Variance Dynamics in Term Structure Models
Job Market Paper

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Abstract

I design a novel specification test for diagnosing the adequacy of affine term structure models to describe the observed yield variance dynamics, and derive the associated limit theory necessary for carrying out the test. The test statistic utilizes model–free estimators of instantaneous variances based on intraday data as well as model-free prices of variance swaps. Hence, it enables a direct testing of variance dynamics, independent of any specific modeling assumptions. I implement the test statistic in Eurodollar futures and options markets and find that affine term structure models cannot accommodate the yield variance dynamics observed in the data, especially during the crisis period of 2008–2010. However, a logarithmic affine specification of variances provides a remarkably improved fit.

Keywords: Affine Term Structure Models, Interest rates, Stochastic volatility

JEL: C52, C58, E43, G12, G13

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1. Introduction

Low–dimensional affine term structure models stand as a cornerstone in the term structure literature. These models provide a very successful fit to the cross–sectional properties of bond yields, see, e.g., Dai and Singleton (2002). However, their implications for yield variances are still not clear. Although these models can generate the patterns of unconditional yield volatility, see, e.g., Dai and Singleton (2003), the conditional volatilites implied by standard affine multifactor models can be uncorrelated or even negatively correlated with the time–series of conditional volatilities estimated via a GARCH approach, see, e.g., Collin–Dufresne, Goldstein, and Jones (2009), and Jacobs and Karoui (2009).

These findings may raise concerns about the presence of some sort of misspecification in affine term–structure models. This is an important problem because systematic biases caused by incorrect specification of underlying models may mislead inference about yield volatility dynamics, yield risks, and their respective pricing implied by the models. Therefore, guidance for diagnosing the presence of such misspecification may be useful particularly for applications that demand correct model implications for yield volatility dynamics.

In this paper, I design a novel specification test for diagnosing the adequacy of affine term structure models to describe the observed yield variance dynamics, and derive the associated limit theory necessary for carrying out the tests. I implement the test statistic in Eurodollar futures and options markets and explore whether the affine term–structure model class provides a satisfactory characterization of the observed yield variance dynamics. The proposed test statistic relies on two measures that are highly sensitive to variance states: model–free prices of variance swaps and model–free estimators of instantaneous volatility based on intraday data. Hence, it enables a direct testing for variance dynamics, being independent from any specific modelling assumptions.

This novel test statistic is useful in several dimensions. First, it enables a direct testing of variance implications of the affine model class without making any reference to *unspanned stochastic volatility*, i.e. a form of bond market incompleteness that variance risk can not be hedged with bonds, see, e.g., Collin–Dufresne and Goldstein (2002) and Joslin (2017). Second, one may test the variance specification at any given point in time, which can be informative about the potential sources of model failures. Third, the developed test statistic serves as a diagnostic tool for the affine specification under the pricing measure and does not hinge upon any parametric assumptions regarding the dynamics under the physical measure beyond what is implied by the no–arbitrage condition. In that aspect, it serves as a valid diagnostic tool for a large set of models including the so–called *essentially affine* class of Duffee (2002), *extended affine* classification of Cheridito, Filipović, and Kimmel (2007) and the models with a *non–affine* drift under the physical measure, see, e.g., Duarte (2003)[1]. Fourth, no specific time series model for the conditional variances is required under this approach. GARCH–type models are often employed in the literature to judge the feasibility of the model–implied variances, see, e.g., Collin–Dufresne, Goldstein, and Jones (2009), Jacobs and Karoui (2009). Instead, the proposed test statistic relies on non–parametric measures of spot volatility.

One approach to mitigate the harmful effects of misspecification of variances is to incorporate datasets that provide valuable information about variances in the estimation. Along this line, several recent studies explore potential benefits of incorporating the interest rate derivatives data in model estimations. For instance, Almeida, Graveline, and Joslin (2011) document that relying on derivatives data together with underlying interest rates in estimation improves the precision of the risk neutral parameter estimates of the underlying affine models. However, they find that even when estimated with derivatives data, standard three–factor affine models fail to match the cross–section of conditional yield volatilities. Cieslak and Povala (2016) employ realized covariances of zero–coupon rates as well as the options data in their proposed affine model's estimation. They document an improved fit of the affine model to the conditional yield variances and the cross–section of yields with maturities above two years but with the cost of extra

[1]See Piazzesi (2010) for a review on nonlinearities in term structure literature.

state variables.

Another strand of the literature along the lines of stochastic volatility in interest rate markets investigates whether the term structure of yields is related to the yield variances. An implication of affine term structure models is that the states driving the yield variances are a linear combination of yields. Collin–Dufresne and Goldstein (2002) document low explanatory power of swap rates in explaining returns on at–the–money straddles, which are particularly sensitive to the volatility risk. Accordingly, they propose a sub–family of the affine term structure models and term it as *unspanned stochastic volatility* models. However, there is mixed evidence on the empirical performance of these models for capturing the cross–sectional and time–series properties of yields, see, e.g., Bikbov and Chernov (2009), Collin–Dufresne, Goldstein, and Jones (2009) and Joslin (2017).

I highlight one fundamental variance implication of the arbitrage–free affine term structure models that earlier literature has not focused on: the instantenous variances of the discount rates are linearly spanned by the contamporaneous term–structure of variance swap rates. This relation is the focus of the testing mechanisms devised in this paper. Intuitively, any natural testing mechanism of this property requires relating the measures of instantenous variances to the cross–section of variance swap rates. However, both the instantenous variances and the variance swap rates are unobserved for discount rates.

Testing mechanism designed in this paper requires two crucial steps. The first one is the model–free recovery of the volatility realizations. This paper relies on the localized versions of the realized variance estimators based on intraday data for inferring instantaneous variances. In particular, for each day in the sample, I form model–free estimators of the spot volatility by using 10–minute observations on implied discount rates. The second step is the construction of the model–free measure of variance swap rates on implied discount rates. Building on previous work of Neuberger (1994), Demeterfi, Derman, Kamal, and Zou (1999), Carr and Wu (2009), and Mele and Obayashi (2013), I show that variance swap rates on implied discount rates can be synthesized using a portfolio of put and call options in a model–free way. Specifically, I provide the contract designs of such variance swaps and the derivation of model–free prices of these variance swaps with a portfolio of put and call options. In that context, *model–free* refers to the circumstance that the theoretical value of the variance swap rates can be replicated by a portfolio of European style options, without hinging upon any parametric assumptions. Based on these results, I approximate the variance swap rates on implied discount rates each day by using a panel of daily option prices.

I perform diagnostic tests for the popular three–factor affine term structure models for Eurodollar futures and options data. More specifically, I examine the variance implications of three–factor models with one and two stochastic volatility factors, denoted by $A_1(3)$ and $A_2(3)$ respectively.[2] The results from testing the variance dynamics of the model with one stochastic volatility factor ($A_1(3)$) are striking. For the 6–month maturity variance, the test statistic implies large violations of the affine specification, especially during the crisis period of 2008 to 2010. Moreover, the affine model struggles occasionally with capturing the variance dynamics before 2007. Although the afffine model with two stochastic volatility factors ($A_2(3)$) performs better especially before 2007, it is strongly rejected during the crisis period.

Given the failure of the affine specification with one and two volatility factors, I explore whether an affine specification of logarithmic variances is supported by the data. There is substantial empirical evidence in the equity and foreign exchange literature that the distributions of the logarithms of daily realized variances are approximately Gaussian, see for example Andersen, Bollerslev, Diebold, and Labys (2000, 2001, 2003). Accordingly, Andersen, Bollerslev, and Diebold (2007) study volatility forecasting via modelling the logarithms of realized variances and find substantial improvements in the forecasting perfomance (see, also, Andersen, Bollerslev, and Meddahi (2005) for volatility forecast evaluations). These results regarding equity and foreign exchange markets pro-

[2]The $A_1(3)$ and $A_2(3)$ models refer to the affine–model classifications where the covariance of the entire state–space system are driven solely by one state and two states respectively.

vide a foundation for exploring the affine specification on logarithmic variances. I design a test statistic to judge the logaritmic affine specification of variances and provide the associated limit theory. I find that the logarithmic affine specification with two factors provides a remarkably improved fit for the variance dynamics of implied discount rates at several maturities including the short end of the yield curve. In stark contrast to the case of affine variance specification, the test statistic exhibits much more moderate values consistently throughout the sample, including the crisis period.

The findings of this paper indicate the necessity of further extensions to the volatility modelling in affine term structure models. Such extensions may be especially important for applications that demand correct model implications for yield volatility dynamics, such as interest rate volatility hedging, and yield volatility forecasting. Broadly, presence of misspecification in the underlying models may lead to biases in estimation and may distort the model-implied inference about volatility risks, yield risks, and their respective pricing. Therefore, preventing potential biases caused by misspecification may offer a fresh perspective on bond risk premia as well as variance risk premia dynamics.

The remainder of this paper is organized as follows. Section 2 introduces the class of affine term–structure models explored in this paper. Moreover, this section clarifies the links between Eurodollar futures variances and variance swap rates in the context of affine term structure models. Section 3 provides the methodologies underlying the model–free construction of spot variances and the variance swap rates together with the associated theory. In Section 4, I develop the diagnostic tests for both the affine and logarithmic affine variance specifications and derive the limiting theory. Section 5 describes the Eurodollar futures and options data set, and outlines the calculations of the instantanous variances and the variance swap rates based on the intraday and the options data. Section 6 presents the main empirical results. Section 7 concludes.

2. The Model

2.1. The Affine Model Specification

I consider the class of term structure models where the spot interest rate, r_t, of a hypothetical AA quality bond is an affine function of the N–dimensional Markov state variable, X_t, that follows an affine diffusion process under the risk–neutral measure. Following Duffie and Kan (1996) and Dai and Singleton (2000), the state variable, X_t, solves the stochastic differential equation (SDE),

$$dX_t = \mathcal{K}\left(\Theta - X_t\right)dt + \Sigma\sqrt{S_t}dW_t^{\mathbb{Q}}, \tag{2.1}$$

where $W_t^{\mathbb{Q}}$ is an N–dimensional vector of independent Brownian Motions under the risk–neutral measure $\mathbb{Q}$, $\mathcal{K}$ and Σ are $N\times N$ matrices, Θ is an $N\times 1$ vector and S_t is a diagonal $N \times N$ matrix with diagonal components given by $[S_t]_{ii} = \alpha_i + \beta_i^\top X_t$, with α_i being a scalar and β_i an $N \times 1$ vector.

The short rate is an affine function of X such that

$$r_t = \delta_0 + \sum_{i=1}^{N} \delta_i X_t^i = \delta_0 + \delta_x^\top X_t, \tag{2.2}$$

where δ_x is an N–dimensional vector.

Given the risk–neutral dynamics of X and the specification of the short rate, the time t price of a zero–coupon bond with maturity at $t+\tau$, with τ being measured in years, is given by

$$P_t(\tau) = \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+\tau} r_s ds}\right]. \tag{2.3}$$

Following Dai and Singleton (2000), I impose the following parameter constraints to guarantee admissibility for the model specification in Equation (2.1) under $\mathbb{Q}$:

1. $\sum_{j=1}^{M} \mathcal{K}_{i,j}\Theta_j > 0, \qquad 1 \leq i \leq M,$
2. $\mathcal{K}_{i,j} = 0, \qquad 1 \leq i \leq M,\ M+1 \leq j \leq N,$
3. $\mathcal{K}_{i,j} \leq 0, \qquad 1 \leq i \neq j \leq M,$

4. $S_t^{ii} = X_t^i, \qquad 1 \le i \le M,$

5. $S_t^{jj} = \alpha_j + \Sigma_{k=1}^{M}[\beta_j]_k X_t^k, \qquad M+1 \le j \le N,$

6. $\Sigma_{i,j} = 0, \qquad 1 \le i \le M, \; M+1 \le j \le N,$

where $\alpha_j \ge 0$, $[\beta_j]_k \ge 0$.

The first three constraints guarantee that the drift of the factors driving the volatility is positive and that the non–volatility factors do not feed into the drift of the volatility factors. Moreover, they guarantee positive feedbacks among volatility factors. The last three conditions ensure the positive semi–definiteness of the covariance matrix of the process. In particular, in this specification, the covariance matrix of N–factors would be determined solely by the M volatility factors, which are autonomous of the $N - M$ conditionally Gaussian states. Hence, under the admissibility conditions (1 to 6), the covariance matrix of the N–factors is affine in the M variance states and is given by:

$$\begin{aligned} \Sigma S_t \Sigma^\top &= \Sigma Diag(\alpha)\Sigma^\top + \sum_{i=1}^{M} \Sigma Diag(\Gamma_{row,i})\Sigma^\top X_t^i \\ &\equiv G_0 + \sum_{i=1}^{M} G_1^i X_t^i. \end{aligned} \tag{2.4}$$

In the rest of the paper, even though I provide the theoretical results for a general $A_M(N)$ model where M states drive the volatility, I will use $A_1(3)$ as an illustrative model where $M = 1$ factor drives the volatility.

Within this setting, one can obtain the zero–coupon bond prices in closed form. Specifically, Duffie and Kan (1996) show that the time–t price of a zero–coupon bond with maturity τ is given by

$$P_t(\tau) = e^{A(\tau)+B(\tau)^\top X_t}, \tag{2.5}$$

where the loadings $A(\tau) \in \mathbb{R}$ and $B(\tau) \in \mathbb{R}^N$ satisfy the Riccati ODEs,

$$\begin{aligned} A^{'}(\tau) &= -\delta_0 + \Theta^\top \mathcal{K}^\top B(\tau) + \frac{1}{2}\sum_{i=1}^{N} \left[B(\tau)^\top \Sigma\right]_i^2 \alpha_i, \qquad (2.6) \\ B^{'}(\tau) &= -\delta_x - \mathcal{K}^\top \beta(\tau) + \frac{1}{2}\sum_{i=1}^{N} \left[B(\tau)^\top \Sigma\right]_i^2 \beta_i, \end{aligned}$$

with initial conditions $A(0) = 0$ and $B(0) = 0$.

Next, I will provide the pricing of the Eurodollar futures contracts.

2.2. Eurodollar futures contracts

Eurodollar futures contracts at time t with maturity τ_f are quoted as

$$F_t^{ED}(\tau_f) = 100\left(1 - f_t^{ED}(\tau_f)\right), \tag{2.7}$$

where $f_t^{ED}(\tau_f)$ is the futures rate for three–month LIBOR rate.

Eurodollar futures are cash settled contracts with final settlement prices being tied to the three–month London Interbank Offer Rate (LIBOR). In particular, the terminal price, i.e., the price at maturity, of a three–month Eurodollar futures contract with time to maturity τ_f is given by

$$F_{t+\tau_f}^{ED}(0) = 100\left(1 - L_{t+\tau_f}(\tau_L)\right), \tag{2.8}$$

where $\tau_L = 90/360$ and $L_{t+\tau_f}(\tau_L)$ is the three–month LIBOR rate at time $t + \tau_f$.[3] The LIBOR is a simple rate and is linked to the zero–coupon bond price,

$$1 + \tau_L L_t(\tau_L) = \frac{1}{P_t(\tau_L)}. \tag{2.9}$$

Consequently, the three–month LIBOR rate at time $t + \tau_f$ can be written as

$$L_{t+\tau_f}(\tau_L) = \frac{1}{\tau_L}\left(\frac{1}{P_{t+\tau_f}(\tau_L)} - 1\right). \tag{2.10}$$

Under the risk–neutral measure, $\mathbb{Q}$, the time t price of a Eurodollar futures contract with time to maturity τ_f follows a martingale process and can be obtained by taking the conditional expectation of its delivery prices under $\mathbb{Q}$,

$$F_t^{ED}(\tau_f) = \mathrm{E}_t^{\mathbb{Q}}\left[100\left(1 - L_{t+\tau_f}(\tau_L)\right)\right]. \tag{2.11}$$

[3]The LIBOR is quoted in annualized terms and the day count convention is 360 days for the USD.

