.$$

Eq. (3.13) implies that

$$|\bar{D}_n - \pi_{n,t,t'}| = E_t^* \left(\frac{\alpha A(w_{t'})}{\alpha + A(w_{t'})} \right) |(\Sigma u)_n|. \quad (\text{A.102})$$

To determine the shape of the term structure, we thus need to derive the dynamics of $\frac{\alpha A(w_t)}{\alpha + A(w_t)}$ under the risk-neutral measure. Eq. (A.101) and Ito's lemma imply that

$$dA(w_t) = \mu_{At}^* dt + \sigma_{At}^\top dB_t^*, \quad (\text{A.103})$$

where

$$\mu_{At}^* \equiv A'(w_t) \mu_{wt}^* + \frac{1}{2} A''(w_t) \sigma_{wt}^\top \sigma_{wt},$$

$$\sigma_{At} \equiv A'(w_t) \sigma_{wt}.$$

Eq. (A.103) and Ito's lemma imply in turn that

$$d \left(\frac{\alpha A(w_t)}{\alpha + A(w_t)} \right) = -d \left(\frac{\alpha^2}{\alpha + A(w_t)} \right) = \mu_t^* dt + \sigma_t^\top dB_t^*, \quad (\text{A.104})$$

where

$$\mu_t^* \equiv \frac{\alpha^2}{(\alpha + A(w_t))^2} \mu_{At}^* - \frac{\alpha^2}{(\alpha + A(w_t))^3} \sigma_{At}^\top \sigma_{At},$$

$$\sigma_t \equiv \frac{\alpha^2}{(\alpha + A(w_t))^2} \sigma_{At}.$$

Using the definitions of $(\mu_{At}^*, \sigma_{At})$, we can write μ_t^* as

$$\mu_t^* = \frac{\alpha^2}{(\alpha + A(w_t))^2} \left[A'(w_t) \mu_{wt}^* + \left(\frac{1}{2} A''(w_t) - \frac{A'(w_t)^2}{\alpha + A(w_t)} \right) \sigma_{wt}^\top \sigma_{wt} \right]. \quad (\text{A.105})$$

Suppose that $\gamma = r = 0$. Rewriting (A.20) as

$$A' = -A^2 - \frac{(\alpha + A)^2}{z} \quad (\text{A.106})$$

and differentiating, we find

$$A'' = -2 \left(A + \frac{\alpha + A}{z} \right) A'. \quad (\text{A.107})$$

Using (A.31), (A.106), (A.107), and $\mu_{w_t}^* = 0$, we can write (A.105) as

$$\mu_t^* = \frac{\alpha^5 A(w_t)}{(\alpha + A(w_t))^5} \left(A(w_t)^2 + \frac{(\alpha + A(w_t))^2}{z} \right) u^\top \Sigma u > 0.$$

Since $\mu_t^* > 0$, (A.102) implies that the term structure is upward-sloping. When t' goes to ∞ , the term structure converges to $E^* \left(\frac{\alpha A(w_{t'})}{\alpha + A(w_{t'})} \right) |(\Sigma u)_n|$, where the expectation is under the stationary distribution that prevails under the risk-neutral measure. Retracing the calculations in Proposition 3.6, we find that the stationary distribution is concentrated at zero. Hence $A(w_{t'})$ is infinite and the limit of the term structure is as in the proposition. When w_t increases, the slope of the term structure increases because $\pi_{n,t,t} = \pi_{n,t}$ decreases while $\pi_{n,t,\infty}$ does not change.

