CAPITAL SUPPLY UNCERTAINTY, CASH HOLDINGS, AND INVESTMENT*

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Abstract

We develop a model of investment, financing, and cash management decisions in which firms face capital supply uncertainty and have to search for investors when in need of capital. We characterize optimal policies explicitly and show that the smooth-pasting conditions used in prior contributions are necessary, but may not be sufficient, for an optimum. Instead of the standard Miller and Orr (1966) double-barrier policy for financing and payout, firms may optimally raise outside funds before exhausting internal resources and the optimal payout policy may feature several payout regions, with both smooth and discrete dividend payments. In the model, firms with high investment costs are qualitatively as well as quantitatively different in their behaviors from firms with low investment costs. Finally, investment and payout do not always increase with slack, challenging the use of investment-cash flow sensitivities or payout ratios as measures of financing constraints.

Keywords: Capital supply uncertainty; cash management; lumpy investment; inventory models.

JEL Classification Numbers: D83; G24; G31; G32; G35.

Introduction

Following Modigliani and Miller (1958), standard models of investment decisions under uncertainty assume that capital markets are frictionless so that firms are always able to secure funding for positive net present value projects and cash reserves are irrelevant.[1] This traditional view has recently been called into question by a large number of empirical studies.[2] These studies show that firms often face uncertainty regarding their future access to capital markets and that this uncertainty has important feedback effects on corporate decisions. They also reveal that the resulting liquidity risk has led firms to accumulate large amounts of cash, with an average cash-to-assets ratio for U.S. industrial firms that has increased from 10.5% in 1980 to 23.2% in 2006 (see Bates, Kahle, and Stulz, BKS 2009).

While it may be clear to most economists that capital supply frictions can affect corporate policies, it is much less clear exactly how they do so. In this paper, we develop a dynamic model of cash management, financing, and investment decisions in which the Modigliani and Miller assumption of infinitely elastic supply of capital is relaxed and firms have to search for investors when in need of funds. With this model, we seek to understand when and how capital supply uncertainty affects real investment. We are also interested in determining the effects of capital markets frictions on firms' financing and cash management policies, i.e. on the decision to pay out or retain earnings and the decision to issue securities.

Our paper makes three main contributions. First, we show that when capital supply is uncertain the optimal policy choices of the firm are in stark contrast with the theoretical predictions of canonical inventory models applied to liquid assets. In particular, a striking feature of the model is that it may not be optimal for firms to follow the standard Miller and Orr (1966) double-barrier policy. Second, our analysis of optimal policies breaks some new grounds on the mathematical analysis of inventory models. Notably, we demonstrate that the smooth-pasting conditions used to characterize optimal polices in prior contributions are necessary, but not sufficient, for an optimum and provide a full characterization of optimal polices. Third, we show that accounting for capital supply uncertainty and lumpy investment changes the predictions of models of financing constraints in ways that are consistent with stylized facts concerning firms' cash management behavior.

[1] See for example McDonald and Siegel (1986), Dixit (1989), Abel and Eberly (1996, 2011), Bar-Ilan and Strange (1996), Boyarchenko (2004), Guo et al. (2005), Manso (2008), or Carlson et al. (2010). Dixit and Pindyck (1994) and Stokey (2009) provide excellent surveys of this literature.

[2] See for example, Gan (2007), Becker (2007), Massa, Yasuda, and Zhang (2010), Duchin, Ozbas, and Sensoy (2010), Campello, Graham, and Harvey (2010), and Lemmon and Roberts (2010). See also the early contributions of Kashyap, Stein and Wilcox (1993) and Kashyap, Lamont, and Stein (1994).

In order to aid in the intuition of the model, consider the following two settings in which capital supply uncertainty and search frictions are likely to be especially important:[3]

1. *Public equity offerings and capital injections for private firms*: Firms first sell their equity to the public through an initial public offering (IPO). One of the main features of IPOs is the book building process, whereby the lead underwriter and firm management search for investors until it is unlikely that the issue will fail. Yet, the risk of failure is often not eliminated and a number of IPOs are withdrawn every year. For example, Busaba, Benveniste, and Guo (2001) show that between the mid-1980s and mid-1990s almost one in five IPOs was withdrawn. Evidence from more recent periods suggests that this fraction has increased to over one in two in some years (see Dunbar and Foerster, 2008). Search frictions are also important for firms that remain private but need new capital injections. Indeed, when a private company decides to raise private equity capital, it must search for investors such as angel investors, venture capital firms, or institutional investors. Even when initial investors are found, the firm will need to search for new investors in every subsequent financing round.

2. *Financial crises and economic downturns*: Search frictions are also important for large, publicly traded firms when capital becomes scarce, e.g. during a financial crisis or an economic downturn. The recent global financial crisis has provided a crisp illustration of the potential effects of capital supply (or liquidity) dry ups on corporate behavior, with a number of studies (e.g. Duchin, Ozbas, and Sensoy (2010)) documenting a significant decline in corporate investment following the onset of the crisis (controlling for firm fixed effects and time variation in investment opportunities). A survey of 1,050 CFOs by Campello, Graham, and Harvey (2010) also indicates that the contraction in capital supply during the recent financial crisis led firms to burn through more cash to fund their operations and to bypass attractive investment opportunities.

A prerequisite for our study is a model that captures in a simple fashion the effects of frictions on firms' policy choices. In this paper, we base our analysis on a dynamic inventory model in which firms' investment, cash management and financing strategies are jointly and endogenously determined. Specifically, we consider a firm with assets in place that generate stochastic cash flows as well as a finite number of opportunities to expand operations (i.e. growth options). In the model, the firm faces two types of frictions: capital supply frictions and lumpiness in investment. Notably, we consider that the firm has to

[3] We thank Darrell Duffie for suggesting these applications.

search for investors when in need of capital. Therefore, it faces uncertainty regarding its future ability to raise funds. In addition, we assume that the firm has to pay a lump-sum cost when investing. Indeed, as noted in Caballero and Engel (1999), "minor upgrades and repairs aside, investment projects are intermittent and lumpy rather than smooth" (see also Doms and Dunne (1998) and Cooper, Haltiwanger, and Power (1999). The firm maximizes its value by making three interrelated decisions: How much cash to retain and pay out, when to invest, and whether to finance investment with internal or external funds. Using this model, we show that capital supply frictions lead firms to value financial slack. Second, and more importantly, we demonstrate that the interplay between lumpiness in investment and capital supply frictions implies that barrier policies for investment, payout, and financing decisions may not be optimal. We then use these results to shed light on existing empirical facts and to generate a rich set of testable predictions.

To understand the effects of capital markets frictions on corporate policies, consider first the case of a firm with assets in place but no growth option. For such a firm cash holdings only serve to cover potential operating losses and, thus, avoid inefficient closure. We show that in this case, the marginal value of cash is decreasing so that optimal policies are always of barrier type. Specifically, we show that there exists a target level for cash holdings such that (i) the optimal payout policy is to distribute dividends to maintain cash holdings at or below the target level, and (ii) when cash holdings are below the target, it is optimal to retain earnings and search for investors so as to increase cash holdings to the target.