By using the link between the LIBOR and zero–coupon bond prices in Equation (2.10) and the exponential affine form of zero–coupon bond prices in Equation (2.5), $f_t^{ED}(\tau_f)$ is given by

$$f_t^{ED}(\tau_f) \quad = \quad \frac{1}{\tau_L} e^{-A(\tau_L)} \mathrm{E}_t^{\mathbb{Q}} \left[e^{-B(\tau_L)^\top X_{t+\tau_f}} \right] - \frac{1}{\tau_L} \tag{2.12}$$

The risk–neutral expectation in Eq. (2.12) can be obtained by using the conditional moment generating function. Duffie, Pan, and Singleton (2000), henceforth DPS, derive closed–form expressions for a class of transforms of affine jump–diffusion processes. One implication of their analysis is that the conditional moment generating function of $X_t + \tau_f$ given X_t can be obtained in closed form within the affine setting of the short rate, Equation (2.2), and the affine diffusion dynamics of X_t under $\mathbb{Q}$, Equation (2.1). In particular, by setting $\delta_0 = 0$, $\delta_x = 0_{N\times 1}$, a closed–form expression for the conditional moment generating function of $X_t + \tau_f$ given X_t can be obtained by Equations (2.4-2.6) in DPS. Accordingly, the future LIBOR rate takes the form

$$f_t^{ED}(\tau_f) = \frac{1}{\tau_L} e^{-A(\tau_L) + A^f(\tau_f) + B^f(\tau_f)^\top X_t} - \frac{1}{\tau_L}, \tag{2.13}$$

where $A^f(\tau_f)$ and $B^f(\tau_f)$ solve the same Riccati ODEs as in Equation (2.6) with $\delta_0 = 0$, $\delta_x = 0_{N\times 1}$, and initial conditions $A^f(0) = 0$ and $B^f(0) = -B(\tau_L)$.

Tranforming the futures rate, $f_t^{ED}(\tau_f)$ in Equation (2.13), into its equivalent three–month discount rate leads to an exponential linear form in state variables. Specifically, the time t implied three–month gross rate is given by

$$\begin{aligned} \Psi_t^{\tau_f} \quad &= \quad e^{-A(\tau_L) + A^f(\tau_f) + B^f(\tau_f)^\top X_t} \\ &= \quad 1 + \tau_L f_t^{ED}(\tau_f). \end{aligned} \tag{2.14}$$

An application of Itô's lemma reveals the dynamics of $\Psi_t^{\tau_f, \tau_L}$,

$$\frac{d\Psi_t^{\tau_f}}{\Psi_t^{\tau_f}} = \mu_\Psi \left(X_t, \tau_f \right) dt + \sigma_\Psi \left(X_t, \tau_f \right) dW_t^{\mathbb{Q}}, \tag{2.15}$$

with the instantaneous variance

$$\begin{aligned} V_t^{\Psi^{\tau_f}} &:= \sigma_\Psi\left(X_t,\tau_f\right)\sigma_\Psi\left(X_t,\tau_f\right)^\top \\ &= B^f(\tau_f)^\top\left(G_0+\sum_{i=1}^{M}G_1^iX_t^i\right)B^f(\tau_f) \\ &= B^f(\tau_f)^\top G_0B^f(\tau_f)+\sum_{i=1}^{M}B^f(\tau_f)^\top G_1^iB^f(\tau_f)X_t^i \\ &\equiv \Phi_0^{\tau_f}+\Phi^{\tau_f\top}X_t^G, \end{aligned} \tag{2.16}$$

where $\Phi_0^{\tau_f}\equiv B^f(\tau_f)^\top G_0B^f(\tau_f)$ and $\Phi^{\tau_f}=\left(\Phi_1^{\tau_f},...,\Phi_M^{\tau_f}\right)^\top$ with $\Phi_i\equiv B^f(\tau_f)^\top G_1^iB^f(\tau_f)$. Since S_t is affine in the variance state vector, $X_t^G\equiv(X_t^1,X_t^2,...,X_t^M)'$, the instantanenous variance, $V_t^{\Psi_t^{\tau_f}}$, is also affine in X_t^G. This result establishes the fundamental link between the M variance states X_t^G and the cross section of instantenous variances of implied discount rates. Virtually in all low dimensional affine term structure models of the form $A_M(N)$ (see the canonical forms in Collin–Dufresne, Goldstein, and Jones (2009) and Joslin (2017)), the time variation in the instantenous volatility of the implied discount rates is driven solely by the M variance states X^G.

In the rest of the paper, implied three–month gross rate, $\Psi_t^{\tau_f}$, will be the main quantity of interest, rather than the Eurodollar futures prices, $F_t^{ED}(\tau_f)$. The reason for this is that given the affine model specification under the risk neutral measure in Equations (2.1-2.2), $\Psi_t^{\tau_f}$ is an exponential affine function of the underlying states, see Equation (2.14). Hence, $\frac{d\Psi_t^{\tau_f}}{\Psi_t^{\tau_f}}$ has spot variance dynamics that are affine only in the volatility states $X^G=(X^1,X^2,...,X^M)'$.

2.3. The Conditional Mean of $\int_t^{t+\tau_v}V_u^{\Psi^{\tau_f}}du$

I now analyze the implications of the underlying $A_M(N)$ model for the first moment of integrated variance to enable testing based only on volatility sensitive quantities. In order to derive the conditional expectation of integrated variance, it is useful to first start with the conditional expectation of X_u^G. The conditional expectation of X_u^G is given by

$$\mathrm{E}^{\mathbb{Q}}\left[X_u^G|\mathcal{F}_t\right]=\Theta^G\left(I-e^{-\mathcal{K}^G(u-t)}\right)+e^{-\mathcal{K}^G(u-t)}X_t^G. \tag{2.17}$$

Now, using the result above, the risk neutral conditional mean of integrated variance over $[t, t + \tau_v]$ for a **general $\mathbf{A_M(N)}$** model is given by

$$\mathrm{E}^{\mathbb{Q}}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du \,\middle|\, \mathcal{F}_t\right] = \Lambda_0^{\tau_f,\tau_v} + \Lambda_1^{\tau_f,\tau_v\top} X_t^G, \tag{2.18}$$

with

$$\Lambda_0^{\tau_f,\tau_v} = \left(\Phi_0^{\tau_f} + \Phi^{\tau_f\top}\Theta^G\right)\tau_v - \Phi^{\tau_f\top}\left(\mathcal{K}^{G^{-1}} - \mathcal{K}^{G^{-1}}e^{-\mathcal{K}^G\tau_v}\right)\Theta^G,$$
$$\Lambda_1^{\tau_f,\tau_v} = \Phi^{\tau_f\top}\left(\mathcal{K}^{G^{-1}} - \mathcal{K}^{G^{-1}}e^{-\mathcal{K}^G\tau_v}\right)^\top.$$

A detailed proof is provided in Appendix A.

Hence, the conditional expectation of the integrated variance is tied to the contemporaneous variance states via an affine mapping. In all $A_M(N)$ style models, the time variation in the risk neutral expectation of the integrated variance is solely determined by the time variation of the M variance states. This feature is in common with the spot variance $V_t^{\Psi^{\tau_f}}$, see Equation (2.16).

Note that the conditional expectation of the integrated variance given in Equation (2.18) is essentially equal to the variance swap rate on the implied discount rate, $\Psi_t^{\tau_f}$, denoted henceforth by $SW_t^{\tau_v}$. A variance swap is a forward contract on the future integrated variance, with a payoff at expiration given by the integrated variance over the contract horizon minus the variance swap rate. Variance swaps do not require a payment to enter, consequently they represent the risk neutral expected value of future integrated volatility. In Section 3.2, I show that the variance swap rate on the implied discount rate $\Psi_t^{\tau_f}$ can be obtained in a model–free fashion by using out–of–the money put and call options on $\Psi_t^{\tau_f}$.

Several studies in the term structure literature focus on low–dimensional affine term structure models where a single factor drives the yield variation, such as $A_1(3)$ and $A_1(4)$. It has been documented that these models, when estimated with bond yields as well as with bond derivaties, capture the time variation in yield volatility reasonably well and the price dynamics in both fixed income and fixed income derivatives, see, e.g., Almeida, Graveline, and Joslin (2011).[4] Hence, I close this section by illustrating the two key quantities for the $A_1(3)$ model, $V_t^{\Psi^{\tau_f}}$ and $SW_t^{\tau_v}$, that enable the testing of variance specification.

$\mathbf{A_1(3)}$ Model: In this canonical form, one of the state variables, here X^1, determines the spot volatility of all three state variables. Under the admissibility conditions, the $A_1(3)$ specification leads to Equation (2.16) being an affine function of X^1 only. In particular,

- The instantaneous variance, $V_t^{\Psi^{\tau_f}}$ is given by

$$V_t^{\Psi^{\tau_f}} = \Phi_0^{\tau_f} + \Phi^{\tau_f} X_t^1.$$

- The integrated variance is given by

$$SW_t^{\tau_v} = \mathrm{E}^{\mathbb{Q}}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du \,\middle|\, \mathcal{F}_t\right] = \Lambda_0^{\tau_f,\tau_v} + \Lambda_1^{\tau_f,\tau_v} X_t^1, \tag{2.19}$$

where

$$\Lambda_0^{\tau_f,\tau_v} = \Phi_0^{\tau_f}\tau_v + \Phi_1^{\tau_f}\left(\theta_1\tau_v - \frac{\theta_1}{\kappa_{11}}\left(1 - e^{-\kappa_{11}\tau_v}\right)\right),$$

and

$$\Lambda_1^{\tau_f,\tau_v} = \frac{\Phi_1^{\tau_f}}{\kappa_{11}}\left(1 - e^{-\kappa_{11}\tau_v}\right).$$

Hence, under the $A_1(3)$ specification, both the instantenous variance $V_t^{\Psi^{\tau_f}}$ and the variance swap rate $SW_t^{\tau_v}$ are determined by the factor X_t^1, which is the only factor driving the covariance of all three factors.

2.4. Affine Variance Spanning Condition

The analytical solutions for the variance swap rate in Equation (2.18) and for the instantenous variance in Equation (2.16) set the baseline for a variance spanning condition implied by the $A_M(N)$ model. In this section, I derive this variance spanning condition.

A variance swap at time t with maturity $t + \tau_v$ is a contract with payoff at expiration given by the integrated variance over the horizon of the contract minus the variance swap

[4] See also Collin–Dufresne, Goldstein, and Jones (2009), Bikbov and Chernov (2009) among others.

rate. In principle, a variance swap contract can be designed for any maturity τ_v, which yields a term–structure of variance swap rates. The result in Equation (2.18) establishes the fundamental link between the term–structure of the variance swap rates and the variance states X^G.

The $M \times 1$ vector of variance swap rates is denoted by $VS_t = (SW_t^{\tau_1}, SW_t^{\tau_2}, ..., SW_t^{\tau_M})'$. Defining the $M \times 1$ vector $\Xi_0 = (\Lambda_0^{\tau_1}, \Lambda_0^{\tau_2}, ..., \Lambda_0^{\tau_M})'$ and the $M \times M$ matrix $\Xi_1 = \left(\Lambda_1^{\tau_f,\tau_1}, \Lambda_1^{\tau_f,\tau_2}, ..., \Lambda_1^{\tau_f,\tau_M}\right)^\top$, Equation (2.18) can be written as a system of equations:

$$VS_t = \Xi_0 + \Xi_1 X_t^G. \tag{2.20}$$

We can invert this system in order to express the variance states X_t^G as an affine function of the swap rates VS_t. This yields,

$$X_t^G = \Xi_1^{-1}\Xi_0 + \Xi_1^{-1} VS_t = \zeta + \Xi_1^{-1} VS_t. \tag{2.21}$$

Now, since the instantaneous variation of the implied discount rate $V_t^{\Psi^{\tau_f}}$ is an affine function of the variance states X^G (shown in Equation (2.16)), for any τ_f, one can find a set of constants $\alpha_j^{\tau_f}, j = 0, ..., M$ such that

$$V_t^{\Psi^{\tau_f}} = \alpha_0^{\tau_f} + \sum_{j=1}^{M} \alpha_j^{\tau_f} SW_t^{\tau_j}. \tag{2.22}$$

Hence, the instantenous variation $V_t^{\Psi^{\tau_f}}$ is tied to the term–structure of the contemporaneous variance swap rates via an affine mapping in Equation (2.22). Note that this is a strict realization by realization identity which holds at all times.

The class of low–dimensional $A_M(N)$ style affine term structure structure models, where the covariance of all states is driven solely by the M variance factors, all imply an affine mapping of the spot variance in M swap rates. In particular, for the model of $A_1(3)$, the variance spanning condition boils down to

$$V_t^{\Psi^{\tau_f}} = \alpha_0^{\tau_f} + \alpha_1^{\tau_f} SW_t^{\tau_1}. \tag{2.23}$$

The variance spanning condition in Equation (2.22) is valid for the unspanned stochastic volatility (USV) models, which impose parametric restrictions on the $A_M(N)$ canonical form such that the variance factors do not affect the cross section of yields (see Joslin (2017)). In principle, such USV models are nested under the $A_M(N)$ style affine term structure models.

The spot variance on the left hand side of Equation (2.22) is a latent process and is not directly observable. Most of the existing studies in low-dimensional affine term structure models focus on the conditional volatility yield dynamics backed out from the estimated underlying model. Such parametric conditional variance estimates are sensitive to the misspecification of the underlying model and can in fact even be unrelated to time series of GARCH–type estimates of the conditional variances, see, e.g., Collin–Dufresne, Goldstein, and Jones (2009).

While the parametric model in Equation (4.31) specifies the dynamics under the risk–neutral measure, $\mathbb{Q}$, there are implications for the data generating measure, $\mathbb{P}$ under the assumption of no–arbitrage. In particular, the no–arbitrage condition implies that the instantaneous diffusive variance is invariant to changes of the probability measure. Hence, the spot variance, $V_t^{\Psi^{\tau_f}}$, stays the same under both the data generating measure, $\mathbb{P}$, and the (parameterized) equivalent martingale measure, $\mathbb{Q}$. Thanks to the availability of the high frequency data, the instantaneous variance $V_t^{\Psi^{\tau_f}}$ can be estimated in a model free way. Specifically, these estimators enable non–parametric inference, i.e., they are not based on a parametric specification of the data generating law, X_t, and accordingly, do not rely on the data generating law of $\Psi_t^{\tau_f,\tau_L}$. In this paper, I employ these estimators to measure spot variance $V_t^{\Psi^{\tau_f}}$.

The variance swap rates on the right hand side of Equation (2.22) can also be measured in a model free way by using a portfolio of put and call options. Building on the previous work of Neuberger (1994) and Demeterfi, Derman, Kamal, and Zou (1999), Carr and Wu (2009), and Mele and Obayashi (2013), I show that this quantity can be synthesized using a portfolio of put and call options on $\Psi_t^{\tau_f,\tau_L}$ in a model–free way. I provide the contract designs of such variance swaps and the derivation of model–free prices of these variance swaps with a portfolio of put and call options in Section 3.2.