Suppose next that $\gamma = 1$ and $z > 0$. Since $A(w_t) = \frac{1}{w_t}$ and $\mu_{w_t}^* = (r - \rho)w_t$,

$$\mu_t^* = \frac{\alpha^2}{\left(\alpha + \frac{1}{w_t} \right)^2} \left(\frac{\rho - r}{w_t} + \frac{\alpha^3}{\left(\alpha + \frac{1}{w_t} \right)^3 w_t^3} u^\top \Sigma u \right) > 0.$$

Since $\mu_t^* > 0$, the term structure slopes up. The rest of the proof is as in the case $\gamma = r = 0$. \blacksquare

Proof of Proposition 6.2: Positive supply does not change the asset demands (3.4) and (3.10) of hedgers and arbitrageurs. We write these demands in terms of the long-lived assets, using the mapping derived in Lemma 4.1. (That is, we use (4.6) to replace $\bar{D} - \pi_t$ by $\mu_{S_t} + \bar{D} - rS_t$, and (4.7) and (4.8) to replace (x_t, y_t) by (X_t, Y_t) .) Equations (3.4) and (3.10) become

$$X_t = \frac{[(\sigma_{S_t} + \sigma)^\top (\sigma_{S_t} + \sigma)]^{-1} (\mu_{S_t} + \bar{D} - rS_t)}{\alpha} - (\sigma_{S_t} + \sigma)^{-1} \sigma u, \quad (\text{A.108})$$

$$Y_t = \frac{[(\sigma_{S_t} + \sigma)^\top (\sigma_{S_t} + \sigma)]^{-1} (\mu_{S_t} + \bar{D} - rS_t)}{A(w_t)}, \quad (\text{A.109})$$

respectively. Substituting (X_t, Y_t) into the market-clearing equation

$$X_t + Y_t = s, \quad (\text{A.110})$$

we find that expected returns are

$$\mu_{S_t} + \bar{D} - rS_t = \frac{\alpha A(w_t)}{\alpha + A(w_t)} (\sigma_{S_t} + \sigma)^\top b_t, \quad (\text{A.111})$$

where

$$b_t \equiv (\sigma_{S_t} + \sigma)s + \sigma u. \quad (\text{A.112})$$

Substituting $\mu_{S_t} + \bar{D} - rS_t$ from (A.111) into (A.109), we find that the arbitrageurs' position in equilibrium is

$$Y_t = \frac{\alpha}{\alpha + A(w_t)} (\sigma_{S_t} + \sigma)^{-1} b_t. \quad (\text{A.113})$$

Substituting c_t from (A.12), $\mu_{S_t} + \bar{D} - rS_t$ from (A.111), and Y_t from (A.113) into (A.54), we find that the dynamics of arbitrageur wealth are given by (A.29) with

$$\mu_{wt} = \left(r - q(w_t)^{-\frac{1}{\gamma}} \right) w_t + \frac{\alpha^2 A(w_t)}{(\alpha + A(w_t))^2} b_t^\top b_t, \quad (\text{A.114})$$

$$\sigma_{wt} = \frac{\alpha}{\alpha + A(w_t)} b_t. \quad (\text{A.115})$$

Using (A.114) and (A.115), we find the following counterparts of (A.56) and (A.57):

$$\mu_{S_t} = \left(r - q(w_t)^{-\frac{1}{\gamma}} \right) w_t S'(w_t) + \frac{\alpha^2}{(\alpha + A(w_t))^2} b_t^\top b_t \left(A(w_t) S'(w_t) + \frac{1}{2} S''(w_t) \right), \quad (\text{A.116})$$

$$\sigma_{S_t} = \frac{\alpha}{\alpha + A(w_t)} b_t S'(w_t)^\top. \quad (\text{A.117})$$

Substituting σ_{S_t} from (A.117) into (A.112) and solving for b_t , we find

$$b_t = \frac{\sigma(s+u)}{1 - \frac{\alpha}{\alpha + A(w_t)} S'(w_t)^\top s}. \quad (\text{A.118})$$