Consider next a firm with both assets in place and growth options. For such a firm, cash holdings generally serve two purposes: Reducing the risk of inefficient closure and financing investment. Our analysis demonstrates that when the cost of investment is low (i.e. for "growth firms" with profitable options), it is again optimal to follow a barrier strategy whereby the firm retains earnings and invests if cash reserves reach some target level or upon obtaining outside funds. We show however that when the cost of investment is high (i.e. for "value firms"), it is no longer optimal to follow a barrier strategy. Notably, we find that when cash reserves are high, the firm optimally retains earnings and invests if cash reserves reach some target level or upon obtaining outside funds. However, if cash reserves go down to a critical level following a series of operating losses, the firm abandons the option of financing investment internally as it becomes too expensive to accumulate enough cash to invest. At this point, the marginal value of cash drops to one and it is optimal to make a lump-sum payment to shareholders. After making this payment, the firm retains earnings again but finances investment exclusively with outside funds. Importantly, we also show that these results do not depend on the number of growth options available to the firm or on the severity of financial frictions.

Our theory of investment with capital supply frictions differs from prior contributions in two important respects. First, with very few exceptions, the literature on inventory models applied to liquid assets assumes that the cumulated net cash consumption has continuous path and, therefore, does not allow for lumpy investment. Second, unlike prior contributions in which liquid assets holdings may be subject to jumps (see e.g. Alvarez and Lippi (2009, 2012) or Bar-Ilan, Perry, and Stadje (2004)), our paper provides a complete characterization of optimal decision rules without restricting the class of strategies available to the decision maker. These unique features allow us to provide several insights on corporate policies that do not arise from previous models.

We highlight the main implications. First, we show that capital supply frictions and lumpiness in investment can give rise to convexity of firm value and lead firms to follow financing and payout policies that differ from the Miller and Orr double barrier policy. Notably, firms may optimally raise funds before exhausting internal resources and the optimal payout policy may feature several payout regions, with both smooth and discrete dividend payments. We also demonstrate that constrained firms with low cash reserves will not finance investment internally and may decide to pay dividends early. By contrast, constrained firms with high cash reserves may finance investment internally and will retain earnings. Therefore, in our model investment and payout do not always increase with slack, challenging the use of investment-cash flow sensitivities or payout ratios as measures of financing constraints.

Second, we show that the choice between internal and external funds for financing investment does not follow a strict pecking order, in that any firm can use both internal and external funds to finance investment. We find however that firms usually wait until external financing arrives before investing, consistent with the studies of Opler, Pinkowitz, Stulz, and Williamson (OPSW, 1999), BKS (2009), and Lins, Servaes, and Tufano (2010). We also find that (i) the probability of investment with internal funds increases with asset tangibility and agency costs and decreases with cash flow volatility and market depth, and (ii) when financing investment with external funds, firms should increase their cash reserves, consistent with Kim and Weisbach (2008) and McLean (2010). Finally, we show that negative capital supply shocks should hamper investment even if firms have enough slack to finance investment, consistent with Gan (2007), Becker (2007), Lemmon and Roberts (2010).

The present paper relates to several strands of literature. First, it relates to the literature on inventory models applied to liquid assets. Classic contributions in this literature include

Baumol (1952), Miller and Orr (1966), and Tobin (1968). Recent contributions include Bar-Ilan (1990), Décamps, Mariotti, Rochet, and Villeneuve (2011), and Bolton, Chen, and Wang (2011). This literature generally assumes that liquid assets holdings have continuous paths. Notable exceptions are Alvarez and Lippi (2009, 2012) and Bar-Ilan, Perry, and Stadje (2004).[4] One important difference between our paper and these contributions is that jumps in our model correspond to endogenous investment or financing decisions. Another key difference is that these papers restrict their attention to barrier strategies and only derive necessary conditions for optimality within that class of strategies. By contrast, our paper provides a complete characterization of optimal decision rules and shows that barrier strategies may not be optimal when investment is lumpy.

Second, our paper relates to the large literature on investment under uncertainty, in which it is generally assumed that firms can instantaneously tap capital markets at no cost to finance investment (see footnote 1 and the references therein). In these models, there is no role for cash holdings, investment is financed exclusively with outside funds, and firms may raise funds infinitely many times to cover temporary losses.

Third, our paper relates to the literature on financing constraints. While early models in this literature ignore the role of cash holdings (see Gomes (2001)), recent contributions recognize that it may be optimal for firms to hoard cash when facing financing constraints (see Décamps, Mariotti, Rochet, and Villeneuve (2011) or Morellec, Nikolov, and Zucchi (2013)). In these recent contributions, firms can always access capital markets. Depending on whether the costs of external finance are high or low, firms either never raise funds or are never liquidated. In addition, when the cost of external finance is low, there is a strict inside/outside funds dominance in that firms only raise funds when their cash buffer is completely depleted. In stark contrast with these papers, the optimal policy in our model may not take the form of a double-barrier policy. In addition, firms raise external funds in discrete amounts on a regular basis and some firms can be liquidated even when issuance costs are low. Also, with firms can use both internal and external funds to finance investment in our model, investment is financed exclusively with internal funds in other contributions.

The remainder of the paper is organized as follows. Section 1 presents the model for a firm holding one growth option. Section 2 derives the firm's optimal financing, investment, and payout policies. Section 3 extends the model to finitely many growth options. Section 4 discusses the implications of the model. The proofs are gathered in the Appendix.

[4] Another exception is the study of Décamps and Villeneuve (2007), that examines the dividend policy of a firm that owns a single growth option and has no access to outside funds.

1 Model and assumptions

Throughout the paper, agents are risk neutral and discount cash flows at rate $\rho > 0$. Time is continuous and uncertainty is modelled by a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions. We start our analysis by solving a model in which the firm has assets in place and a single growth option and is not required to pay any search costs when looking for outside investors and issuance costs when raising funds. We show in Section 3 that our results naturally extend to the case where the firm has a finite number of growth options and pays both issuance or search costs.

We consider that before investment, assets in place generate a continuous stream of cash flows dX_t satisfying

$$dX_t = \mu_0 dt + \sigma dB_t,$$

where the process B_t is a Brownian motion and (μ_0, σ) are constant parameters representing the mean and volatility of cash flows. The growth option allows the firm to increase cash flows to $dX_t + (\mu_1 - \mu_0)\, dt$, where $\mu_1 > \mu_0$. The cost of investment is constant and denoted by K. The firm has full flexibility in the timing of investment.

Management acts in the best interest of shareholders and chooses not only the firm's investment policy but also its financing, payout, and liquidation policies. Notably, we allow management to retain earnings inside the firm and denote by C_t the amount of cash that the firm holds at any time $t \geq 0$, i.e. its cash buffer (in what follows, we use indifferently the terms cash buffer, cash holdings, and cash reserves). Cash holdings earn a constant rate of interest $r < \rho$ inside the firm and can be used to fund investment or to cover operating losses if other sources of funds are costly and/or unavailable. The wedge $\delta \equiv \rho - r > 0$ should be interpreted as a carry cost of cash.