I develop a testing procedure in Section 3 for the variance spanning condition in Equation (2.22) implied by the class of parametric $A_M(N)$ models. Such tests serve as a diagnostic tool for the affine specification of the model with M variance factors under the risk neutral measure. The test statistic employs model–free measures of spot variance $V_t^{\Psi^{\tau_f}}$ and variance swap rates $SW_t^{\tau_j}$, which I introduce in the next section.

3. Testing the Affine Variance Spanning Condition

This section introduces a specification test for evaluating the affine variance spanning condition in Equation (2.22) and states the asymptotic results for the test statistic. The structure of this section is as follows: I start with defining the model–free measures (non–parametric estimators) of the spot variance $V_t^{\Psi^{\tau_f}}$ and variance swap rates $SW_t^{\tau_j}$. Then, I develop the limit theory necessary to devise the formal specification tests for the variance spanning condition. Last, I provide the test statistic and the associated limit theory. The developed test statistic is a diagnostic tool for the variance dynamics implied by the underlying model under the pricing measure, and does not restrict $\mathbb{P}$ dynamics of the states over what is implied by the no–arbitage condition. In that sense, it relies on the hypothesis that the model is correctly specified under the pricing measure, free from the parametric specifications about the dynamics of the states under the physical measure.

3.1. Non–Parametric Inference for Spot Volatility

I start by defining the non–parametric estimators of spot variance $V_t^{\Psi^{\tau_f}}$ that are used in this paper. Thanks to the availability of high–frequency data, *realized variance* and *realized power variance* estimators have been extensively studied both empirically and theoretically, see Andersen and Bollerslev (1998), Andersen, Bollerslev, Diebold and Labys (2001, 2003), Barndorf–Nielsen and Shephard (2001, 2002a,b, 2003, 2004) among many others. The advantage of these estimators is that the pathwise realizations of volatility at specific times can be recovered non–parametrically. They employ in–fill asymptotics; they are consistent as the observations are sampled more frequently and feature an asymptotic variance that can be estimated by using the observed prices of the underlying asset.

The details about the non–parametric estimator to recover the volatility realizations used in this paper are as follows. The time unit is normalized to a year. Suppose we have equidistant high–frequency observations $y_s^{\tau_f} = log(\Psi_s^{\tau_f})$ over $t-h$ to t with grid size $\delta = \frac{h}{n}$ (e.g. 5 min.). Then, the annualized spot volatility estimator $\widehat{V_t}^{\Psi,k_n,n}$ is given by

$$\widehat{V_t}^{\Psi,K_n,n} = \frac{1}{K_n\delta}\sum_{j\in\mathcal{I}}\left(\Delta_j y^{\tau_f}\right)^2, \tag{3.24}$$

with $\mathcal{I} = -K_n+1, ..., 0$ intraday increments $\Delta_j\Psi = y^{\tau_f}_{t+j\delta} - y^{\tau_f}_{t+(j-1)\delta}$. The block size $K_n < n$ is a deterministic sequence of integers. This estimator is the localized counterpart of the realized variance estimator.[5]

$\widehat{V_t}^{\Psi,K_n,n}$ is consistent for $V_t^{\Psi^{\tau_f}}$ as $K_n \to \infty$ and $K_n\delta \to 0$. For the construction of the test statistic devised in the later sections, the spot volatility needs to be estimated only at finite sample of points.

In the empirical implementation, 10 minutes observations of the Eurodollar futures prices just prior to the close at 2pm (CDT) are used to calculate the spot variance estimator $\widehat{V_t}^{\Psi,K_n,n}$ for each day. Note that Eurodollar futures prices are transformed to Ψ for the corresponding maturities via Equation (2.14). To be precise, in the empirical implementation, the unit of time is a year, where $h = \frac{1}{360}$ refers to a day. The trading part of the day is divided into n intervals and K_n observations prior to the close of the day are employed for the construction of the spot variance estimator $\widehat{V_t}^{\Psi,K_n,n}$.

[5]In settings with jumps, we can construct spot counterparts of the integrated truncated variation estimators of Mancini (2001), see also Jacod and Protter (2012).

3.2. Model–free Construction of Variance Swap Rates on $\Psi_t^{\tau_f}$

There is an extensive literature focusing on the pricing of equity volatility, see Demeterfi, Derman, Kamal, and Zou (1999), Bakshi and Madan (2000), Carr and Madan (2009) among many others. In fact, the theoretical results provided for the model–free replication of the theoretical price of the variance swaps are used by the CBOE to calculate the popular VIX index. The VIX index has become a benchmark over years for measuring and trading US equity market volatility. Although pricing of future volatility is well–understood in equity markets, it is still in its infancy for fixed income markets. Design and pricing of variance swaps in fixed income markets is a delicate issue because of the stochastic interest rates.

In this section, I provide contract designs for variance swaps on the implied discount rate Ψ^{τ_f} and provide the theory regarding the model–free construction of these variance swap rates. In that context, *model–free* refers to the circumstance that the theoretical value of the variance swap rates can be replicated by a portfolio of European style options, without hinging upon any parametric assumptions.[6]

A variance swap is a forward contract such that at maturity one party pays the quadratic variation over the contract horizon and the other party pays a fixed rate in exchange, which is termed as the variance swap rate. Variance swaps require no initial payment. In particular, I consider a variance swap based on Ψ^{τ_f} with the contract horizon from t to $t+\tau_v$. The payoff at maturity on the long side of the swap is equal to

$$\frac{1}{\tau_v}\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}}\,du - SW_t^{\tau_v}, \tag{3.25}$$

where $SW_t^{\tau_v}$ is the fixed variance swap rate determined at time t. The value of the swap rate $SW_t^{\tau_v}$ is determined at time t. A variance swap costs zero to enter. Consequently, under the assumption that the short rate is uncorrelated with integrated volatility,[7] the

[6] Excluding mild assumptions such as absence of arbitrage and the frictionless markets

[7] See the literature on unspanned stochastic volatility

variance swap rate is given by

$$SW_t^{\tau_v} = \frac{1}{\tau_v}\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}}\,du\right]. \tag{3.26}$$

The variance swap rate $SW_t^{\tau_v}$ can be replicated by the continuum of European out–of–the money put and call options. A proof for the following result is provided in Appendix C.

$$SW_t^{\tau_v} \;=\; \frac{2}{\tau_v}\int_0^{\Psi_t^{\tau_f}} \frac{Put_t(\tau_v, K_\Psi)}{{K_\Psi}^2 P_t(\tau_v)} dK_\Psi + \frac{2}{\tau_v}\int_{\Psi_t^{\tau_f}}^{\infty} \frac{Call_t(\tau_v, K_\Psi)}{{K_\Psi}^2 P_t(\tau_v)} dK_\Psi + \epsilon_t^{\tau_v}, \tag{3.27}$$

where $Put_t(\tau_v, K_\Psi)$ and $Call_t(\tau_v, K_\Psi)$ are time t prices of the out–of–the–money European put and call options with strike K_Ψ and with expiration at time $t+\tau_v$, written on the simple implied three–month rate $\Psi_t^{\tau_f}$. I document, with simulations based on empirically relevant parameter values for the Eurodollar futures market, that the effect of the last part is not significant for practical purposes. Hence, I exclude this term in the implementation and obtain the variance swap rate in a model–free fashion by a portfolio of out–of–the–money put and call options. The simulation results are presented in Appendix C.2.

Equation (3.27) means that on each day one can synthesize the variance swap rate with a horizon τ_v by using the options (with corresponding maturity) available on that day. Since Equation (3.27) holds for any τ_v, variance swap rates for various horizons, i.e. term structure of swap rates, can be constructed depending on the availability of the option data. Consequently, variance swap rates are approximated by a discretization of the portfolio of the continuum of options such that

$$\widehat{SW}_t^{\tau_v} \;=\; \frac{2}{\tau_v}\sum_i \frac{\tilde{\mathcal{O}}_t(\tau_v, K_\Psi^i)}{P_t(\tau_v)}\frac{\Delta K_\Psi}{{K_\Psi^i}^2} - \frac{1}{\tau_v}\left(\frac{F_t^{\Psi,\tau_v}}{K_\Psi^0} - 1\right)^2, \tag{3.28}$$

where $\mathcal{O}_t(\tau_v, K_\Psi^i)$ is the time–t price of an out–of–the money option with strike price K_Ψ and maturity $t+\tau_v$. $P_t(\tau_v)$ is the time–t price of a zero–coupon bond with maturity τ_v. F_t^{Ψ,τ_v} is the forward level approximated from the option prices (via the strike where the absolute difference between the call and put is smallest), and K_{Ψ}^{0} is the first available strike below the forward level F_t^{Ψ,τ_v}. The last part in Equation (3.28) is a correction for the error introduced by the substition of K_{Ψ}^{0} instead of the forward price. $\Delta_{K_{\Psi}^{i}}$ is set as $\frac{K_{\Psi}^{i-1}+K_{\Psi}^{i+1}}{2}$ for all strikes, excluding the smallest and the largest strikes. For the smallest and the largest strikes, $\Delta_{K_{\Psi}^{i}}$ is set as the distance to the next higher strike and next lower strike respectively.

Eurodollar futures contracts do not have constant maturities over days, instead their maturities follow a seesaw pattern over days. Then, the swap rate $\widehat{SW_t}^{\tau}$ for a fixed time to maturity τ can be calculated from the available options with the two nearest maturities τ_{v_1} and τ_{v_2} via the following linear interpolation

$$\widehat{SW_t}^{\tau} = \frac{1}{\tau}\left(\tau_{v_1}\widehat{SW_t}^{\tau_{v_1}}\frac{\tau_{v_2}-\tau}{\tau_{v_2}-\tau_{v_1}} + \tau_{v_2}\widehat{SW_t}^{\tau_{v_2}}\frac{\tau-\tau_{v_1}}{\tau_{v_2}-\tau_{v_1}}\right), \tag{3.29}$$

where $SW_t{}^{\tau_{v_1}}$ and $SW_t{}^{\tau_{v_2}}$ are variance swap rates with time to maturity τ_{v_1} and τ_{v_2}, respectively.

I construct the variance swap rates by using the settlement prices of options at each day for which the Eurodollar option data is available. The available option data covers the period from January 1, 2004 to July 13, 2010. At each day in the sample, I construct a term–structure of variance swap rates on the implied discount rate Ψ for fixed maturities of three, six, nine months, one year and one and a half years by using Equation (3.29). As the data is not directly available on the implied discount rates and options on them but on Eurodollar futures and options, we need to transform the prices of Eurodollar futures and options to the corresponding measures on the implied discount rates. I provide the details regarding the construction variance swap rates on the implied discount rates later in Section 5.1.

This paper is not the first one to analyze the variance contracts in fixed income markets. Mele and Obayashi (2013) provide the theory underlying the pricing of variance contracts on Treasury futures. Choi, Mueller, and Vedolin (2017) study variance risk premiums in the Treasury market. Grishchenko, Song, and Zhou (2015) examine the role of variance risk premiums based on interest rate swaps and swaptions in predicting Treasury excess returns. The theoretical and empirical results regarding the variance swaps documented in this paper complement the work above as the focus here is on the pricing of the variance in Eurodollar markets (on implied discount rates).

4. Test Statistics and the Limiting Theory

This section formalizes the set–up underlying the econometric analysis of the paper. Our interest is on the process, Ψ^{τ_f} defined on a probability space $\left(\Omega^{(0)}, \mathcal{F}^{(0)}, \left(\mathcal{F}_t{}^{(0)}\right)_{t\geq 0}, \mathbb{P}^{(0)}\right)$ follows the general dynamics under $\mathbb{P}^{(0)}$:

$$\frac{d\Psi_t^{\tau_f}}{\Psi_t^{\tau_f}} = \mu_t^{\Psi^{\tau_f}} dt + \sigma_t^{\Psi^{\tau_f}} dW_t, \tag{4.30}$$

where W is an N–dimensional Wiener process, $\mu_t^{\Psi^{\tau_f}}$ and $\sigma_t^{\Psi^{\tau_f}}$ are càdlàg and $\mathcal{F}_t$–adapted processes. The spot variance process is $V_t^{\tau_f} = \sigma_t^{\Psi^{\tau_f}} \sigma_t^{\Psi^{\tau_f}\top}$ takes its values in the set $\mathcal{D}^+ \equiv (0, \infty)$. See Assumption 1 in Appendix D for precise assumptions regarding regularity conditions on $\Psi_t^{\tau_f}$.

Under the assumption of no–arbitrage, there exists a risk–neutral probability measure $\mathbb{Q}$ (see, e.g., Duffie (2001)) which is locally equivalent to $\mathbb{P}^{(0)}$. To be specific, Ψ^{τ_f} under $\mathbb{Q}$ follows

$$\frac{d\Psi_t^{\tau_f}}{\Psi_t^{\tau_f}} = \mu_t^{\Psi,\mathbb{Q}} dt + \sigma_t^{\Psi^{\tau_f}} dW_t^{\mathbb{Q}}, \tag{4.31}$$

where $W_t^{\mathbb{Q}}$ is an N–dimensional $\mathbb{Q}$–Wiener process. Similarly, $\mu_t^{\Psi,\mathbb{Q}}$ is càdlàg and adapted.

Note that I stress that I do not assume any parametric functional forms on $\mu_t^{\Psi^{\tau_f}}$, $\mu_t^{\Psi,\mathbb{Q}}$ and $\sigma_t^{\Psi^{\tau_f}}$, i.e. recall that $\mu_t^{\Psi^{\tau_f}}$, $\mu_t^{\Psi,\mathbb{Q}}$ and $\sigma_t^{\Psi^{\tau_f}}$ carry an affine parametric form in the underlying (latent) states under low–dimensional affine term–structure models. Moreover, the setting in this section allows the instantenous variance process $V_t^{\Psi_f^{\tau}}$ to be an Itô semi–martingale with general forms of vol–of–vol and jumps.

We have observations from options written on Ψ^{τ_f} at integer times (i.e. days) $t =$

$1, ..., T$, within a time span $[0, T]$. In particular, at a given time t, we have a cross–section of out–of–the money option prices $\{\mathcal{O}_t(\tau_j, K_{\Psi j}); j = 1, ..., N_K\}$ for some integer N_K. For a tenor of τ, there are N_K^τ number of options, and hence $\sum_\tau N_K^\tau = N_K$.

Suppose for each tenor τ, $K_m(\tau)$ and $K_M(\tau)$ represent the minimum and the maximum strikes. To this end, I assume that the strikes between $K_m(\tau)$ and $K_M(\tau)$ are equidistant, with grid size $\Delta_K = \frac{K_M(\tau)-K_m(\tau)}{N_K^\tau}$.

Option prices are assumed to be observed with error:

$$\tilde{\mathcal{O}}_t(\tau, K_\Psi) = \mathcal{O}_t(\tau, K_\Psi) + \varepsilon_t^{\tau, K_\Psi}, \tag{4.32}$$

where the observation errors, $\varepsilon_t^{\tau, K_\Psi}$, are defined on space $\Omega^{(1)}$. We endow the space $\Omega^{(1)}$ with the product Borel σ–field $\mathcal{F}^{(1)}$ and the filtration $\mathcal{F}^{(1)} = \sigma(\varepsilon_s^{\tau, K_\Psi} : s \leq t)$ and the probability $\mathbb{P}^{(1)}(w^{(0)}, dw^{(1)})$. Then, the extended probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is given by

$$\Omega = \Omega^{(0)} \times \Omega^{(1)}, \qquad \mathcal{F} = \mathcal{F}^{(0)} \times \mathcal{F}^{(1)}, \qquad \mathcal{F}_t = \cap_{s>t}(\mathcal{F}_s^{(0)} \times \mathcal{F}_s^{(1)}),$$

$$\mathbb{P}\left(dw^{(0)}, dw^{(1)}\right) = \mathbb{P}^{(0)}\left(dw^{(0)}\right)\mathbb{P}^{(1)}\left(w^{(0)}, dw^{(1)}\right).$$

Observation errors are assumed to be conditionally centered and to display stochastic volatility which can also depend on the tenor:

$$\mathrm{E}\left[\varepsilon_t^{\tau, \mathcal{K}^\Psi} | \mathcal{F}^{(0)}\right] = 0, \quad \text{and} \quad \mathrm{E}\left[\varepsilon_t^{\tau, \mathcal{K}^\Psi 2} | \mathcal{F}^{(0)}\right] = \eta_{t, K_\Psi, \tau}. \tag{4.33}$$

See Appendix 2 for additional assumptions regarding the observation errors.