Substituting $(\mu_{S_t}, \sigma_{S_t})$ from (A.116) and (A.117) into (A.111), we find

$$\begin{aligned} \left(r - q^{-\frac{1}{\gamma}} \right) w S' + \frac{\alpha^2}{2(\alpha + A)^2} b_t^\top b_t S'' + \bar{D} - rS &= \frac{\alpha A}{\alpha + A} \sigma^\top b_t \\ \Leftrightarrow \left(r - q^{-\frac{1}{\gamma}} \right) w S' + \frac{\alpha^2 (s+u)^\top \Sigma (s+u)}{2(\alpha + A - \alpha S'^\top s)^2} S'' + \bar{D} - rS &= \frac{\alpha A \Sigma (s+u)}{\alpha + A - \alpha S'^\top s}, \end{aligned} \quad (\text{A.119})$$

where the second step follows from (A.118). A solution $S(w_t)$ to (A.119) must be of the form (6.2). Substituting (6.2) into (A.119), we find that $g(w_t)$ solves the ODE

$$\left(r - q^{-\frac{1}{\gamma}}\right) w g' + \frac{\alpha^2 (s+u)^\top \Sigma (s+u)}{2(\alpha + A - \alpha g'(s+u)^\top \Sigma s)^2} g'' - r g = \frac{\alpha^2 (g'(s+u)^\top \Sigma s - 1)}{\alpha + A - \alpha g'(s+u)^\top \Sigma s}. \quad (\text{A.120})$$

Substituting (σ_{S_t}, b_t) from (A.117) and (A.118) into (A.111) and using (6.2), we find that expected returns $\frac{dE_t(dR_t)}{dt} = \mu_{S_t} + \bar{D} - r S_t$ are given by (6.3).

Proceeding as in the proof of Proposition 3.3 we find the following counterparts of the Bellman equations (3.16) and (3.17):

$$\begin{aligned} \rho q &= \gamma q^{1-\frac{1}{\gamma}} + \left(r - q^{-\frac{1}{\gamma}}\right) q' w + r q (1 - \gamma) \\ &\quad + \frac{1}{2} \left(q'' + \frac{2q'\gamma}{w} - \frac{2q'^2}{q} + \frac{q(1-\gamma)\gamma}{w^2} \right) \frac{\alpha^2 (s+u)^\top \Sigma (s+u)}{\left(\alpha + \frac{\gamma}{w} - \frac{q'}{q} - \alpha g'(s+u)^\top \Sigma s\right)^2}, \end{aligned} \quad (\text{A.121})$$

$$\rho q_1 = \log(\rho) + \frac{r - \rho}{\rho} + (r - \rho) q'_1 + \frac{1}{2} \left(q''_1 + \frac{2q'_1}{w} + \frac{1}{\rho w^2} \right) \frac{\alpha^2 (s+u)^\top \Sigma (s+u)}{\left(\alpha + \frac{1}{w} - \alpha g'(s+u)^\top \Sigma s\right)^2}. \quad (\text{A.122})$$

Solving for equilibrium for $\gamma \neq 1$ amounts to solving the system of (A.120) and (A.121), where A in (A.120) is set to $\frac{\gamma}{w} - \frac{q'}{q}$. Solving for equilibrium for $\gamma = 1$ amounts to solving (A.120), where A is set to $\frac{1}{w}$. ■

Proof of Proposition 6.3: We first solve the optimization problem of a hedger. The Bellman equation is

$$\bar{\rho} V = \max_{\bar{c}_t, x_t} \left\{ u(\bar{c}_t) + V_v \mu_{vt} + \frac{1}{2} V_{vv} \sigma_{vt}^\top \sigma_{vt} + V_w \mu_{wt} + \frac{1}{2} V_{ww} \sigma_{wt}^\top \sigma_{wt} + V_{vw} \sigma_{vt}^\top \sigma_{wt} \right\}, \quad (\text{A.123})$$

where $u(\bar{c}_t) = e^{-\frac{\alpha}{r} \bar{c}_t}$, (μ_{vt}, σ_{vt}) are the drift and diffusion of the hedger's wealth v_t , and (μ_{wt}, σ_{wt}) are the drift and diffusion of arbitrageur wealth. The drifts and diffusions are