The firm can increase its cash buffer either by retaining earnings or by raising funds in capital markets. A key difference between our setup and previous contributions is that we explicitly take into account capital supply frictions by considering that it takes time to secure outside funding and that capital supply is uncertain. Specifically, we assume that the firm needs to search for investors in order to raise funds and that, conditional on searching, it meets investors at the jump times of a Poisson process N_t with arrival rate $\lambda \geq 0$. Under these assumptions, the cash reserves of the firm evolve according to

$$dC_t = \left(rC_{t-} + \mu_0 + 1_{\{T \leq t\}}(\mu_1 - \mu_0)\right) dt + \sigma dB_t + f_t dN_t - dD_t - 1_{\{t=T\}}K,$$

where T is a stopping time representing the time of investment, f_t is a nonnegative predictable process representing the funds raised upon finding investors, and D_t is a non-decreasing adapted process with $D_{0-} = 0$ representing the cumulative dividends paid to shareholders. In our model, T, D, and f are endogenously determined.

Because capital supply is uncertain, new investors may be able to capture part of the surplus generated at refinancing dates. That is, we consider that once management and investors meet, they bargain over the terms of the new issue to determine the cost of capital or, equivalently, the proceeds from the stock issue. We assume that the allocation of the financing surplus between incumbent shareholders and new investors results from Nash bargaining. Denoting the bargaining power of new investors by $\eta \in [0, 1]$, we therefore have that the amount π^* that new investors can extract at financing dates satisfies

$$\pi^* = \underset{\pi \geq 0}{\operatorname{argmax}}\, \pi^\eta \left[\mathcal{S}V(c) - \pi\right]^{1-\eta} = \eta\, \mathcal{S}V(c).$$

where $V(c)$ gives the value of the firm as a function its cash holdings and the nonnegative operator

$$\mathcal{S}V(c) = \sup_{f \geq 0} \left(V(c + f) - f - V(c)\right)$$

gives the maximal financing surplus. This specification implies that whenever $\eta \neq 0$ issuance costs are stochastic and time varying. In the model, the bargaining power of new investors can be related to the supply of funds in capital markets by assuming for example that $\eta \equiv a/(a + \lambda)$ for some $a > 0$. In this case, the fraction of the surplus captured by outside investors and the cost of capital decrease with capital supply.

The firm can be liquidated if its cash buffer reaches zero following a series of negative cash flow shocks. Alternatively, it can choose to abandon its assets at any time by distributing all of its cash reserves. We consider that the liquidation value of the assets of a firm with mean cash flow rate μ_i is given by $\ell_i = \frac{\varphi \mu_i}{\rho}$ where the constant $1 - \varphi \in [0, 1]$ represents a haircut related to the partial irreversibility of investment. When this constant is one, investment is completely irreversible and the liquidation value of assets is zero. When this constant is zero, investment is costlessly reversible. In the analysis below, we denote by τ_0 the stochastic liquidation time, refer to φ as the tangibility of assets, and consider that $\varphi < 1$.

Because management can decide to pay a liquidating dividend at any point in time and capital supply is uncerain, liquidation is automatically triggered when the cash buffer reaches zero. The problem of management is therefore to maximize the present value of future dividends by choosing the firm's payout (D), financing (f), and investment (T) policies. That is, management solves:

$$V(c) = \sup_{(f,D,T)} E_c \left[\int_0^{\tau_0} e^{-\rho t} [dD_t - (f_t + \eta \mathcal{S} V(C_{t-})) dN_t] + e^{-\rho \tau_0} \left(\ell_0 + 1_{\{\tau_0 > T\}} (\ell_1 - \ell_0) \right) \right].$$

The first term in this expression represents the present value of payments to shareholders until liquidation net of the claim of new investors on future cash flows. The second term represents the present value of the cash flow to shareholders in liquidation.

Since firm value appears in the objective function via the surplus generated at refinancing dates, the above optimization problem is akin to a rational expectations problem: When bargaining over the terms of financing, outside investors have to correctly anticipate the strategy that the firm will use in the future. We show in the Appendix that introducing bargaining in the model and solving the corresponding rational expectations equilibrium is equivalent to reducing the arrival rate of investors from λ to $\lambda^* = \lambda(1-\eta)$ in an otherwise similar model where investors have no bargaining power.

2 Model solution

2.1 Value of the firm with no growth option

To facilitate the analysis of the optimization problem, we start by deriving the value $V_i(c)$ of a firm with mean cash flow rate μ_i and no growth option. When $i = 1$, this function also gives the value of the firm after the exercise of the growth option.

When there is no growth option, management only needs to determine the firm's payout, liquidation, and financing policies. In line with previous models in the literature (see e.g. Décamps et al. (2011) and the references therein), we conjecture and later verify that there exists some level C_i^* for the cash buffer below which the marginal value of cash is strictly higher than one and above which it is equal to one. Accordingly, the optimal payout policy should take the form of a barrier policy whereby the firm makes dividend payments to maintain its cash holdings at or below C_i^*. In addition, since there are no issuance costs other than those generated by the bargaining friction, we expect that below C_i^* it is optimal for the firm to search for new investors so as to increase its cash buffer back to the target. Finally, since the marginal value of cash is strictly higher than one below the target, our conjecture implies that the firm only liquidates if its cash holdings reach zero.

Denote by $v_i(c;b)$ the value of a firm that follows a barrier policy as above with target level $b \geq 0$. Standard arguments show that in the region $(0,b)$ where the firm retains earnings and searches for investors, $v_i(c;b)$ is twice continuously differentiable and satisfies

$$\rho v_i(c;b) = v_i'(c;b)(rc+\mu_i) + \frac{\sigma^2}{2} v_i''(c;b) + \lambda^* \left[v_i(b;b) - b + c - v_i(c;b)\right], \tag{1}$$

with $\lambda^* = \lambda(1-\eta)$ and the boundary condition

$$v_i(0;b) = \ell_i \tag{2}$$

at the point where the firm runs out of cash. The left-hand side of the differential equation (1) represents the required rate of return for investing in the firm. The first and second terms on the right hand side capture the effects of cash savings and of cash flow volatility. The third term reflects the effect of capital supply uncertainty. This last term is the product of the instantaneous probability of meeting investors and the fraction of the surplus that accrues to incumbent shareholders when raising the cash buffer to the given target level.

Consider next the payout region $[b,\infty)$ over which the firm distributes any cash holdings in excess of the given target level. In this region, we have

$$v_i(c;b) = v_i(b;b) + c - b, \qquad c \geq b, \tag{3}$$

and the fact that the firm pays dividends in a minimal way to maintain its cash holdings at or below b requires that

$$\lim_{c\uparrow b} v_i'(c,b) = 1. \tag{4}$$

This boundary condition allows to uniquely determine the value of the firm at the payout boundary and completes the characterization of firm value under a given barrier strategy.