Now, we are at a stage to establish the asymptotic distribution of the estimators. I first start with the limiting distribution of the high–frequency based variance estimator $\widehat{V}_t^{\Psi, K_n, n}$ given in Equation (3.24).

Theorem 4.1. *Let Ψ be a stochastic process solution to (4.31). Let Assumption 1 holds. If $K_n \to \infty$, $\delta \to 0$ and $K_n\sqrt{\delta} \to 0$, then for all t in $(0, T]$, we have*

$$\frac{\widehat{V}_t^{\Psi, K_n, n} - V_t^{\Psi^{\tau_f}}}{\sqrt{\frac{2}{K_n}\left(\widehat{V}_t^{\Psi, K_n, n}\right)^2}} \xrightarrow{\mathcal{L}-s} \mathcal{U}, \tag{4.34}$$

where conditionally on $\mathcal{F}$, $\mathcal{U}$ is an $\mathcal{N}(0,1)$ variable.

See Alvarez, Panloup, Pontier, and Savy (2012) for a proof.[8]

Theorem 4.2. *Let Assumption 2 holds. $N_K^{\tau_v}$ is the number of options on a given day t with tenor τ_v and with equally spaced strikes of grid size Δ_K between $[K_m(\tau_v), K_M(\tau_v)]$. Let $\widehat{SW}_t^{\tau_v}$ is given as in Equation (3.28) with $F_t^{\Psi,\tau_v} = K_\Psi^0$. Moreover, assume there exists*

$$\widehat{\eta}_{t,K_\Psi,\tau_v} \xrightarrow{\mathbb{P}} \eta_{t,K_\Psi,\tau_v}, \tag{4.35}$$

uniformly on $[K_m(\tau_v), K_M(\tau_v)]$. Then, as $\Delta_K \to 0$, we have

$$\frac{\widehat{SW}_t^{\tau_v} - SW_t^{\tau_v,m,M}}{\sqrt{\Delta_K \widehat{\mathcal{W}}_t^{N_K}}} \xrightarrow{\mathcal{L}-s} \mathcal{N}(0,1), \tag{4.36}$$

where

$$SW_t^{\tau_v,m,M} = \frac{2}{\tau_v} \int_{K_m(\tau_v)}^{K_M(\tau_v)} \frac{\mathcal{O}_t(\tau_v, K_\Psi)}{\mathcal{K}_\Psi{}^2} dK_\Psi, \tag{4.37}$$

and

$$\widehat{\mathcal{W}}_t^{N_K} = \frac{4}{P_t(\tau_v)^2} \sum_{j=1}^{N_K^{\tau_v}} \frac{\widehat{\eta}_{t,K_j,\tau_v}}{K_\Psi^{j\;4}} \Delta_K. \tag{4.38}$$

See Appendix D for a proof.

The limiting distribution of a test statistic for the affine variance spanning condition is provided next. Specifically, the underlying null hypothesis is the affine variance spanning condition

$$V_t^{\Psi^{\tau_f}} = \alpha_0^{\tau_f} + \sum_{j=1}^{M} \alpha_j^{\tau_f} SW_t^{\tau_j}.$$

.

Theorem 4.3. *Let Assumption 1 and 2 hold. Suppose $T \to \infty$, $N_K \to \infty$ with $\frac{N_K}{T} \to 0$, $K_n \to \infty$, $K_n\sqrt{\delta} \to 0$, with $\frac{K_n}{N_K} \to \rho$ and $\frac{K_n}{T} \to 0$.*

$$\widehat{Z}_t = \frac{\widehat{V}_t^{\Psi,K_n,n} - \hat{\alpha}_0^{\tau_f} - \sum_{j=1}^{M} \hat{\alpha}_j^{\tau_f} \widehat{SW}_t^{\tau_j}}{\sqrt{\frac{2}{K_n}\left(\widehat{V}_t^{\Psi,K_n,n}\right)^2 + \Delta_K \sum_{j=1}^{M} (\hat{\alpha}_j^{\tau_f})^2 \widehat{\mathcal{W}}_t^{N_K,j}}} \xrightarrow{\mathcal{L}-s} \mathcal{N}(0,1), \tag{4.39}$$

[8] In a setting with price jumps, similar results have been established for the localized versions of the truncated variation estimators (see, e.g., Jacod and Protter (2012), Andersen, Fusari, and Todorov (2015)).

where $(\hat{\alpha}_0, \hat{\alpha}_1, ..., \hat{\alpha}_M)^\top = \underset{\{\alpha_j^{\tau_f}\}_{j=0}^M}{\operatorname{argmin}} \sum_{t=1}^T \left(\widehat{V}_t^{\Psi,K_n,n} - \alpha_0^{\tau_f} - \sum_{j=1}^M \alpha_j^{\tau_f} \widehat{SW}_t^{\tau_j} \right)^2$ *and* $\widehat{\mathcal{W}}_t^{N_K}$ *is given as in Theorem 4.2.*

See Appendix D for a proof.

I adjust the variance spanning condition in Equation (2.22) for logarithmic variances as

$$\log V_t^{\Psi^{\tau_f}} = \alpha_{0,log}^{\tau_f} + \sum_{j=1}^M \alpha_{j,log}^{\tau_f} \log SW_t^{\tau_j},$$

where $\alpha_{j,log}^{\tau_f}, j = 0, ..., M$ is a set of constants. I term this relation as the logarithmic variance spanning condition. Next, I provide the limiting distribution of a test statistic under the null of the logaritmic affine spanning condition.

Theorem 4.4. *Let Assumption 1 and 2 hold. Suppose* $T \to \infty$, $N_K \to \infty$ *with* $\frac{N_K}{T} \to 0$, $K_n \to \infty$, $K_n\sqrt{\delta} \to 0$, *with* $\frac{K_n}{N_K} \to \rho$ *and* $\frac{K_n}{T} \to 0$. *Then, we have*

$$\widehat{Z}_t^{log} = \frac{\log \widehat{V}_t^{\Psi,K_n,n} - \hat{\alpha}_{0,log}^{\tau_f} - \sum_{j=1}^M \hat{\alpha}_{j,log}^{\tau_f} \log \widehat{SW}_t^{\tau_j}}{\sqrt{\frac{2}{K_n} + \Delta_K \sum_{j=1}^M (\hat{\alpha}_j^{\tau_f})^2 \widehat{\Pi}_t^{N_K,j}}} \xrightarrow{\mathcal{L}-s} \mathcal{N}(0,1), \tag{4.40}$$

where

$$(\hat{\alpha}_{0,log}, \hat{\alpha}_{1,log}, ..., \hat{\alpha}_{M,log})^\top = \underset{\{\alpha_{j,log}^{\tau_f}\}_{j=0}^M}{\operatorname{argmin}} \left(\log \widehat{V}_t^{\Psi,K_n,n} - \alpha_{0,log}^{\tau_f} - \sum_{j=1}^M \alpha_{j,log}^{\tau_f} \log \widehat{SW}_t^{\tau_j} \right)^2,$$

and $\widehat{\Pi}_t^{N_K,j} = \frac{\widehat{\mathcal{W}}_t^{N_K,j}}{(\widehat{SW}_t^{\tau_j})^2}$.

See Appendix D for a proof.

5. Data and Preliminary Analyses

The analysis in this paper relies on Eurodollar futures and options data traded at the CME from January 1, 2004 to July 13, 2010. Eurodollar futures contracts are cash settled contracts with the delivery based on the 3–month LIBOR. These contracts are issued quarterly with maturities ranging from three months to ten years. Consequently, at each date there are up to fourty contracts available with contract months March, June, September and December. Note that Eurodollar futures do not have constant maturities. Accordingly in order to get one time series, I roll the futures data at the end of the month preceeding the contract month. Therefore, the time–to–maturities follow a seesaw pattern over days for each maturity.

The CME offers both quarterly and serial option contracts. The contract months for quarterly options are March, June, September and December and they exercise into the Eurodollar futures contracts with the corresponding maturities. Although the CME recently introduced quarterly options with maturities up to four years, the analysis here is based on the options with maturities up to two years because these contracts are the most liquid ones. The serials' contract months are the two non–quarterly front months and they exercise into the corresponding quarterly Eurodollar futures contract immediately following the serial. Note that these contracts are much less liquid than the quarterly options. Consequently, I discard all the serial options from the sample. Consistent with the rolling procedure implemented for the Eurodollar futures, options are rolled at the end of the month preceeding the contract month.

5.1. Construction of the Swap Rates

In order to construct the term–structure of the variance swap rates in a model free way, I use the daily settlement prices of the CME standard quarterly options from January 1, 2004 to July 13, 2010. Daily data on Eurodollar futures and options is obtained from the CME.[9] I apply the following filters to the option data before constructing the variance swap rates: I eliminate 1) the options with zero settlement prices 2) the options with zero strikes 3) the options with zero time–to–maturity 4) the options with zero open interest. Since the model–free construction of variance swap rates employs only out–of–the money options, I discard all in–the–money options.

In order to construct the variance swap rates on the implied discount rate Ψ^{τ_f}, we need to transform the Eurodollar futures and options prices to the corresponding values for the implied discount rate. Appendix B outlines the procedures regarding these

[9] I use settlement prices of options, which prevents issues related to stale trading or microstructure noise. CME calculates these prices based on the Globex trades between 13.59 and 14.00.

tranformations.

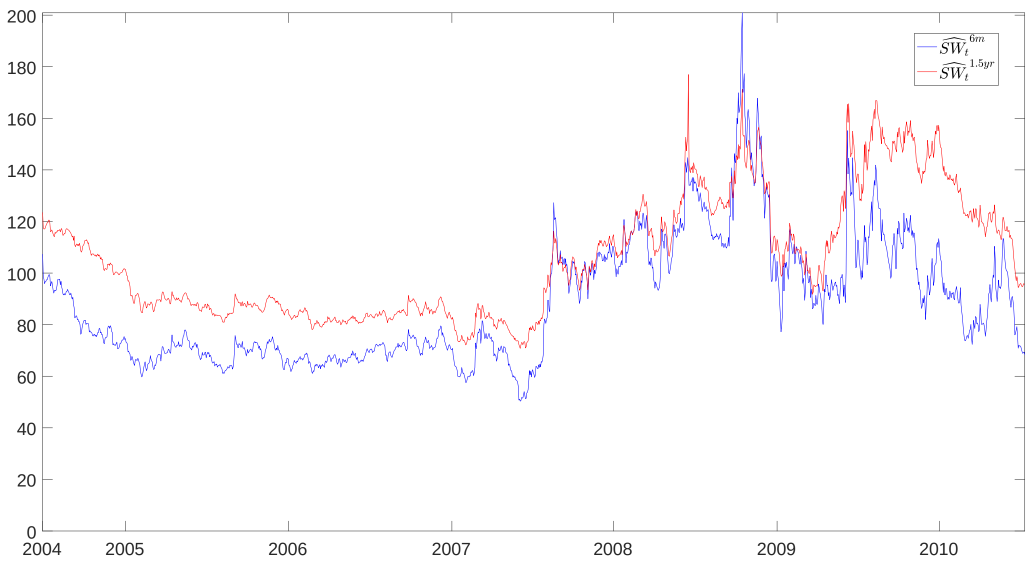
I choose horizons of 6–month and 1 year and 6–months for the synthetic variance swap rates. The remaining horizons are excluded because of liquidity concerns. On a given date, I choose the two closest maturities to a given horizon (i.e. 6–months) and construct the variance swap rates in a model free way by implementing the discretization (3.28) for these two maturities and linearly interpolate the horizon to obtain fixed horizons of interest via Equation (3.29).

Figure 1 plots the time series of the square root of the variance swap rates for the maturities of 6 months and 1.5 years. The figure depicts all values in annualized basis point units. The variance swap rates for 1.5 year maturity is usually larger than the variance swap rates for 6 months maturity. However, in times of distress such as the crisis period during 2007 and 2008, the swap rates with the short maturity occasionally go above the swap rates with the longer maturity. These findings could be explained by the markets pricing the long term contracts with the expectation that volatility would fall back from the crisis levels. Such behaviors regarding the term structure of variance swap rates are consistent with the findings in the literature with regards to the variance swap rates in equity markets, see, for example Dew-Becker, Giglio, Le, and Rodriguez (2017).

5.2. Construction of Spot Variance

In order to construct a non–parameteric estimator of the spot variance, I obtain the intraday series of Eurodollar Futures prices from TickData for the period from January 1, 2004 to July 13, 2010. Note that Eurodollar futures are traded both open outcry (pit trading) and electronically (Globex); I use intraday trades data from both the electronic and the pit trading sessions. In line with the pit trading hours, I start the intraday record at 7:20 am (CDT) and end it at 2:00 pm (CDT). I sample Eurodollar futures trades at a 10–minutes frequency over a 6 hours 40 minutes trading period and convert them into the corresponding implied discount rates based on the procedure outlined in Appendix B. After implementing the necessary transformations, the returns on $\log \Psi^{\tau_f}$

Figure 1: **Variance Swap Rates**



The figure plots the time series of the square root of the variance swap rates per annum in basis points. The red line represents the daily variance swap rate for 6–months maturity. The blue line represents the daily swap rate for 1.5–years maturity. The sample covers the period from 06/29/2004 to 07/13/2010.

are computed. Note that, via the rolling procedure, it is possible to obtain single series for up to fourty maturities. The maturity structure of each of these series has a seesaw pattern over days.