$$\mu_{vt} = r v_t - \bar{c}_t + X_t^\top (\mu_{S_t} + \bar{D} - r S_t) + u^\top \bar{D}, \quad (\text{A.124})$$

$$\sigma_{vt} = (\sigma_{S_t} + \sigma) X_t + \sigma u, \quad (\text{A.125})$$

for the hedger, and

$$\mu_{wt} = r w_t - c_t + Y_t^\top (\mu_{S_t} + \bar{D} - r S_t), \quad (\text{A.126})$$

$$\sigma_{wt} = (\sigma_{S_t} + \sigma) Y_t, \quad (\text{A.127})$$

for arbitrageurs. Substituting (6.4) and (A.124)-(A.127) into (A.123), we can write the latter equation as

$$\begin{aligned}
-\bar{\rho}e^{-[\alpha v_t + F(w_t)]} &= \max_{\bar{c}_t, X_t} \left\{ -e^{-\frac{\alpha}{r}\bar{c}_t} + \alpha e^{-[\alpha v_t + F(w_t)]} \left[rv_t - \bar{c}_t + X_t^\top (\mu_{St} + \bar{D} - rS_t) + u^\top \bar{D} \right] \right. \\
&- \frac{1}{2} \alpha^2 e^{-[\alpha v_t + F(w_t)]} \left[X_t^\top (\sigma_{St} + \sigma)^\top + u^\top \sigma^\top \right] [(\sigma_{St} + \sigma)X_t + \sigma u] \\
&+ F'(w_t) e^{-[\alpha v_t + F(w_t)]} \left[rw_t - c_t + Y_t^\top (\mu_{St} + \bar{D} - rS_t) \right] \\
&+ \frac{1}{2} [F''(w_t) - F'(w_t)^2] e^{-[\alpha v_t + F(w_t)]} Y_t^\top (\sigma_{St} + \sigma)^\top (\sigma_{St} + \sigma) Y_t \\
&\left. - \alpha F'(w_t) e^{-[\alpha v_t + F(w_t)]} \left[X_t^\top (\sigma_{St} + \sigma)^\top + u^\top \sigma^\top \right] (\sigma_{St} + \sigma) Y_t \right\}. \tag{A.128}
\end{aligned}$$

The first-order conditions with respect to \bar{c}_t and X_t are

$$\frac{\alpha}{r} e^{-\frac{\alpha}{r}\bar{c}_t} = \alpha e^{-[\alpha v_t + F(w_t)]}, \tag{A.129}$$

$$\mu_{St} + \bar{D} - rS_t = \alpha (\sigma_{St} + \sigma)^\top [(\sigma_{St} + \sigma)X_t + \sigma u] + F'(w_t) (\sigma_{St} + \sigma)^\top (\sigma_{St} + \sigma) Y_t, \tag{A.130}$$

respectively. Eqs. (A.129) and (A.130) imply that

$$\bar{c}_t = rv_t + \frac{rF(w_t)}{\alpha} - \frac{r \log(r)}{\alpha}, \tag{A.131}$$

$$X_t = \frac{[(\sigma_{St} + \sigma)^\top (\sigma_{St} + \sigma)]^{-1} (\mu_{St} + \bar{D} - rS_t)}{\alpha} - (\sigma_{St} + \sigma)^{-1} \sigma u - \frac{F'(w_t)}{\alpha} Y_t, \tag{A.132}$$

respectively. Combining (A.132) with (A.55) and (A.109), we find that expected returns are

$$\mu_{St} + \bar{D} - rS_t = \frac{\alpha A(w_t)}{\alpha + A(w_t) - F'(w_t)} (\sigma_{St} + \sigma)^\top b_t, \tag{A.133}$$

and the arbitrageurs' position is

$$Y_t = \frac{\alpha}{\alpha + A(w_t) - F'(w_t)} (\sigma_{St} + \sigma)^{-1} b_t. \tag{A.134}$$