In order to derive a closed form solution for $v_i(c;b)$, we first need to introduce some notation. Denote by $C_{i,t}$ the uncontrolled cash buffer process associated with μ_i, let $\tau_{i,x}$ be the stopping time at which this process reaches the level $x \in \mathbb{R}$ for the first time, and define a pair of bounded functions by setting

$$L_i(c;b) = E_c\left[e^{-(\rho+\lambda^*)\tau_{i,0}} 1_{\{\tau_{i,0}\leq\tau_{i,b}\}}\right], \tag{5}$$

$$H_i(c;b) = E_c\left[e^{-(\rho+\lambda^*)\tau_{i,0}\wedge\tau_{i,b}}\right] - L_i(c;b) = E_c\left[e^{-(\rho+\lambda^*)\tau_{i,b}} 1_{\{\tau_{i,b}\leq\tau_{i,0}\}}\right]. \tag{6}$$

$L_i(x; b)$ is a discount factor that gives the present value of one dollar to be paid in liquidation, should it occur before the uncontrolled cash buffer reaches $b > 0$ or finding new investors. Similarly, $H_i(c; b)$ gives the present value of one dollar to be received when the uncontrolled cash buffer reaches $b > 0$, should this occur before liquidation or finding new investors. Closed form expressions for these two functions are provided in the Appendix.

Using the above notation, together with basic properties of diffusion processes as found for example in Stockey (2009, Chapter V), it can be shown that over the region $(0, b)$ the unique solution to equations (1), (2) and (3) satisfies

$$\begin{aligned} v_i(c; b) = v_i(b; b)H_i(c; b) + \ell_i L_i(c; b) \\ + \Pi_i(c; b) - \Pi_i(b; b)H_i(c; b) - \Pi_i(0; b)L_i(c; b) \end{aligned} \tag{7}$$

with the function

$$\Pi_i(c; b) = \frac{\lambda^*}{\rho + \lambda^*}\left(v_i(b; b) - b + c + \frac{\mu_i + rc}{\rho + \lambda^* - r}\right).$$

To better understand this solution, recall that over $(0, b)$ the firm retains earnings. This implies that over this region firm value is the sum of the present value of the payments that incumbent shareholders obtain if cash holdings reach either the payout trigger or the liquidation point before external financing can be secured (first line), plus the present value of the claim that they receive if the firm gets an opportunity to raise funds from outside investors before its cash holding reach either endpoints of the region (second line).

Equations (1), (2), (3) and (4), or equivalently (4) and (7), characterize the value of a given barrier strategy. To determine the optimal target level C_i^*, we further require that the value of the firm be twice continuously differentiable over the whole positive real line by imposing the high-contact condition (see e.g. Dumas (1991))

$$\lim_{c \uparrow C_i^*} v_i''(c; C_i^*) = 0 \tag{8}$$

at the dividend distribution boundary. Given (1) and (4), the high contact condition implies

$$\lim_{c \uparrow C_i^*} v_i(c; C_i^*) = \frac{\mu_i}{\rho} + C_i^* - \left(1 - \frac{r}{\rho}\right) C_i^*. \tag{9}$$

Thus, the value of the firm at the optimal target level equals the first best value of the firm minus the present value of the cost of keeping the optimal amount of cash.

In the Appendix we show that there exists a unique target level that solves (9) and, relying on a verification theorem for the Bellman equation (32) associated with our optimization problem, we prove that the corresponding barrier strategy is optimal among all strategies. The following proposition summarizes these results and provides some basic comparative statics on the optimal target level.

Proposition 1 (Firm value without growth option) *The value of a firm with mean cash flow rate μ_i and no growth option is $V_i(c) = v_i(c; C_i^*)$, where C_i^* is the unique solution to (9). The optimal target C_i^* increases with cash flow volatility σ^2 and decreases with capital supply λ and asset tangibility φ.*

2.2 Value of the firm with a growth option

We now turn to the analysis of corporate policies when the firm has the option to increase the mean cash flow rate by paying a lump sum cost K. The growth option changes the firm's policy choices and value only if the project has positive net present value. The following proposition provides a necessary and sufficient condition for this to be the case.

Proposition 2 *The option to invest has positive net present value if and only if the cost of investment is lower than K^* defined by:*

$$\frac{\mu_1 - \mu_0}{\rho} = K^* + \left(1 - \frac{r}{\rho}\right)(C_1^* - C_0^*), \tag{10}$$

where the constant C_i^* is the optimal level of cash reserves for a firm with mean cash flow rate μ_i and no growth option.

The intuition for this result is clear. The left hand side of equation (10) represents the expected present value of the increase in cash flows following the exercise of the growth option. The right hand side represents the total cost of investment, which incorporates both the direct cost of investment and the change in the carry cost of the optimal cash balance. In the following, we consider that the cost of investment is below K^*. If this was not the case, the firm would simply follow the barrier strategy described in Proposition 1.

2.2.1 Optimality of a barrier strategy

Following the logic of the previous section, it is natural to conjecture that for a firm with a growth option there exists a cutoff level $C_U^* \geq K$ for the cash buffer such that it is optimal

Insert Figure 1 Here

to retain earnings and search for investors when $c < C_U^*$ and to invest when the cash buffer reaches C_U^* or upon obtaining outside funds. As before, the firm is liquidated if a sequence of operating losses depletes its cash reserves before financing can be secured. The expected shape and properties of the value function under such a policy are illustrated in Figure 1.

To determine when such a barrier strategy is optimal, we start by calculating the value $u(c;b)$ of a firm that follows a barrier strategy with some investment trigger $b \geq K$. Standard arguments show that in the region $(0, b)$ over which the firm retains earnings and searches for investors, $u(c;b)$ is twice continuously differentiable and satisfies the differential equation

$$\rho u(c;b) = u'(c;b)(rc + \mu_0) + \frac{\sigma^2}{2}u''(c;b) + \lambda^* \left[V_1(C_1^*) - C_1^* - K + c - u(c;b)\right] \tag{11}$$

subject to the value matching conditions

$$u(0;b) = \ell_0, \tag{12}$$

$$u(c;b) = V_1(c - K), \qquad c \geq b, \tag{13}$$

at the point where the firm runs out of cash and in the region where it uses its cash reserves to invest. This equation is similar to that derived in the previous section for the value of the firm after investment. The only difference is that the financing surplus is now given by $V_1(C_1^*) - C_1^* - K + c - u(c;b)$ since, upon obtaining outside funds, the firm invests and simultaneously readjusts its cash buffer to the level C_1^* that is optimal after investment.