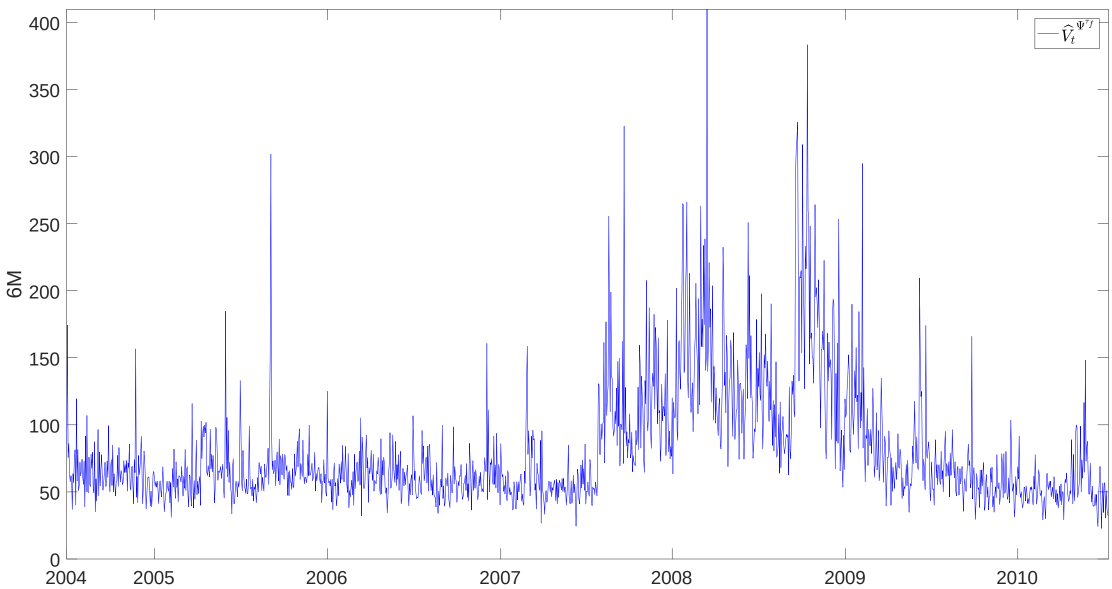
Before constructing the non–parametric variance estimator, I apply the following filters to clean the intraday data: I eliminate the days with no trading activity, half–trading days, and the days with early market closures which typically happens before holidays. Moreover, I exclude the days where there is no price change during the last two thirds of the day.[10] Consequently, we have 40 intraday changes on each trading day over approximately 1600 trading days for the series included in the sample. I construct the non–parametric variance estimator for each series (with maturities up to 4 years) by implementing Equation (3.24), where $n = 40$ and K_n is set as 30. Note that these series are in annualized terms since $h = \frac{1}{360}$. To obtain a spot variance series with a fixed

[10]These events result in low trading activities for these days.

maturity of $\mathcal{S}$ days, I choose the two spot variance series with the closest maturities to $\mathcal{S}$ and linearly interpolate the two spot variance estimators on each day.

Figure 2 depicts the daily values for the square root of the non–parametric variance estimator for 6–months $\log(\Psi)$ constructed from high–frequency data. The figure shows the series in annualized basis point units. Casual inspection shows that volatility follows higher levels during the crisis period from mid 2007 to the end of 2008. After 2009, it goes back, on average, to the levels observed before the crisis period. Moreover, the 6–months volatility in Eurodollar markets shows occasional spikes, which is in line with corresponding findings in Treasury markets, see for example, Andersen and Benzoni (2010).

Figure 2: **Non–parametric Spot Volatility Estimate with 6–month maturity**



This figure plots the time series the non–parametric estimate of the instantenous variance of 6–month implied discount rate (per annum in basis points). The sample covers the period from 06/29/2004 to 07/13/2010.

6. Evidence on Testing the Affine Variance Spanning Condition

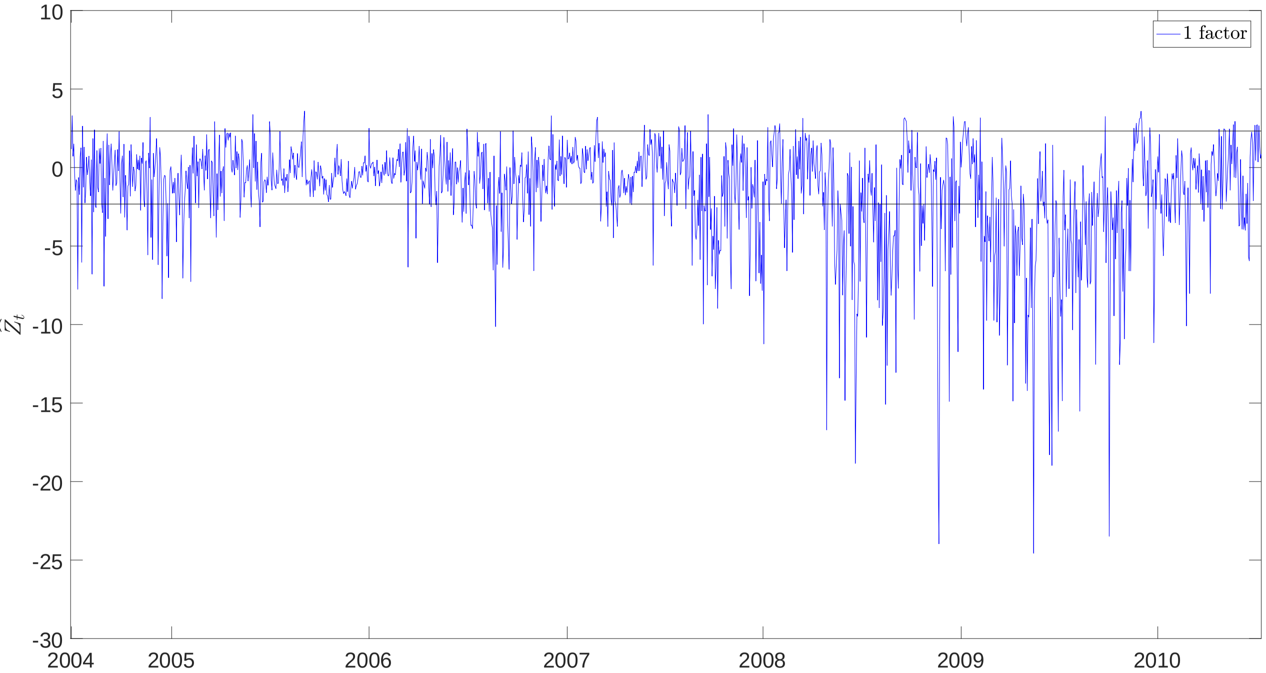
I now present my main empirical findings. This section studies whether the affine variance spanning condition in Equation (2.22), implied by $A_M(N)$ style affine term structure models, is satisfied in the Eurodollar data. In particular, I test the affine variance spanning condition by implementing the test statistic $\widehat{Z}_t$ (see Theorem 4.3). Recall that the test statistic relies on the high–frequency based non–parametric spot variance estimator as well as on the model–free measures of variance swap rates. I focus on the results for the 6–month maturity spot variance, that is $V_t^{\Psi^{\tau_f}}$ where $\tau_f = \frac{1}{2}$. The findings for the 3–months, 1 year, 1.5 years and 3 years maturity spot variances are provided in the Appendix. I rely on the model–free measures of the variance swap rates for 6–months and 1.5–years maturities to construct the test statistic.

I test the variance spanning implication for the two most often used models, $A_1(3)$ and $A_2(3)$, in the low–dimensional term structure literature. I inititally focus on the diagnostic analysis of the variance spanning condition under the $A_1(3)$ model, where the covariance of all states is solely driven by a single factor. Recall that the variance spanning condition in that case boils down to a specification with one factor only (see Equation (2.23)). Consequently, the diagnostic analysis of the variance spanning condition under $A_1(3)$ model relies on the non–parametric measure of the variance swap rate with 6–months maturity only.

Figure 3 depicts the daily times series of the test statistic $\widehat{Z}_t$ from Theorem 4.3 for the variance spanning condition implied by the $A_1(3)$ model. The test statistic is employed dynamically at each point in time via a rolling window of length of 120 days. Remarkably, the test statistic exhibits very large negative values (considerably below the 1st percentile value) over the crisis period of 2008 to 2010, which indicates the failure of the affine variance spanning condition under the one volatility factor model. Moreover, the $A_1(3)$ model struggles occasionally during the calm period of 2004–2007. From an econometric perspective, the $A_1(3)$ model is rejected.

Casual inspection shows persistent behavior of the test statistic over periods where

Figure 3: **Test Statistic for the Affine Variance Spanning Condition with $M = 1$ factor; 6 month maturity**



The plot shows the time series of the test statistics

$$\widehat{Z}_t = \frac{\widehat{V}_t^{\Psi,K_n,n} - \hat{\alpha}_0^{\tau_f} - \hat{\alpha}_1^{\tau_f} \widehat{SW_t}^{\tau_1}}{\sqrt{\frac{2}{K_n}\left(\widehat{V}_t^{\Psi,K_n,n}\right)^2 + \Delta_K (\hat{\alpha}_1^{\tau_f})^2 \widehat{\mathcal{W}}_t^{N_K,1}}}, \tag{6.41}$$

where $(\hat{\alpha}_0, \hat{\alpha}_1)^\top = argmin_{\alpha_j^{\tau_f};j=0,1} \sum_{t=1}^T \left(\widehat{V}_t^{\Psi,K_n,n} - \alpha_0^{\tau_f} - \alpha_1^{\tau_f} \widehat{SW_t}^{\tau_1}\right)^2$ and $\widehat{\mathcal{W}}_t^{N_K}$ is given as in Theorem 4.2. $\widehat{SW_t}^{\tau_1}$ represents the variance swap rate on the 6–month implied discount rate and it is synthesized from Eurodollar options via a transformation in a model–free way every day in the sample and $\hat{V}_t^{\Psi,K_n,n}$ is the spot variance estimate for the implied discount rate with 6–month maturity obtained from high frequency data non–parametrically as in Eqn. (3.24). The test statistic $\widehat{Z}_t$ is constructed every day by using rolling windows of length 120 days. The trading period over a day is 6 hours 40 mins which leads to n=40. K_n is fixed to 30. The sample covers the period from January 1, 2004 to July 13, 2010.

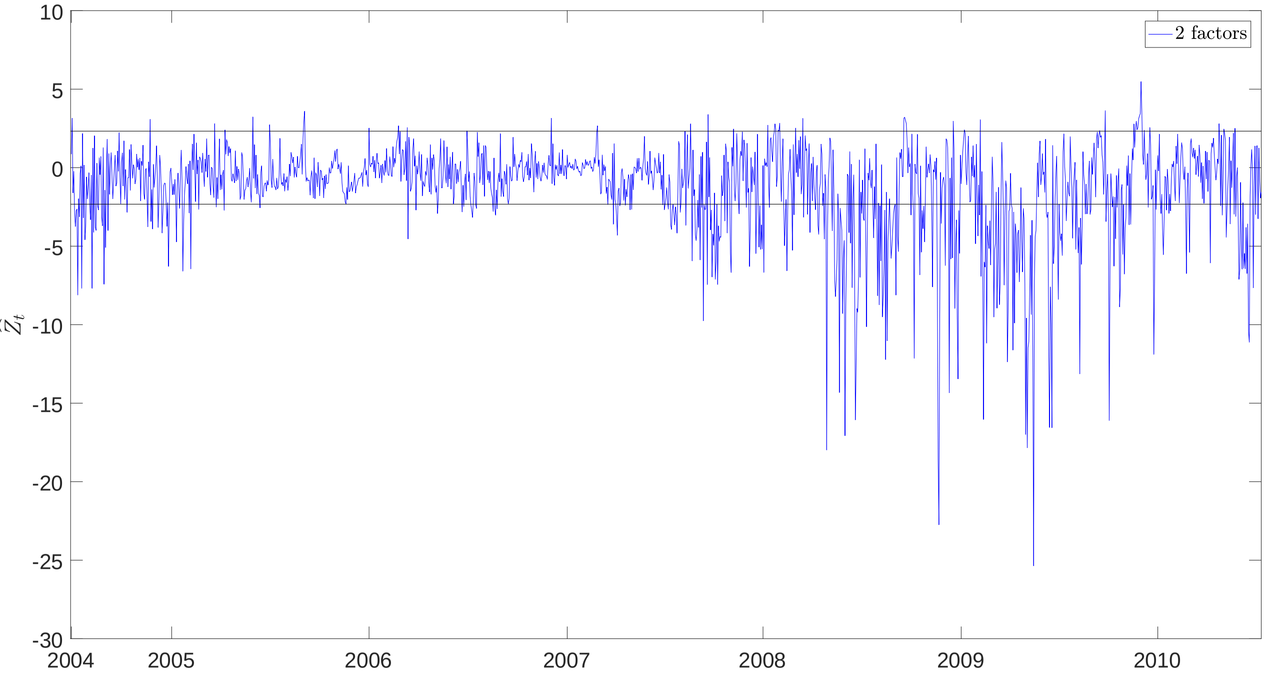
the spanning condition under the $A_1(3)$ model fails. This may indicate the necessity of a second volatility factor. Given the failure of the $A_1(3)$ model, a natural next step would be to explore extensions with additional volatility factors. Accordingly, I test the variance spanning condition implied by the $A_2(3)$ model, where the covariances of all states are driven by two factors. Recall that the variance spanning condition in that case boils down to a specification with two variance swap rates. The test statistic in that case relies on the 6–months variance swap rate and the 1.5–years variance swap rate.

Figure 4 illustrates the results for the test statistic for the $A_2(3)$ model. Not surprisingly, the additional factor improves the performance of the affine model in satisfying the variance spanning condition, especially during the tranquil period of 2004–2007. However, the test statistic still exhibits large negative values during the 2008–2010 crisis period. Consequently, from an econometric perspective, the affine specification with two volatility factors is strongly rejected during the periods of major market disruptions. The results for testing the affine variance spanning condition under the $A_2(3)$ model for 3–months, 1–year, 1.5–years and 3–years maturities are provided in Appendix E. Consistent with the results with regards to 6–month maturity, the test statistic exhibits often large negative spikes during the crisis period, implying also the failure of the affine specification also with two volatility factors.

6.1. Logarithmic Affine Variance Spanning Condition

There is substantial empirical evidence in the equity and foreign exchange literature that the distributions of the logarithms of daily realized variances are approximately Gaussian, see for example Andersen, Bollerslev, Diebold and Labys(2000, 2001, 2003). Accordingly, Andersen, Bollerslev, and Diebold (2007) study volatility forecasting via modelling the logarithms of realized variances and find a substantial improvement in the forecasting perfomance (see, also, Andersen, Bollerslev, and Meddahi (2005) for volatility forecast evaluations). Chernov, Gallant, Ghysels, and Tauchen (2003) employs log–linear specifications of volatility for modelling the distributions of equity returns. Exponential–OU processes are moreover studied in Todorov and Tauchen (2011), Todorov, Tauchen,

Figure 4: **Test Statistic for the Affine Variance Spanning Condition with $M = 2$ factors**



The plot shows the time series of the test statistics

$$\widehat{Z}_t = \frac{\widehat{V}_t^{\Psi,K_n,n} - \hat{\alpha}_0^{\tau_f} - \hat{\alpha}_1^{\tau_f}\widehat{SW_t}^{\tau_1} - \hat{\alpha}_2^{\tau_f}\widehat{SW_t}^{\tau_2}}{\sqrt{\frac{2}{K_n}\left(\widehat{V}_t^{\Psi,K_n,n}\right)^2 + \Delta_K(\hat{\alpha}_1^{\tau_f})^2\widehat{\mathcal{W}}_t^{N_K,1} + \Delta_K(\hat{\alpha}_2^{\tau_f})^2\widehat{\mathcal{W}}_t^{N_K,2}}},$$

where $(\hat{\alpha}_0, \hat{\alpha}_1, , \hat{\alpha}_2)^\top = argmin_{\alpha_j^{\tau_f};j=0,1} \sum_{t=1}^{T}\left(\widehat{V}_t^{\Psi,K_n,n} - \alpha_0^{\tau_f} - \alpha_1^{\tau_f}\widehat{SW_t}^{\tau_1} - \alpha_2^{\tau_f}\widehat{SW_t}^{\tau_2}\right)^2$ and $\widehat{\mathcal{W}}_t^{N_K,j}$ is given as in Theorem 4.2. $\widehat{SW_t}^{\tau_1}$ and $\widehat{SW_t}^{\tau_2}$ represents the variance swap rate on the 6–month and 18–month implied discount rate respectively and it is synthesized from Eurodollar options via a transformation in a model–free way every day in the sample and $\hat{V}_t^{\Psi,K_n,n}$ is the spot variance estimate for the implied discount rate with 6–month maturity obtained from high frequency data non–parametrically as in Eqn. (3.24). The test statistic $\widehat{Z}_t$ is constructed every day by using rolling windows of length 120 days. The trading period over a day is 6 hours 40 mins which leads to n=40. K_n is fixed to 30. The sample covers the period from January 1, 2004 to July 13, 2010.

and Grynkiv (2014). The results in the literature regarding equity and foreign exchange markets provide a foundation for exploring the affine specifications based on logarithmic variances. Consequently, I explore whether the affine specification of logarithmic variance is satisfied in data. More precisely, for any maturity τ_f, I adjust the variance spanning condition in Equation (2.22) for logarithmic variances as follows

$$\log V_t^{\Psi^{\tau_f}} = \alpha_{0,log}^{\tau_f} + \sum_{j=1}^{M} \alpha_{j,log}^{\tau_f} \log SW_t^{\tau_j}, \tag{6.42}$$

where $\alpha_{j,log}^{\tau_f}; j = 0, ..., M$ is a set of constants. I term this relation as the logarithmic affine variance spanning condition.