To determine asset prices, we use (A.133) and (A.134), and proceed as in Proposition 6.2. The

counterparts of (A.116)-(A.118) are

$$\mu_{St} = \left(r - q(w_t)^{-\frac{1}{\gamma}}\right) w_t S'(w_t) + \frac{\alpha^2}{(\alpha + A(w_t) - F'(w_t))^2} b_t^\top b_t \left(A(w_t) S'(w_t) + \frac{1}{2} S''(w_t)\right), \quad (\text{A.135})$$

$$\sigma_{St} = \frac{\alpha}{\alpha + A(w_t) - F'(w_t)} b_t S'(w_t)^\top, \quad (\text{A.136})$$

$$b_t = \frac{\sigma(s+u)}{1 - \frac{\alpha}{\alpha + A(w_t) - F'(w_t)} S'(w_t)^\top s}, \quad (\text{A.137})$$

respectively, and the counterparts of (A.119) and (A.120) as

$$\left(r - q^{-\frac{1}{\gamma}}\right) w S' + \frac{\alpha^2 (s+u)^\top \Sigma (s+u)}{2(\alpha + A - F' - \alpha S'^\top s)^2} S'' + \bar{D} - rS = \frac{\alpha A \Sigma (s+u)}{\alpha + A - F' - \alpha S'^\top s},$$

and

$$\left(r - q^{-\frac{1}{\gamma}}\right) w g' + \frac{\alpha^2 (s+u)^\top \Sigma (s+u)}{2(\alpha + A - F' - \alpha g'(s+u)^\top \Sigma s)^2} g'' - r g = \frac{\alpha (F' + \alpha g'(s+u)^\top \Sigma s - \alpha)}{\alpha + A - F' - \alpha g'(s+u)^\top \Sigma s}, \quad (\text{A.138})$$

respectively. Expected returns $\frac{dE_t(dR_t)}{dt} = \mu_{St} + \bar{D} - rS_t$ are given by (6.5).

Using (A.129), (A.131), and (A.132), we can write the terms in the first line of the hedger Bellman equation (A.128) as

$$\begin{aligned} & -e^{-\frac{\alpha}{r} \bar{c}_t} + \alpha e^{-[\alpha v_t + F(w_t)]} \left[r v_t - \bar{c}_t + X_t^\top (\mu_{St} + \bar{D} - rS_t) + u^\top \bar{D} \right] \\ & = -r e^{-[\alpha v_t + F(w_t)]} + \alpha e^{-[\alpha v_t + F(w_t)]} \\ & \times \left[\frac{r \log(r)}{\alpha} - \frac{r F(w_t)}{\alpha} + \alpha X_t^\top (\sigma_{St} + \sigma)^\top [(\sigma_{St} + \sigma) X_t + \sigma u] + F'(w_t) X_t^\top (\sigma_{St} + \sigma)^\top (\sigma_{St} + \sigma) Y_t \right]. \end{aligned}$$

Substituting back into (A.128), we can write that equation as

$$\begin{aligned} 0 = & \bar{\rho} - r + r \log(r) - r F(w_t) + \alpha u^\top \bar{D} + \frac{1}{2} \alpha^2 \left[X_t^\top (\sigma_{St} + \sigma)^\top - u^\top \sigma^\top \right] [(\sigma_{St} + \sigma) X_t + \sigma u] \\ & + F'(w_t) \left[r w_t - c_t + Y_t^\top (\mu_{St} + \bar{D} - rS_t) \right] \\ & + \frac{1}{2} \left[F''(w_t) - F'(w_t)^2 \right] Y_t^\top (\sigma_{St} + \sigma)^\top (\sigma_{St} + \sigma) Y_t - \alpha F'(w_t) u^\top \sigma^\top (\sigma_{St} + \sigma) Y_t. \quad (\text{A.139}) \end{aligned}$$