Basic properties of diffusion processes show that the unique solution to (11), (12), (13) is given by

$$u(c;b) = V_1(b-K)H_0(c;b) + \ell_0 L_0(c;b) + \Phi(c) - \Phi(b)H_0(c;b) - \Phi(0)L_0(c;b) \tag{14}$$

with the function

$$\Phi(c) = \frac{\lambda^*}{\rho + \lambda^*}\left(V_1(C_1^*) - C_1^* - K + c + \frac{\mu_0 + rc}{\rho + \lambda^* - r}\right). \tag{15}$$

The interpretation of this solution is similar to that of (7). The first two terms give the present value of the payments that incumbent shareholders receive if the firm invests with internal funds or is liquidated before external funding can be secured. The other terms gives the present value of the payments that they receive if the firm invests with external funds before its cash holdings reach either zero or the given investment trigger.

Equations (13) and (14) provide a complete characterization of the value associated with a given barrier strategy for investment and financing. Since the decision to invest with internal funds can be seen as an optimal stopping problem, it is natural to expect that the optimal trigger is determined by the smooth pasting condition

$$\lim_{c \uparrow C_U^*} u'(c; C_U^*) = 1. \tag{16}$$

We show in the Appendix that this is indeed the case if the investment cost is not too small in that $K > \underline{K}$, for some threshold $\underline{K} < K^*$ for which we provide an explicit characterization. If the investment cost lies below this threshold, then the smooth pasting pasting no longer characterizes the optimal investment trigger and we have the corner solution $C_U^* = K$. In this case, the growth option is so profitable that it becomes optimal to invest as soon as possible and liquidate immediately thereafter. In either case, the barrier policy associated with the investment trigger C_U^* is optimal in the class of barrier policies but it is not necessarily optimal among all strategies. A detailed analysis of the Bellman equation presented in the Appendix allows us to determine the conditions under which this is indeed the case and leads to the following theorem.

Theorem 3 *There exists a constant $K^{**} < K^*$ such that a barrier strategy is optimal if and only if $K \leq K^{**}$. In this case, the value of the firm is given by $V(c) = u(c; C_U^*)$ where C_U^* is the unique solution to (16) when $K > \underline{K}$ and $C_U^* = K$ otherwise.*

2.2.2 Optimality of a band strategy

The function defined by $U(c) \equiv u(c; C_U^*)$ gives the value of the firm under the optimal *barrier* policy and can be constructed for any investment cost. However, Theorem 3 shows that this barrier strategy is *suboptimal* if the investment cost is high, in which case the smooth pasting condition is not sufficient to determine the globally optimal strategy. The intuition for this finding is that with a sufficiently high investment cost, it becomes too expensive for a firm with low cash holdings to accumulate the amount of cash necessary to invest with internal funds. Specifically, we show in the Appendix that the barrier strategy fails to be optimal because, with a high investment cost, there exists a level below C_U^* where the marginal value of cash $U'(c)$ drops to one. At that point, incumbent shareholders would rather abandon the option of financing investment internally and receive dividends, than continue hoarding cash inside the firm. Figure 2 provides an illustration of the marginal value of cash associated with the optimal barrier strategy in this case.

Insert Figure 2 Here

Following this line of argument, we conjecture and later verify that, when the investment cost is high, the optimal strategy includes an intermediate payout region and can be described in terms of thresholds $C_W^* \le C_L^* \le C_H^*$ as follows: When cash holdings are in (C_L^*, C_H^*), the firm retains earnings and invests either upon obtaining outside funds or when its cash holdings reach the level $C_H^* \ge K$. If cash holdings drop to the level C_L^* following a sequence of operating losses, the firm abandons the option of financing investment internally and makes a lump-sum payment $C_L^* - C_W^*$ to shareholders. Finally, if cash holdings are at or below the level C_W^*, the firm retains earnings, pays dividends to keep its cash reserves in $(0, C_W^*)$, and finances investment exclusively with outside funds. As in the case of a barrier strategy, the firm is liquidated only if its cash buffer reaches zero. The shape and properties of the firm value under such a strategy are illustrated in Figure 3.

Insert Figure 3 Here

To verify our conjecture, we start by constructing the value $v(c; b)$ of a firm that follows a strategy as above with thresholds $b = (b_1, b_2, b_3)$ for some arbitrary constants $b_1 \le b_2 \le b_3$ with $b_3 \ge K$. Standard arguments show that in the region $(0, b_1) \cup (b_2, b_3)$ over which the firm retains earnings, $v(c; b)$ is twice continuously differentiable and satisfies:

$$\rho v(c; b) = v'(c; b)(rc + \mu_0) + \frac{\sigma^2}{2} v''(c; b) + \lambda^* \left[V_1(C_1^*) - C_1^* - K + c - w(c; b_2) \right], \tag{17}$$

subject to the value matching conditions

$$v(0; b) = \ell_0, \tag{18}$$

$$v(c; b) = V_1(c - K), \qquad c \ge b_3. \tag{19}$$

This differential equation is the same as (11), but the solutions we are looking for differ because the strategy now includes an intermediate payout region $[b_1, b_2]$ over which the value does not satisfy (17). Instead, the value of the firm over this intermediate region is given by

$$v(c; b) = v(b_1; b) + c - b_1, \qquad b_1 \le c \le b_2. \tag{20}$$

and the fact that once below b_1 the firm distributes dividends so as to maintain its cash holdings at or below this level implies that we have

$$\lim_{c \uparrow b_1} v'(c; b_1) = 1. \tag{21}$$

From these boundary conditions, it is clear that since $v(0; b) = U(0) = \ell_0$ we must necessarily have $v'(0;b) > U'(0)$ for our candidate strategy to dominate the optimal *barrier* strategy of the previous section. We show in the Appendix that given the optimal thresholds, this condition is equivalent to the restriction $K > K^{**}$.

Proceeding as in the two previous cases, it is easily shown that the unique solution to (17), (18), (19), (20) and (21) satisfies

$$\begin{aligned} v(c;b) = {} & v(b_1;b)H_0(c;b_1) + \ell_0 L_0(c;b_1) \\ & + \Phi(c) - \Phi(0)L_0(c;b_1) - \Phi(b_1)H_0(c;b_1) \end{aligned} \tag{22}$$

in the retention interval $(0, b_1)$, and

$$\begin{aligned} v(c;b) = {} & V_1(b_3 - K)H_0(c;b_2,b_3) + v(b_2;b)L_0(c;b_2,b_3) \\ & + \Phi(c) - \Phi(b_2)L_0(c;b_2,b_3) - \Phi(b_3)H_0(c;b_2,b_3) \end{aligned} \tag{23}$$

in the retention interval (b_2, b_3) where the discount factors are defined as in (5) and (6), but with the first hitting time of the level b_2 instead of the liquidation time, and the function $\Phi(c)$ is defined as in (15). The first line in (22) gives the present value of the payments that incumbent shareholders receive if the cash buffer reaches either the liquidation point or the payout trigger b_1 before external financing can be secured. The second line gives the value of the payment that they receive if the firm finds new investors before its cash buffer reaches either zero or b_1. Similarly, the first line in (23) gives the present value of the payments that incumbent shareholders receive if the cash buffer reaches either the intermediate payout trigger b_2 or the internal investment trigger b_3 before external financing can be secured. The second line gives the present value of the claims that they receive if external funds are raised before the cash buffer reaches either b_2 or b_3.