Recall that the test statistic for the logarithmic affine variance spanning condition is as follows

$$\widehat{Z}_t^{log} = \frac{\log \widehat{V}_t^{\Psi, K_n, n} - \hat{\alpha}_{0,log}^{\tau_f} - \sum_{j=1}^{M} \hat{\alpha}_{j,log}^{\tau_f} \log \widehat{SW}_t^{\tau_j}}{\sqrt{\frac{2}{K_n} + \Delta_K \sum_{j=1}^{M} (\hat{\alpha}_j^{\tau_f})^2 \widehat{\Pi}_t^{N_K, j}}}, \tag{6.43}$$

where $(\hat{\alpha}_{0,log}, \hat{\alpha}_{1,log}, ..., \hat{\alpha}_{M,log})^\top$ and $\widehat{\Pi}_t^{N_K,j}$ are as defined in Theorem 4.4. The limiting distribution of $\widehat{Z}_t^{log}$ is provided in Theorem 4.4. Note that this test statistic can be constructed for spot variances with any maturity τ_f. In the main text, the focus is on the 6–months maturity. However, the results for maturities up to 3–years are presented in Appendix E.

Next, I provide empirical evidence on whether the logarithmic variance spanning condition is satisfied in the Eurodollar data. Specifically, the test statistic is constructed every day in the sample by employing high–frequency spot variance measures and the variance swap rates. As with the empirical analyses regarding the variance spanning condition, I focus on the spot variance with 6–months maturity in the main text. I start with testing a simple specification with only $M = 1$ factor and rely on the variance swap rate with the 6–months maturity for this factor.

Figure 5 illustrates the results for the 1–factor logarithmic spanning condition. The test statistic is constructed dynamically with rolling windows of window length 120 days. The model fit improves remarkably, especially during the crisis period of 2008–2010. In contrast to the case with the affine spanning, the test statistic for the logarithmic variance spanning condition exhibits much more moderate values in magnitude, occasionally outside of the region between the 1st and 99th percentile values of the standard normal distribution. Specifically, during the crisis period of 2008-2010, the absolute value of the test statistic based on the affine variance spanning condition is on average 3.2 and exhibits a standard deviation of 3.5 and a kurtosis of 10.1. Moreover, the absolute value of the test statistic reaches extreme values of up to 20 around specific crisis events such as the bankruptcy of Lehman Brothers and the European sovereign debt crisis. On the other hand, the mean value, the standard deviation and the kurtosis decrease to 1.9, 1.5 and 4.2 respectively for the absolute value of the test statistic based on the logarithmic variance spanning

Next, I explore an extension of the logarithmic variance specification involving two factors. In particular, the logarithmic specification in Equation (6.42) is tested for $M = 2$ factors. The first factor is the same as in the case of the one factor specification. The second factor is based on the variance swap rate with 1.5–years maturity. Figure 6 depicts the time series of the test statistic with $M = 2$ factors. The logarithmic specification with two factors improves the performance of the test statistic, however the improvements are not substantial. The average value of the test statistic goes down to 1.52 from 1.63 for the full sample. The standard deviation also decreases to 1.37 from 1.45.

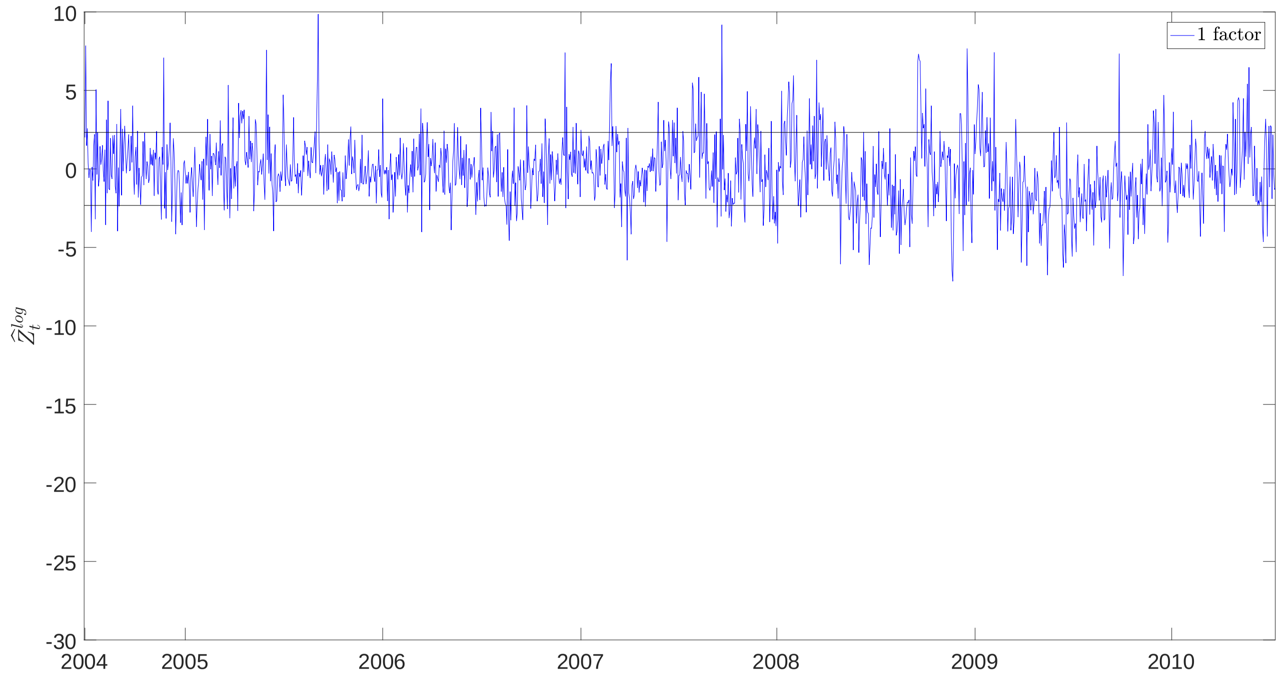
All in all, the logaritmic variance specification in Equation (6.42) provides a remarkably better fit for the spot variances compared to the affine variance specification implied by the affine term structure models. More general specifications with two factors further improve the performance, however the qualitative implications remain unchanged. In particular, the affine variance spanning condition is strongly rejected especially during the crisis periods for both one factor and two factor specifications. Moreover, although the two factor logarithmic specification performs better than the one factor logarithmic specification, the time series behavior of the test statistic does not change substantially by adding the second factor.

7. Conclusion

The recent fixed income literature is centered around the class of low–dimensional multi–factor affine term structure models. One key implication of the affine term structure models is that the yield variances are spanned linearly by the contemporaneous cross–sections of the variance swap rates. This paper designs novel specification tests for evaluating these variance spanning implications.

By relying on model–free measures of the variance swap rates for implied discount rates and the respective high–frequency estimates of the instantaneous variances, I formally test whether the affine variance spanning condition holds for the Eurodollar data. I find strong statistical evidence against the affine variance spanning condition in Eurodollar futures markets. As an alternative to the affine–model specification, I explore whether an affine specification of logarithmic variances provides a more satisfactory characterization of the data. I document that the logarithmic affine specification of variances provides a remarkably improved fit for the variance dynamics in the Eurodollar market.

Figure 5: **Test Statistic for Judging the Affine Restriction - 6M**

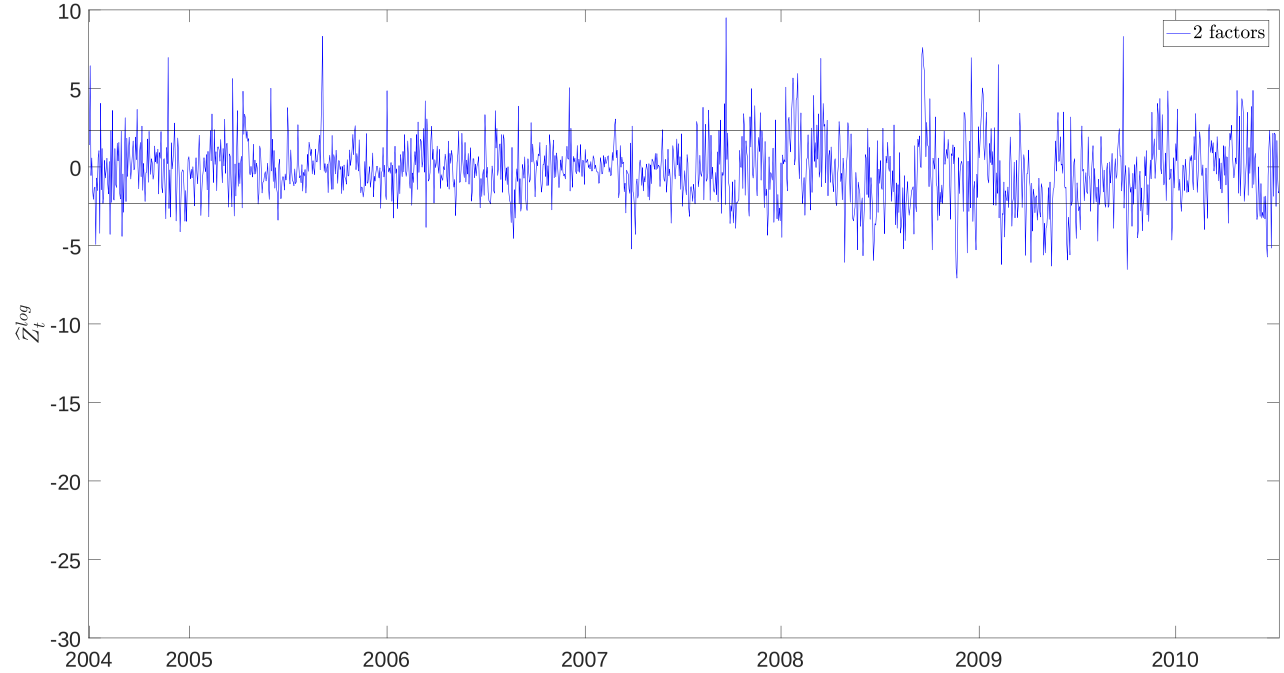


The plot shows the time series of the test statistics

$$\widehat{Z}_t^{log} = \frac{\log \widehat{V}_t^{\Psi,K_n,n} - \hat{\alpha}_{0,log}^{\tau_f} - \hat{\alpha}_{1,log}^{\tau_f} \log \widehat{SW}_t^{\tau_1}}{\sqrt{\frac{2}{K_n} + \Delta_K (\hat{\alpha}_1^{\tau_f})^2 \widehat{\Pi}_t^{N_K,1}}},$$

where $(\hat{\alpha}_{0,log}, \hat{\alpha}_{1,log})^\top = argmin_{\alpha_{j,log}^{\tau_f};j=0,1} \sum_{t=1}^{T} \left(\log \widehat{V}_t^{\Psi,K_n,n} - \alpha_{0,log}^{\tau_f} - \alpha_{1,log}^{\tau_f} \log \widehat{SW}_t^{\tau_1} \right)^2$ and $\widehat{\Pi}_t^{N_K,1}$ is given as in Theorem Theorem 4.4. $' log\widehat{SW}_t^{\tau_1}$ represents logarithm of the variance swap rate on the 6–month implied discount rate and it is synthesized from Eurodollar options via a transformation in a model–free way every day in the sample and $\log \hat{V}_t^{\Psi,K_n,n}$ is the logarithm of the spot variance estimate for the implied discount rate with 6–month maturity obtained from high frequency data non–parametrically as in Eqn. (3.24). The test statistic $\widehat{Z}_t^{log}$ is constructed every day by using rolling windows of length 120 days. The trading period over a day is 6 hours 40 mins which leads to n=40. K_n is fixed to 30. The sample covers the period from January 1, 2004 to July 13, 2010.

Figure 6: **Test Statistic for the Logarithmic Affine Variance Spanning Condition with $M = 1$ factor**



The plot shows the time series of the test statistics

$$\widehat{Z}_t^{log} = \frac{\log \widehat{V}_t^{\Psi,K_n,n} - \hat{\alpha}_{0,log}^{\tau_f} - \hat{\alpha}_{1,log}^{\tau_f} \log \widehat{SW_t}^{\tau_1} - -\hat{\alpha}_{2,log}^{\tau_f} \log \widehat{SW_t}^{\tau_2}}{\sqrt{\frac{2}{K_n} + \Delta_K (\hat{\alpha}_1^{\tau_f})^2 \widehat{\Pi}_t^{N_K,1} + \Delta_K (\hat{\alpha}_2^{\tau_f})^2 \widehat{\Pi}_t^{N_K,2}}},$$

where $(\hat{\alpha}_{0,log}, \hat{\alpha}_{1,log}, \hat{\alpha}_{2,log})^\top = argmin_{\alpha_{j,log}^{\tau_f};j=0,1} \sum_{t=1}^{T} \left(\log \widehat{V}_t^{\Psi,K_n,n} - \alpha_{0,log}^{\tau_f} - \alpha_{1,log}^{\tau_f} \log \widehat{SW_t}^{\tau_1} - -\alpha_{2,log}^{\tau_f} \log \widehat{SW_t}^{\tau_2} \right)^2$ and $\widehat{\Pi}_t^{N_K,j}$ is given as in Theorem Theorem 4.4. $\log \widehat{SW_t}^{\tau_1}$ and $\log \widehat{SW_t}^{\tau_2}$ represents logarithm of the variance swap rate on the 6–month implied discount rate and on the 18–month implied discount rate respectively and it is synthesized from Eurodollar options via a transformation in a model–free way every day in the sample and $\log \hat{V}_t^{\Psi,K_n,n}$ is the logarithm of the spot variance estimate for the implied discount rate with 6–month maturity obtained from high frequency data non–parametrically as in Eqn. (3.24). The test statistic $\widehat{Z}_t^{log}$ is constructed every day by using rolling windows of length 120 days. The trading period over a day is 6 hours 40 mins which leads to n=40. K_n is fixed to 30. The sample covers the period from January 1, 2004 to July 13, 2010.