To further simplify (A.139), we use

$$\mu_{St} + \bar{D} - rS_t = A(w_t) (\sigma_{St} + \sigma)^\top (\sigma_{St} + \sigma) Y_t,$$

which follows from (A.109), and

$$\begin{aligned}(\sigma_{S_t} + \sigma)X_t + \sigma u &= (\sigma_{S_t} + \sigma)(s - Y_t) + \sigma u = b_t - (\sigma_{S_t} + \sigma)Y_t = \frac{A(w_t) - F'(w_t)}{\alpha}(\sigma_{S_t} + \sigma)Y_t, \\(\sigma_{S_t} + \sigma)X_t - \sigma u &= (\sigma_{S_t} + \sigma)X_t + \sigma u - 2\sigma u = \frac{A(w_t) - F'(w_t)}{\alpha}(\sigma_{S_t} + \sigma)Y_t - 2\sigma u,\end{aligned}$$

which follow from (A.55), (A.112), and (A.137). Substituting into (A.139) and simplifying, we find

$$\begin{aligned}0 &= \bar{\rho} - r + r \log(r) - rF(w_t) + \alpha u^\top \bar{D} + F'(w_t)(rw_t - c_t) \\&\quad + \frac{1}{2}(A(w_t)^2 + F''(w_t))Y_t^\top (\sigma_{S_t} + \sigma)^\top (\sigma_{S_t} + \sigma)Y_t - \alpha A(w_t)u^\top \sigma^\top (\sigma_{S_t} + \sigma)Y_t.\end{aligned}\quad (\text{A.140})$$

Substituting c_t from (A.12), Y_t from (A.134), and b_t from (A.137), we can write (A.140) as

$$\begin{aligned}0 &= \bar{\rho} - r + r \log(r) - rF + \alpha u^\top \bar{D} + \left(r - q^{-\frac{1}{\gamma}}\right) F'w \\&\quad + \frac{\alpha^2(A^2 + F'')(s+u)^\top \Sigma(s+u)}{2(\alpha + A - F' - \alpha g'(s+u)^\top \Sigma s)^2} - \frac{\alpha^2 A(s+u)^\top \Sigma u}{\alpha + A - F' - \alpha g'(s+u)^\top \Sigma s}.\end{aligned}\quad (\text{A.141})$$

Proceeding as in the proof of Proposition 3.3 we find the following counterparts of the arbitrageur Bellman equations (3.16) and (3.17):

$$\begin{aligned}\rho q &= \gamma q^{1-\frac{1}{\gamma}} + \left(r - q^{-\frac{1}{\gamma}}\right) q'w + rq(1-\gamma) \\&\quad + \frac{1}{2} \left(q'' + \frac{2q'\gamma}{w} - \frac{2q'^2}{q} + \frac{q(1-\gamma)\gamma}{w^2} \right) \frac{\alpha^2(s+u)^\top \Sigma(s+u)}{\left(\alpha + \frac{\gamma}{w} - \frac{q'}{q} - F' - \alpha g'(s+u)^\top \Sigma s\right)^2},\end{aligned}\quad (\text{A.142})$$

$$\rho q_1 = \log(\rho) + \frac{r-\rho}{\rho} + (r-\rho)q'_1 + \frac{1}{2} \left(q''_1 + \frac{2q'_1}{w} + \frac{1}{\rho w^2} \right) \frac{\alpha^2(s+u)^\top \Sigma(s+u)}{\left(\alpha + \frac{1}{w} - F' - \alpha g'(s+u)^\top \Sigma s\right)^2}.\quad (\text{A.143})$$

Solving for equilibrium for $\gamma \neq 1$ amounts to solving the system of (A.138), (A.141), and (A.142), where A in (A.138) and (A.141) is set to $\frac{\gamma}{w} - \frac{q'}{q}$. Solving for equilibrium for $\gamma = 1$ amounts to solving the system of (A.138) and (A.141), where A is set to $\frac{1}{w}$. \blacksquare

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