Equations (20), (22), and (23) provide a complete characterization of the value of the firm for a given set of thresholds $b = (b_1, b_2, b_3)$. It remains to determine the triple $C^* = (C_W^*, C_L^*, C_H^*)$ of optimal thresholds. Following the same logic as in the case without growth option, we determine the point C_W^* below which the firm invests exclusively with outside funds by imposing the high-contact condition

$$\lim_{c \uparrow C_W^*} v''(c;C^*) = 0 \tag{24}$$

at the lower end of the intermediate payout region. In addition, since the decision to invest with internal funds and the decision to make an intermediate dividend payment can be seen as a joint optimal stopping problem, it is natural to expect that the two remaining thresholds are determined by the smooth pasting conditions

$$\lim_{c \downarrow C_L^*} v'(c; C^*) = 1, \tag{25}$$

$$\lim_{c \uparrow C_H^*} v'(c; C^*) = V_1'(C_H^* - K). \tag{26}$$

We show in the Appendix that these boundary conditions uniquely determine a triple of thresholds C^* and we prove that the corresponding strategy is globally optimal when the investment cost is above K^{**}. The following theorem summarizes our findings.

Theorem 4 *If the investment cost is such that $K \in (K^{**}, K^*)$, then the value of the firm with a growth option is given by $V(c) = v(c; C^*)$ where the thresholds are the unique solutions to (24), (25) and (26).*

The above results show that when the investment cost is high, the optimal strategy depends on the level of cash reserves. When cash reserves are below C_W^*, it is optimal to finance the capital expenditure exclusively with external funds and to use cash reserves only to cover operating losses. When cash holdings are between C_W^* and C_L^*, the firm pays a specially designated dividend to lower its cash holdings and abandons the option of investing with internal funds. Finally, when cash holdings are above C_L^* the firm finances the capital expenditure using either internal or external funds and the optimal policy is to retain earnings until the firm invests or its cash buffer drops to C_L^*.

Interestingly, the change in the optimal financing policy that occurs at C_L^* implies that we have $V'(C_L^*) = 1$. Since the marginal value of cash is constantly above or equal to one, it follows that firm value is not globally concave. In fact, we show in the Appendix that the value of the firm is strictly convex over the whole interval (C_L^*, C_H^*) when the investment cost is high. To understand this feature, recall that in our model cash holdings generally serve two purposes: Reducing the risk of inefficient closure and financing investment. However, when the investment cost is high and cash reserves are below C_L^*, it is optimal to invest exclusively with outside funds and the value of cash holdings only comes from their mitigating effect on liquidation risk. At the point C_L^* the marginal value of cash is equal to one and it is optimal to start paying dividends. Above this point, the possibility to finance investment internally introduces a second motive for holding cash and pushes the marginal value of cash above one, leading firm value to be convex. Figure 4 provides an illustration of the marginal value of cash under the globally optimal band strategy when the cost of investment is high.

Insert Figure 4 Here

3 Finitely many growth options

A key and novel feature of the optimal policy for a firm with a growth option is that it may include an intermediate payout region where shareholders optimally abandon the option of investing with internal funds. This feature is unexpected, but we contend that it is in fact universal in models with fixed costs and capital supply frictions.

Insert Figure 5 Here

To make this point, we consider in this section a firm with assets in place and $N \geq 1$ growth options that arrive sequentially over time. The initial mean cash flow rate of the firm is μ_0 and the exercise of the i'th growth option allows to increase the mean cash flow rate from μ_{i-1} to $\mu_i > \mu_{i-1}$ by paying a constant cost K_i. To prevent the simultaneous exercise of multiple growth options, we assume that the firm can hold at most one growth option at a time and that, after exercising each growth option, the firm enters a waiting phase in which the next growth option arrives at an exponentially distributed random time with intensity λ_o. As in the benchmark model, management seeks to maximize shareholders' wealth and has full flexibility over the investment, payout, and financing policies of the firm. The sequential arrival and exercise of the growth options is illustrated in Figure 5. Lastly, we do not consider for simplicity the case when λ_i and φ_i may depend on i. We provide comparative static results on this more general case in Propositions 6 and 9 below.

To solve this extension of the model, we use the fact that there are finitely many investment opportunities and proceed backwards in time starting from the last period where the firm has exhausted its growth potential. Let $V_{o,i}(c)$ denote the value of the firm as a function of its cash holdings in the period where it holds the i'th growth option, and $V_{n,i}(c)$ denote the value of the firm in the waiting period following the exercise of this growth option.

After the exercise of the last growth option, the value of the firm is $V_{n,N}(c) = V_N(c)$, where the later is the value of a firm with a mean cash flow rate μ_N and no growth option that was derived in section 2.1. Similarly, in the period prior to the exercise of the last growth option, the value of the firm is given by $V_{o,N}(c) = V(c)$ where the later is the value of a firm with a single growth option that was derived in section 2.2. To proceed further in this backward recursion, we now have to solve the problem of the firm in the waiting period between the exercise of a growth option and the arrival of the next one.

3.1 Optimal policy in the waiting period

In the waiting period following the exercise of the i'th growth option, the firm may retain earnings to avoid inefficient closure and to exercise not only the next growth option but potentially each of the $N-i$ growth options that it stands to receive. Following the logic of the previous section, we therefore conjecture that the optimal strategy in the waiting period can be characterized in terms of an optimal target level and up to $N-i$ intermediate payout intervals, whose upper ends correspond to the points where the firm decides to temporarily stop hoarding cash to finance a future investment opportunity.

In order to describe the class of all such strategies, let $s = (a, b, x)$ where $x \geq 0$ is a constant that represents the target level for the cash holdings of the firm when raising outside funds and $a, b \in \mathbb{R}^n_+$ for some $n \in [0, N-i]$ are vectors with

$$a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_n \leq b_n \leq x$$

that specify the earnings retention intervals and the intermediate dividend distribution intervals associated with the strategy. Specifically, for every given s as above, the set

$$\mathcal{R}(s) = (0, a_1) \cup (b_1, a_2) \cup \cdots \cup (b_{n-1}, a_n) \cup (b_n, x) = \bigcup_{k=0}^{n} \mathcal{R}_k(s)$$

gives the region over which the firm retains earnings and searches for new investors while its complement in $[0, x]$, that is

$$\mathcal{D}(s) = \bigcup_{k=1}^{n} \mathcal{D}_k(s) = \bigcup_{k=1}^{n} [a_k, b_k],$$

gives the collection of intermediate dividend distribution intervals (i.e. the “bands”). When its cash holdings are above the target x, the firm makes a lump sum payment $c - x$. When its cash holdings are in $\mathcal{R}_k(s)$, the firm retains earnings, distributes dividends to remain in the same interval, and searches for new investors in order to adjust its cash holdings to the target level x. If its cash holdings fall to the lower endpoint of the interval before outside funds can be secured, then the firm is liquidated if $k = 0$ and otherwise stops hoarding cash towards the exercise of one of its future growth options. In the latter case, the firm makes a lump sum payment to shareholders given by

$$b_k - a_k = |\mathcal{D}_k(s)| = \inf(\mathcal{R}_k(s)) - \sup(\mathcal{R}_{k-1}(s)) \geq 0,$$

Insert Figure 6 Here

in order to bring its cash buffer down to the next earnings retention interval and then follows an entirely similar payout, financing and liquidation strategy. Figure 6 provides an illustration of the value of a firm as a function of its cash holdings in the waiting period.