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A. Proof for the First Moment of $\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du$ - Equation (2.18)

$$
\begin{aligned}
\mathrm{E}^{\mathbb{Q}}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du \,\middle|\, \mathcal{F}_t\right] &= \int_t^{t+\tau_v} \mathrm{E}^{\mathbb{Q}}\left[V_u^{\Psi^{\tau_f}} \middle| \mathcal{F}_t\right] du \\
&= \int_t^{t+\tau_v} \mathrm{E}^{\mathbb{Q}}\left[\Phi_0^{\tau_f} + \Phi^{\tau_f\top} X_t^G \middle| \mathcal{F}_t\right] du \\
&= \Phi_0^{\tau_f}(\tau_v) + \left(\int_t^{t+\tau_v} \Phi^{\tau_f\top}\left[\Theta^G\left(I - e^{-\mathcal{K}^G(u-t)}\right) + e^{-\mathcal{K}^G(u-t)} X_t^G\right] du\right).
\end{aligned}
$$

$$
\begin{aligned}
SW_t^{\tau_v} &\equiv \mathrm{E}^{\mathbb{Q}}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du \,\middle|\, \mathcal{F}_t\right] \\
&= \Phi_0^{\tau_f}(\tau_v) + \int_t^{t+\tau_v} \Phi^{\tau_f\top} \mathrm{E}^{\mathbb{Q}}\left[X_u^G \,\middle|\, \mathcal{F}_t\right] du.
\end{aligned}
$$

Transform X_u such that K^G is diagonal: $U^{-1}X_u^G = Y_u^G$, $UK^GU^{-1} = D$, $U^{-1}\Theta^G = \Theta^{DG}$.

Then,

$$
\begin{aligned}
\mathrm{E}^{\mathbb{Q}}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du \,\middle|\, \mathcal{F}_t\right] &= \Phi_0^{\tau_f}\tau_v + \int_t^{t+\tau_v} \Phi_f^{\tau\top} U U^{-1} \mathrm{E}^{\mathbb{Q}}\left[X_u^G \,\middle|\, \mathcal{F}_t\right] du \\
&= \Phi_0^{\tau_f}\tau_v + \int_t^{t+\tau_v} \Phi^{\tau_f\top} U \mathrm{E}^{\mathbb{Q}}\left[Y_u^G \,\middle|\, \mathcal{F}_t\right] du \\
&= \Phi_0^{\tau_f}\tau_v + \int_t^{t+\tau_v} \Phi^{\tau_f\top} U\left[(I - e^{-D(u-t)})\Theta^{DG} + e^{-D(u-t)} Y_t^G\right] du \\
&= \Phi_0^{\tau_f}\tau_v + \Phi^{\tau_f} U \Theta^{DG}\tau_v - \Phi^{\tau_f} U\left(D^{-1} - D^{-1}e^{-D\tau_v}\right)\Theta^{DG} \\
&\quad + \Phi^{\tau_f} U\left(D^{-1} - D^{-1}e^{-D\tau_v}\right) Y_t^G \\
&= \Phi_0^{\tau_f} + \Phi^{\tau_f}\Theta^G\tau_v - \Phi^{\tau_f}\left(\mathcal{K}^{G^{-1}} - \mathcal{K}^{G^{-1}} e^{-\mathcal{K}^G\tau_v}\right)\Theta^G \\
&\quad + \Phi^{\tau_f}\left(\mathcal{K}^{G^{-1}} - \mathcal{K}^{G^{-1}} e^{-\mathcal{K}^G\tau_v}\right) X_t^G \\
&\equiv \Lambda_0^{\tau_f,\tau_v} + \Lambda_1^{\tau_f,\tau_v} X_t^G.
\end{aligned}
$$

B. Transformations of Option Prices

In this section, I derive the option prices written on $\Psi_t^{\tau_f}$ by transforming the option prices written on Eurodollar futures prices, $F_t^{\tau_L}(\tau_f)$. In particular, I will mainly make use of the transformation

$$\Psi_t^{\tau_f} = 1 + \tau_L - \frac{\tau_L}{100} F_t^{\tau_L}(\tau_f). \tag{B.1}$$

In particular, the price of a put option written on Ψ^{τ_f} is given by

$$\begin{aligned}
Put_t(T-t, K_\Psi) &= \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left(K_\Psi - \Psi_T^{\tau_f}\right)^+\right] \\
&= \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left(\frac{\tau_L F_T^{\tau_L}(\tau_f)}{100} - \frac{\tau_L K_F}{100}\right)^+\right] \\
&= \frac{\tau_L}{100}\mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left(F_T^{\tau_L}(\tau_f) - K_F\right)^+\right] \\
&= \frac{\tau_L}{100} Call_t(T-t, K_F),
\end{aligned} \tag{B.2}$$

where the second equality follows from substituting (B.1).

Similarly, the price of a call option written on Ψ^{τ_f} is

$$\begin{aligned}
Call_t(T-t, K_\Psi) &= \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left(\Psi_T^{\tau_f} - K_\Psi\right)^+\right] \\
&= \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left(\frac{\tau_L K_F}{100} - \frac{\tau_L F_T^{\tau_L}(\tau_f)}{100}\right)^+\right] \\
&= \frac{\tau_L}{100}\mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left(K_F - F_T^{\tau_L}(\tau_f)\right)^+\right] \\
&= \frac{\tau_L}{100} Put_t(T-t, K_F),
\end{aligned} \tag{B.3}$$

where the second equality follows from substituting (B.1). Moreover, by implementing change of variables via (B.1), I obtain

$$\begin{aligned}
\int_0^{\Psi_t^{\tau_f}} \frac{Put_t(T-t, K_\Psi)}{K_\Psi^2} dK_\Psi &= \frac{\tau_L}{100}\int_{K_{F_L}}^{K_{F_U}} \frac{Call_t(T-t, K_F)}{\left(1 + \tau_L - \frac{\tau_L}{100}K_F\right)^2} \times \frac{-\tau_L}{100} dK_F \\
&= \frac{\tau_L^2}{100^2}\int_{K_{F_L}}^{K_{F_U}} \frac{Call_t(T-t, K_F)}{\left(1 + \tau_L - \frac{\tau_L}{100}K_F\right)^2} dK_F.
\end{aligned} \tag{B.4}$$

where $K_{F_U} = 100 \times \left(\frac{1+\tau_L}{\tau_L}\right)$ and $K_{F_L} = 100 \times \left(\frac{1+\tau_L - \Psi_t^{\tau_f}}{\tau_L}\right) = F_t^{ED}(\tau_f)$.

Similarly,

$$\int_{\Psi_t^{\tau_f}}^{\infty} \frac{Call_t(T-t, K_\Psi)}{K_\Psi^2} dK_\Psi = \frac{\tau_L^2}{100^2}\int_{-\infty}^{F_t^{\tau_L}(\tau_f)} \frac{Put_t(T-t, K_F)}{\left(1 + \tau_L - \frac{\tau_L}{100}K_F\right)^2} dK_F, \tag{B.5}$$

where the equality follows from substituting (B.1).

C. Replication of variance swap rates from options

In this section, I provide the theory underlying the model–free construction of variance swap rates on $\Psi_t^{\tau_f}$. There are two steps involved. First one is to link the price of the integrated variance to the price of the log contract. The second step is to replicate the price of this contract in a model–free way with a position in a portfolio of options.

C.1. Theory

Consider the variance swap with the following payoff at maturity over the period from t to T,

$$\int_t^T V_u^{\Psi^{\tau_f}} du - SW_t^{T-t}, \tag{C.6}$$

where SW_t^{T-t} is the fixed variance swap rate determined at time t and it is the value for which the contract has zero value at initiation. Under the assumption that the short rate is uncorrelated with the integrated variance,[11] the variance swap rate is given by

$$SW_t^{T-t} = \mathrm{E}_t^{\mathbb{Q}} \left[\int_t^T V_u^{\Psi^{\tau_f}} du \right]. \tag{C.7}$$

From here, we can infer SW_t^{T-t} in a model–free way in a few steps only.

First apply Itô's lemma to $ln\Psi^{\tau_f}$ and obtain

$$ln\frac{\Psi_T^{\tau_f}}{\Psi_t^{\tau_f}} = \int_t^T \frac{d\Psi_t^{\tau_f}}{\Psi_t^{\tau_f}} - \frac{1}{2}\int_t^T V_u^{\Psi^{\tau_f}} du. \tag{C.8}$$

Following the results in Carr and Madan (2009), I apply a Taylor expansion of $ln\Psi_T^{\tau_f}$ about the point $\Psi_t^{\tau_f}$ and obtain the spanning result,

$$\begin{aligned} ln\Psi_T^{\tau_f} &= ln\Psi_t^{\tau_f} + \frac{\Psi_T^{\tau_f} - \Psi_t^{\tau_f}}{\Psi_t^{\tau_f}} - \int_0^{\Psi_t^{\tau_f}} \frac{(K_\Psi - \Psi_T^{\tau_f})^+}{K_\Psi^2} dK_\Psi \\ &- \int_{\Psi_t^{\tau_f}}^{\infty} \frac{(\Psi_T^{\tau_f} - K_\Psi)^+}{K_\Psi^2} dK_\Psi. \end{aligned} \tag{C.9}$$

[11] See the literature on unspanned stochastic volatility

Combining Eq. (C.8) and Eq. (C.9), we have

$$\begin{aligned}\frac{1}{2}\int_t^T V_u^{\Psi^{\tau_f}} du &= \int_0^{\Psi_t^{\tau_f}} \frac{(K_\Psi - \Psi_T^{\tau_f})^+}{K_\Psi^2} dK_\Psi + \int_{\Psi_t^{\tau_f}}^{\infty} \frac{(\Psi_T^{\tau_f} - K_\Psi)^+}{K_\Psi^2} dK_\Psi \\ &+ \int_t^{t+T} \left[\frac{1}{\Psi_u^{\tau_f}} - \frac{1}{\Psi_t^{\tau_f}}\right] d\Psi_u^{\tau_f}. \end{aligned} \qquad (C.10)$$

Multiplying both parts by $e^{-\int_t^T r_s ds}$ and taking expectations under $\mathbb{Q}$ yields

$$\begin{aligned}\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^T V_u^{\Psi^{\tau_f}} du\right] &= \frac{2}{P_t\,(T-t)} \int_0^{\Psi_t^{\tau_f}} \frac{Put_t(T-t, K_\Psi)}{K_{\Psi^2}} dK_\Psi \qquad (C.11) \\ &+ \frac{2}{P_t\,(T-t)} \int_{\Psi_t^{\tau_f}}^{\infty} \frac{Call_t(T-t, K_\Psi)}{K_{\Psi^2}} dK_\Psi \\ &+ \frac{2}{P_t\,(T-t)} \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left\{\int_t^T \left[\frac{1}{\Psi_u^{\tau_f}} - \frac{1}{\Psi_t^{\tau_f}}\right] d\Psi_u^{\tau_f}.\right\}\right],\end{aligned}$$

where $Put_t(T-t, K_\Psi)$ and $Call_t(T-t, K_\Psi)$ are time t prices of the out–of–the–money European put and call options with strike K_Ψ and with expiry at time T, written on the simple implied three–month rate $\Psi_t^{\tau_f}$. The last component is the discounted risk–neutral expectation of the total return from a dynamic trading strategy holding $2\left[\frac{1}{\Psi_u^{\tau_f}} - \frac{1}{\Psi_t^{\tau_f}}\right]$ in Ψ^{τ_f} at time u, scaled by the inverse of the price of the zero–coupon bond, $P_t\,(T-t)$. I document, with simulations based on empirically relevant parameter values for the Eurodollar futures market, that the effect of the last part is very small with approx. 3%. Hence, I exclude this term in the implementation and obtain the variance swap rate in a model–free fashion by a portfolio of out–of–the–money put and call options.

C.2. Monte Carlo Analyzes

The model–free, risk–neutral expectation of the integrated variance in Eq. (C.11) relies on the validity of the assumptions that a) the short rate is uncorrelated with the integrated variance and b) that the last term in Eq. (C.11) is sufficiently small. In this section, I provide simulation evidence for these two items.

As the benchmark model for our Monte Carlo experiment I choose the one–stochastic volatility three–factor, $A_1(3)$, model with parameters from Bikbov and Chernov (2009, their Table 1) estimated via maximum likelihood methods based on Eurodollar futures and options market data.

Assume that X follows affine dynamics under both the $\mathbb{P}$ and $\mathbb{Q}$ measure, and that the short rate is affine in X. Note that I apply the change of measure to the short rate dynamics under the physical measure and use their estimates of the prices of risk and obtain the model under the risk–neutral measure.

I simulate the dynamics of Ψ^{τ_f} for one year with a 360 day count convention of. I present our results for three–month integrated variance. Table **??** exhibits the simulation results for the components of Eq. (C.11)

- $\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^T V_u^{\Psi^{\tau_f}} du\right]$,
- $MA_{\mathbb{Q}^F}{=}\frac{2}{P_t(T-t)}\mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left\{\int_t^T\left[\frac{1}{\Psi_u^{\tau_f}}-\frac{1}{\Psi_t^{\tau_f}}\right]d\Psi_u^{\tau_f}\right\}\right]$,

as well as

- $\mathrm{E}_t^{\mathbb{Q}^F}\left[\int_t^T V_u^{\Psi^{\tau_f}} du\right] = \frac{1}{P_t(T)}\mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_u^{\Psi^{\tau_f}} du\right]$,
- bias in IV= $(\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^T V_u^{\Psi^{\tau_f}} du\right] - \mathrm{E}_t^{\mathbb{Q}^F}\left[\int_t^T V_u^{\Psi^{\tau_f}} du\right])/\mathrm{E}_t^{\mathbb{Q}^F}\left[\int_t^T V_u^{\Psi^{\tau_f}} du\right])$,
- $MA_{\mathbb{Q}^F}/\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^T V_u^{\Psi^{\tau_f}} du\right]$.

In particular, for each quantity above, I present the average estimate over $S = 50,000$ simulations runs. First, note that the difference between $\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^T V_u^{\Psi^{\tau_f}} du\right]$ and $\mathrm{E}_t^{\mathbb{Q}^F}\left[\int_t^T V_u^{\Psi^{\tau_f}} du\right]$ is very small with a percentage difference of 1.82%, which is in line with our assumption that $e^{-\int_t^T r_s ds}$ and $\int_t^T V_u^{\Psi^{\tau_f}} du$ are uncorrelated. Remember that these two quantities are equal when $e^{-\int_t^T r_s ds}$ and $\int_t^T V_u^{\Psi^{\tau_f}} du$ are uncorrelated.

Moreover, observe that the effect of the last component in Eq. (C.11) is small in variance swap rate, with $MA_{\mathbb{Q}^F}/\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^T V_u^{\Psi^{\tau_f}} du\right]$ being approx. 3%. Accordingly, I exclude this model dependent component in our construction of variance swap rate and obtain the variance swap rate in a model–free way based on Eurodollar options.

D. Assumptions and Proofs of Section 3

Assumption 1. *The process $\Psi_t^{\tau^f}$ in Equation (4.30) satisfies:*

- $\mu_t^{\Psi^{\tau_f}}$ *and* $\sigma_t^{\Psi^{tau_f}}$ *are locally bounded.*
- *There exists a sequence of stopping times* T_p *increasing to* ∞*. For each integer* p *there exists a bounded process* $V_t^{\Psi^{\tau_f}} = V_t^{(p)}$ *for all* $t \leq T_p$*, and satisfying* $E\left[|V_t^{(p)} - V_s^{(p)}|^2\right] |\mathcal{F}_s^{(0)} \leq K_p|t-s|$ *for* $s < t$ *and* K_p *is a positive constant.*

Assumption 2. *The measurement error process* $\varepsilon_t^{\tau,\mathcal{K}_\Psi}$ *satisfies the following conditions:*

- $E\left[\varepsilon_t^{\tau,\mathcal{K}_\Psi}|\mathcal{F}^{(0)}\right] = 0,$
- $E\left[\varepsilon_t^{\tau,\mathcal{K}_\Psi\,2}|\mathcal{F}^{(0)}\right] = \eta_{t,K_\Psi,\tau},$
- *For* $t \neq t'$, $\tau \neq \tau'$ *and* $\mathcal{K}_\Psi \neq \mathcal{K}_\Psi{}'$, $\varepsilon_t^{\tau,\mathcal{K}_\Psi}$ *and* $\varepsilon_{t'}^{\tau',\mathcal{K}_\Psi{}'}$ *are independent conditionally on* $\mathcal{F}^{(0)}$,
- *Conditional on* $\mathcal{F}^{(0)}$, $\varepsilon_t^{\tau,\mathcal{K}_\Psi}$ *has a finite 4th moment almost surely.*

Proof of Theorem 4.2. First I will show that the following Lemma holds.