Let $v_{n,i}(c,s)$ denote firm value under such a strategy. Standard arguments show that in in the retention region $\mathcal{R}(s)$ this function is twice continuously differentiable and satisfies

$$\begin{aligned}\rho v_{n,i}(c;s) &= (rc+\mu_i)v'_{n,i}(c;s) + \frac{\sigma^2}{2}v''_{n,i}(c;s) \\ &\quad + \lambda^* \left[v_{n,i}(x;s) - x + c - v_{n,i}(c;s)\right] + \lambda_o \left[V_{o,i+1}(c) - v_{n,i}(c;s)\right],\end{aligned}$$

subject to the value matching conditions

$$v_{n,i}(0,s) = \ell_i$$

at the point where the firm is liquidated. This differential equation is similar to (1) with the exception of the last term on the second line which accounts for the change in the value of the firm that occurs upon the arrival of the next investment opportunity.

In each of the dividend distribution intervals, the value of the firm is defined by imposing the value matching condition

$$v_{n,i}(c;s) = c - a_k + v_{n,i}(a_k;s), \qquad c \in \mathcal{D}_k(s),$$

and the fact that, once in the interval $\mathcal{R}_k(s)$, the firm distributes dividends to maintain its cash holdings at or below the right endpoint of the interval implies that

$$\lim_{c\uparrow a_k} v'_{n,i}(c;s) = \lim_{c\uparrow x} v'_{n,i}(c;s) = 1.$$

The above equations provide a complete characterization of the value associated with a given band strategy and can be solved using techniques similar to those of section 2.2. To determine the optimal strategy, we further impose the smooth pasting conditions

$$\lim_{c\downarrow b^*_{i,k}} v'_{n,i}(c;s^*_i) = 1, \tag{27}$$

at each of the points where the firm stops hoarding cash towards the exercise of one of its future growth opportunities, and the high contact conditions

$$\lim_{c\uparrow a^*_{i,k}} v''_{n,i}(c;s^*_i) = \lim_{c\uparrow x^*_i} v''_{n,i}(c;s^*_i) = 0. \tag{28}$$

at the target level x_i^* and each of the intermediate target levels $a_{i,k}^*$. In the Appendix, we show that there always exist a unique s_i^* such that these conditions are satisfied and a detailed analysis of the Bellman equation associated with the problem of the firm in the waiting period allows us to prove that the corresponding strategy is optimal among all the strategies available to the firm. The following proposition summarizes our findings.

Theorem 5 *The value of the firm in the waiting period following the exercise of the i'th growth option is given by*

$$V_{n,i}(c) = v_{n,i}(c; s_i^*)$$

and satisfies

$$V_{n,i}(x_i^*) = \frac{1}{\rho + \lambda_o}\left(rx_i^* + \mu_i + \lambda_o V_{o,i+1}\left(x_i^*\right)\right)$$

where the triple s_i^* that determines the optimal earnings retention and dividend distribution intervals is the unique solution to (27) and (28).

Theorem 5 shows that in the waiting period that follows the exercise of the i'th growth option, the optimal strategy may include up to $N - i$ intermediate dividend distribution intervals (bands). But this upper bound is rough as many of these intervals may actually collapse. While it does not seem possible to determine ex-ante the number of intermediate payout intervals, we provide in the Appendix an explicit algorithm that allows to analytically construct these intervals for each given target level. We also prove the following comparative static result.

Proposition 6 *Suppose that the exercise of the i'th growth option changes the tangibility of assets from φ_{i-1} to φ_i and capital supply from λ_{i-1} to λ_i. Then, the dividend distribution region $\mathcal{D}$ in the waiting period following the exercise of the i'th growth option is increasing with respect to φ_i in the inclusion order, and the target cash level x_i^* is monotone decreasing with respect φ_i and λ_i.*

3.2 Optimal strategy for a firm with a growth option

Having constructed the value of the firm in the waiting period, we now consider the optimal policy of a firm that already holds a growth option. As a first step towards the solution to this problem, the next result provides sufficient conditions for the growth option to have a positive net present value.

Proposition 7 *A sufficient condition for the i'th growth option to have positive net present value is that*

$$V_{n,i}(x_i^*) - x_i^* - K_i \geq V_{i-1}(C_{i-1}^*) - C_{i-1}^*,$$

where the function $V_{i-1}(c)$ and the constant C_{i-1}^* denote the value and optimal target level of a firm with mean cash flow rate μ_{i-1} and no growth option.

The intuition for this result is clear. Indeed, the left hand side of the inequality gives the maximal value that the firm can attain by exercising the growth option. The right hand side gives the maximal value that it can achieve by abandoning the growth option. In the later case, the firm not only abandons the next growth option but also all subsequent ones.

To simplify the presentation of our results, we assume below that

$$K_i \leq K_i^* = \min\left\{V_{n,i}(x_i^*) - x_i^* + C_{i-1}^* - V_{i-1}(C_{i-1}^*), \frac{\mu_i - \mu_{i-1}}{r}\right\} \tag{29}$$

for all i. In the single option case, this condition is necessary for a positive net present value but this is not so with multiple options because in that case the net present value of an individual option can no longer be determined on a stand-alone basis. In the present context, this assumption allows us to guarantee that the regions over which the firm invests with internal funds are half-lines instead of unions of disjoint intervals. This assumption can be relaxed at the cost of significantly more involved notation.

When the firm holds a growth option, cash holdings serve two purposes: Reducing the risk of inefficient closure and financing investment. Following the logic of the single option case, we therefore conjecture that the optimal strategy can be described in terms of thresholds $C_{i,W}^* \leq C_{i,L}^* \leq C_{i,H}^*$ with $C_{i,H}^* \geq K_i$. When the investment cost is low, it should never be optimal for the firm to abandon the option of investing with internal funds. We therefore expect the firm to follow a barrier strategy as in section 2.2.1 with $0 = C_{i,W}^* = C_{i,L}^*$. On the contrary, when the investment cost is high, we expect the firm to follow a strategy similar to that of section 2.2.2, with an intermediate dividend distribution interval at the point where the firm optimally abandons the option to invest with internal funds.