Lemma D.1. *Under the conditions of Theorem 4.2, we have*

$$\frac{1}{\sqrt{\Delta_K}}\left(\widehat{SW}_t^{\tau_v} - SW_t^{\tau_v,m,M}\right) \xrightarrow{\mathcal{L}-s} \mathcal{W}_t^{N_K}\mathcal{E},$$

where $\mathcal{E}$ *is an* $\mathcal{N}(0,1)$ *variable.*

Proof. Observe that

$$\begin{aligned}
\widehat{SW}_t^{\tau_v} - SW_t^{\tau_v,m,M} &= \frac{2}{\tau_v}\sum_i^{N_K}\frac{\tilde{\mathcal{O}}_t(\tau_v,K_\Psi^i)}{P_t(\tau_v)K_\Psi^{i\,2}}\Delta K_\Psi - \frac{2}{\tau_v}\int_{K_m(\tau_v)}^{K_M(\tau_v)}\frac{\mathcal{O}_t(\tau_v,K_\Psi)}{P_t(\tau_v)\mathcal{K}_\Psi{}^2}dK_\Psi \\
&= \frac{2}{\tau_v}\sum_i\frac{O_t(\tau_v,K_\Psi^i)}{P_t(\tau_v)K_\Psi^{i\,2}}\Delta K_\Psi - \frac{2}{\tau_v}\int_{K_m(\tau_v)}^{K_M(\tau_v)}\frac{\mathcal{O}_t(\tau_v,K_\Psi)}{P_t(\tau_v)\mathcal{K}_\Psi{}^2}dK_\Psi \\
&+ \frac{2}{\tau_v P_t(\tau_v)}\sum_i^{N_K}\frac{\varepsilon_t^{\tau_v,K_\Psi^i}}{K_\Psi^{i\,2}}\Delta_K.
\end{aligned}$$

As the grid between strikes decreases, for the first part we trivially have

$$\frac{2}{\tau_v}\sum_i\frac{O_t(\tau_v,K_\Psi^i)}{P_t(\tau_v)K_\Psi^{i\,2}}\Delta_K \to \frac{2}{\tau_v}\int_{K_m(\tau_v)}^{K_M(\tau_v)}\frac{\mathcal{O}_t(\tau_v,K_\Psi)}{P_t(\tau_v)\mathcal{K}_\Psi{}^2}dK_\Psi.$$

The consistency of the second term is trivial under Assumption 2.
As the grid between strikes decreases,, we have by central limit theorem

$$\frac{1}{\sqrt{\Delta_K}}\left(\frac{2}{\tau_v P_t(\tau_v)}\sum_i^{N_K}\frac{\varepsilon_t^{\tau_v,K_\Psi^{i\,2}}}{K_\Psi^i}\Delta_K\right) \xrightarrow{\mathcal{L}-s} \mathcal{W}_t^{N_K}\mathcal{N}(0,1), \tag{D.12}$$

Under Assumption 2, we have as the grid size between strikes decreases to zero $\Delta_K \to 0$

$$\mathrm{E}\left[\frac{2}{\tau_v P_t(\tau_v)} \sum_{i}^{N_K} \frac{\varepsilon_t^{\tau_v, K_\Psi^i \, 2}}{K_\Psi^i} \sqrt{\Delta_K} \;\middle|\; \mathcal{F}^{(0)}\right] \to 0.$$

and

$$\begin{aligned}
&\mathrm{E}\left[\frac{4}{\tau_v{}^2 P_t(\tau_v)^2} \sum_{i}^{N_K} \frac{\varepsilon_t^{\tau_v, K_\Psi^i}}{K_\Psi^{i\;2}} \sqrt{\Delta_K} \sum_{i}^{N_K} \frac{\varepsilon_t^{\tau_v, K_\Psi^i}}{K_\Psi^{i\;2}} \sqrt{\Delta_K} \;\middle|\; \mathcal{F}^{(0)}\right] \\
&= \mathrm{E}\left[\frac{4}{\tau_v{}^2 P_t(\tau_v)^2} \sum_{i}^{N_K} \frac{\varepsilon_t^{\tau_v, K_\Psi^i \, 2}}{K_\Psi^{i\;4}} \Delta_K \;\middle|\; \mathcal{F}^{(0)}\right] \to \mathcal{W}_t^{N_K},
\end{aligned}$$

where $\mathcal{W}_t^{N_K} = \frac{4}{\tau_v{}^2 P_t(\tau_v)^2} \int_{K_m(\tau_v)}^{K_M(\tau_v)} \frac{\eta_{t,K_\Psi,\tau}}{K_\Psi^4} dK_\Psi$. □

Morover, given that

$$\widehat{\eta}_{t,K_\Psi,\tau_v} \to \eta_{t,K_\Psi,\tau_v}, \tag{D.13}$$

we trivially have,

$$\widehat{\mathcal{W}}_t^{N_K} = \frac{4}{P_t(\tau_v)^2} \sum_{j=1}^{N_K^{\tau_v}} \frac{\widehat{\eta}_{t,K_j,\tau_v}}{K_\Psi^{j\;4}} \Delta_K \to \mathcal{W}_t^{N_K}. \tag{D.14}$$

in probability, which completes the proof of Theorem 4.2. □

Table 1: **Monte Carlo Simulations**

This table illustrates simulation results for the components of. (C.11) based on the empirically relevant parameters obtained from Bikbov and Chernov (2009) for Eurodollar futures and options market. $MA_{\mathbb{Q}^F} = \frac{2}{P_t(T-t)} \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left\{\int_t^T \left[\frac{1}{\Psi_u^{\tau_f}} - \frac{1}{\Psi_t^{\tau_f}}\right] d\Psi_u^{\tau_f}\right\}\right]$ and bias in IV= $(\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du\right] - \mathrm{E}_t^{\mathbb{Q}^F}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du\right]) / \mathrm{E}_t^{\mathbb{Q}^F}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du\right])$. (T-t) represents three months and τ_f and τ_d are also fixed to three months. The results are based on S=50000 simulations.

$\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du\right]$	1.26E-06
$\mathrm{E}_t^{\mathbb{Q}^F}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du\right]$	1.24E-06
bias in IV	1.18E-02
$MA_{\mathbb{Q}^F}$	-3.89E-08
$MA_{Q^F}/\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^{t+\tau_v} V_u^{\Psi^{\tau_f}} du\right]$	-3.09E-02

E. Test Statistics for Various Maturities

Figure 7: **Test Statistic for Affine Variance Spanning Condition - $M = 2$ Factors**

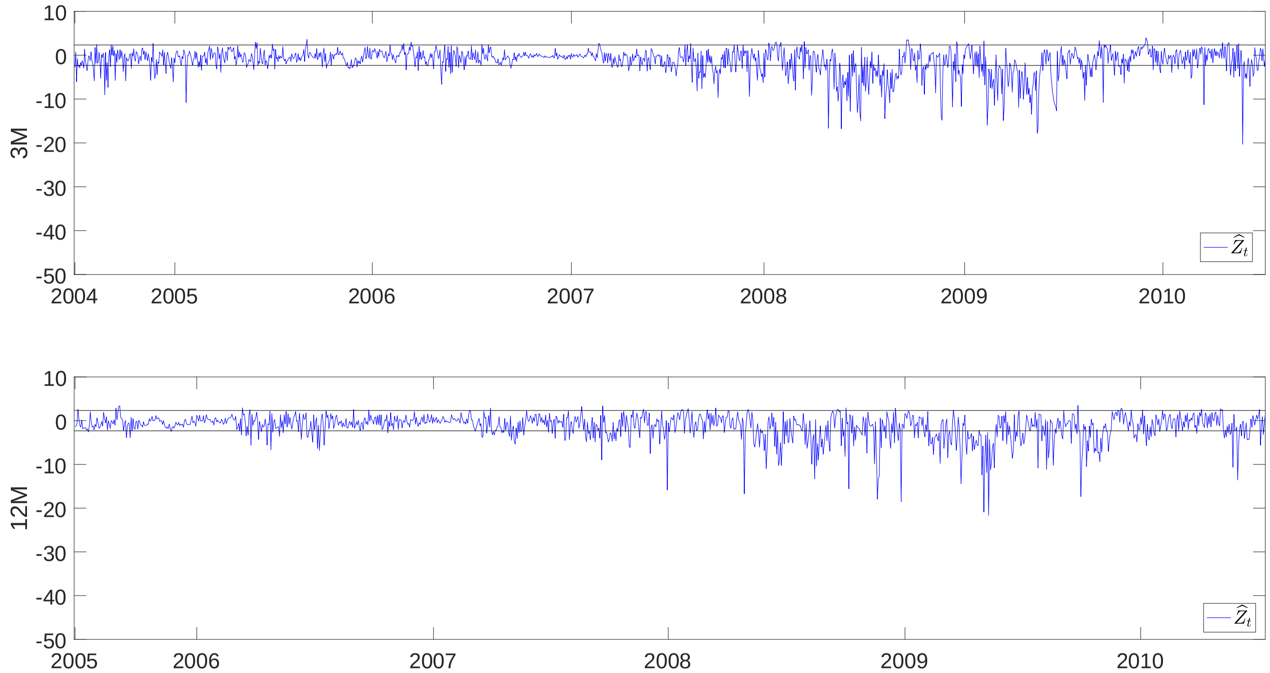
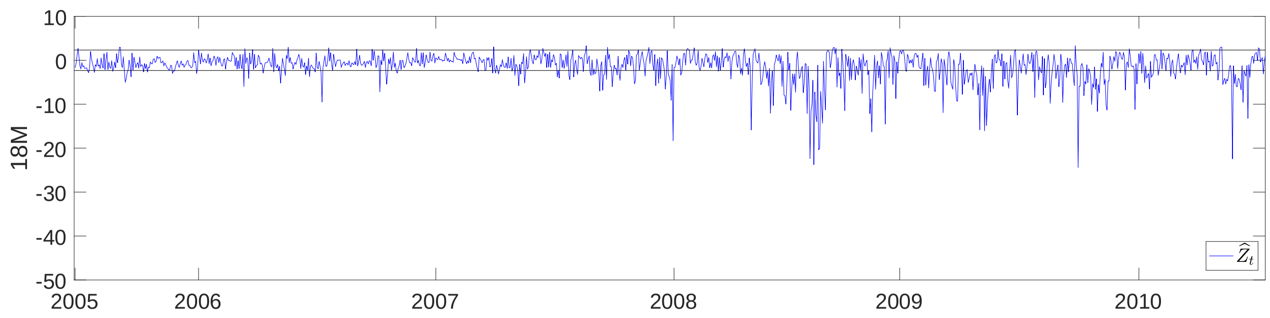


Figure 8: **Test Statistic for Affine Variance Spanning Condition -** $M = 2$ **Factors**



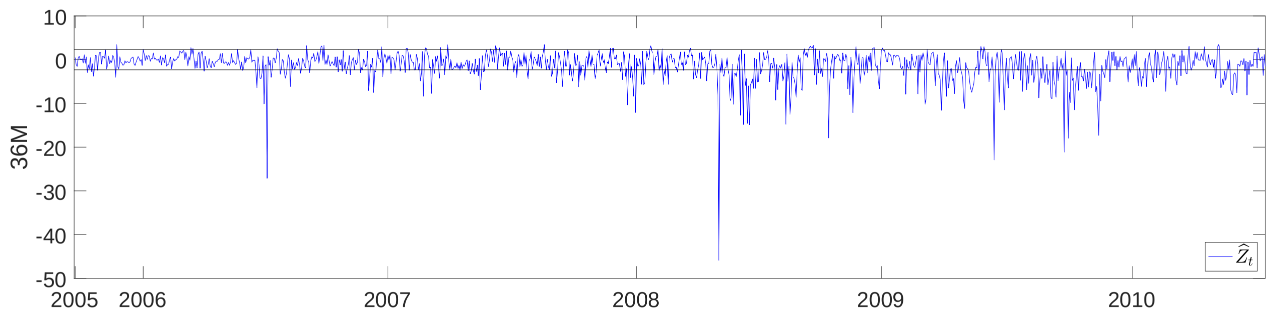


Figure 9: **Test Statistic for Logarithmic Affine Variance Spanning Condition -** $M = 2$ **Factors**

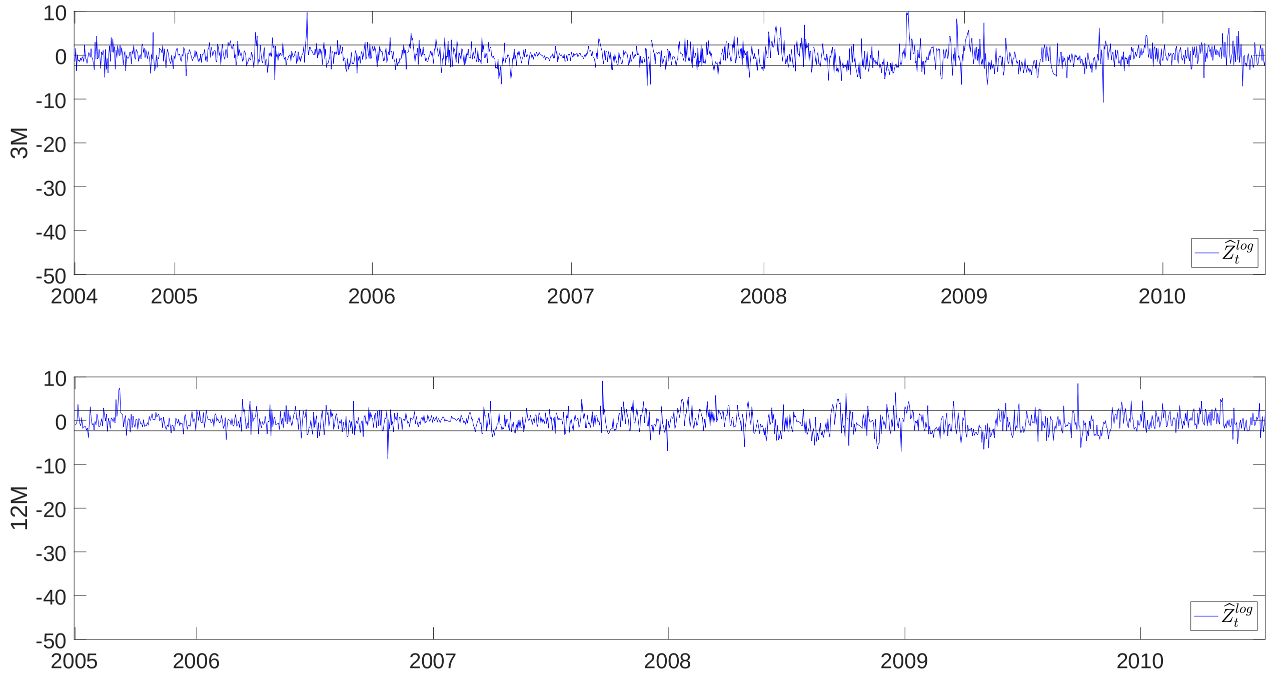


Figure 10: **Test Statistic for Logaritmic Affine Variance Spanning Condition -** $M = 2$ **Factors**