To verify our conjecture, we start by constructing the value $v_{o,i}(c; b)$ of a firm that follows a strategy as above with thresholds $b = (b_1, b_2, b_3)$ where $b_1 \leq b_2 \leq b_3$ and $b_3 \geq K$. Standard arguments imply that in the region $(0, b_1) \cup (b_2, b_3)$ over which the firm retains earnings and searches for investors, $v_{o,i}(c; b)$ is twice continuously differentiable and satisfies

$$\rho v_{o,i}(c; b) = (rc + \mu_{i-1}) v'_{o,i}(c; b) + \frac{\sigma^2}{2} v''_{o,i}(c; b) + \lambda^* \left[V_{n,i}(x_i^*) - x_i^* - K_i + c - v_{o,i}(c; b) \right]$$

subject to the value matching conditions

$$\begin{aligned} &v_{o,i}(0; b) = \ell_{i-1}. \\ &v_{o,i}(c; b) = V_{n,i}(c - K_i), \qquad c \geq b_3. \end{aligned}$$

This differential equation is similar to that of section 2.2.2 with the exception of the last term which reflect the fact that upon finding new investors the firm raises funds to invest and simultaneously adjust its cash holdings to the target level x_i^* that is optimal in the waiting period following the exercise of the growth option.

If the thresholds under consideration are such that $[b_1, b_2] = \{0\}$, then the above equations are sufficient to determine the value of the firm and can be solved in closed form using a modification of (14). Otherwise, the value of the firm satisfies

$$v_{o,i}(c; b) = c - b_1 + v_{o,i}(b_1; b), \qquad c \in [b_1, b_2],$$

and the fact that in the lowest retention region $(0, b_1)$ the firm distributes dividends to maintain its cash holdings at or below b_1 implies that we have

$$\lim_{c \uparrow b_1} v'_{o,i}(c; b) = 1.$$

In this case, the value of the firm under the given strategy can be derived in closed-form using a modification of (22) and (23).

To determine the optimal strategy, we distinguish two cases depending on the level of the investment cost. If the investment cost is sufficiently low, then we set $b_i^* = (0, 0, C_{i,H}^*)$ and determine the optimal investment trigger by imposing the smooth pasting condition

$$\lim_{c \uparrow C_{i,H}^*} v_{o,i}(c; b_i^*) = V'_{n,i}(C_{i,H}^* - K_i). \tag{30}$$

If the investment cost is high then, the optimal investment trigger is still determined by the above smooth pasting condition but this equation now needs to be solved in conjunction with the smooth pasting and high contact conditions

$$\lim_{c \uparrow C^*_{i,W}} v''_{o,i}(c; b^*_i) = \lim_{c \downarrow C^*_{i,L}} v'_{o,i}(c; b^*_i) - 1 = 0 \tag{31}$$

that determine the intermediate payout interval.

In the Appendix, we show that in either case the above equations admit a unique solution and a detailed analysis of the Bellman equation associated with the problem of the firm allows us to confirm our conjecture regarding the optimality of the corresponding strategies.

Theorem 8 *Assume that condition (29) holds. Then there exists a constant $K^{**}_i \leq K^*_i$ such that the value of a firm holding a growth option is*

$$V_{o,i}(c) = v_{o,i}(c; b^*_i)$$

where the thresholds b^*_i are given by the unique solutions to (30) and (31) when $K_i \geq K^{**}_i$ and by the unique solution to (30) such that $C^*_{i,W} = C^*_{i,L} = 0$ otherwise.

Theorem 8 shows that the results derived in the one growth option case naturally extend to a model in which the firm has multiple growth options. In the Appendix, we show that these results also hold if we incorporate search and issuance costs in the model. In this case however, firms only raise funds when the cash buffer is below some threshold C^*_F, where the financing surplus equals 0. We conclude this section with the following proposition, which provides analytic comparative static results on the optimal investment trigger.

Proposition 9 *Suppose that the exercise of the i'th growth option changes the mean cash flow rate from μ_{i-1} to μ_i, the asset tangibility from φ_{i-1} to φ_i, and capital supply from λ_{i-1} to λ_i. Then, the investment trigger $C^*_{i,H}$ is monotone increasing in current capital supply λ_{i-1}, drift μ_{i-1} and asset tangibility φ_{i-1}, and is decreasing in future tangibility φ_i.*

4 Discussion

4.1 General properties of the model

Following Modigliani and Miller (1958), extant theoretical research on costly external finance generally assumes that capital supply is infinitely elastic so that corporate behavior depend solely on firm characteristics. In this literature, it is always optimal for firms to follow a double barrier (financing/liquidation and payout) policy. In addition, firms either never raise funds when the cost of external finance is high or are never liquidated when the cost is low. Finally, when the cost of external finance is low, firms only raise funds when their cash buffer is completely depleted. That is, firms never simultaneously hold cash and raise external funds and they only tap capital market following a series of negative shocks.

In this paper, we take a first step in constructing a dynamic model of investment, financing, and cash management policies with capital supply effects. Using this model, we show that when firms face capital supply uncertainty it is optimal to safeguard against future liquidity needs by hoarding cash. Second, and more importantly, we demonstrate that the interplay between lumpiness in investment and capital supply frictions can give rise to convexity of firm value and lead firms to follow optimal investment, financing, and payout policies that differ from the Miller and Orr double barrier policy.

In our model, firms only follow a double-barrier policy when the cost of investment is low or, equivalently, when the net present value of the project is high. Firms may raise outside funds before exhausting internal resources, some firms may be liquidated even with low issuance costs, and the optimal payout policy may feature several payout regions, with both smooth and discrete dividend payments. Our analysis also demonstrates that constrained firms with low cash holdings will not finance investment internally and may decide to pay dividends early. By contrast, constrained firms with high cash holdings may finance investment internally and will retain earnings. These results are in sharp contrast with those in standard models of financing constraints as they imply that investment and payout do not always increase with slack. Our results also challenge the use of investment-cash flow sensitivities or payout ratios as measures of financing constraints.

Another prediction of our model is that, when financing investment with external funds, the optimal policy is to raise enough funds to finance both the capital expenditure and the potential gap between current cash holdings and the optimal level after investment. That is, firms always increase their cash buffer when raising outside funds. This prediction of the model is consistent with the evidence in Kim and Weisbach (2008) and McLean (2010), who find that firms' decisions to issue equity are essentially driven by their desire to build up cash reserves. Also, while in prior contributions firms always raise the same amount of cash when accessing financial markets, there exists some time series variation in the amount of funds raised in our model since firms raise $(C_i^* - C_t)^+$ upon meeting investors at time t.

Another key implication of the model is to show that, with capital supply uncertainty, the choice between internal and external funds for financing investment does not follow a strict pecking order, in that any given firm can use both internal and external funds. This is in sharp contrast with the financing policy in prior contributions, in which firms exhaust internal funds before issuing securities and finance investment exclusively with internal funds (see for example Décamps et al. (2011), or Bolton, Chen, and Wang (2011, 2012)).

4.2 Additional model implications

The section provides additional implications of our model for investment, cash holdings, and financing decisions using numerical examples. Throughout the section we assume for simplicity that the firm has a single growth option.

4.2.1 Characterizing the firm's optimal strategy

Insert Figure 7 Here

A key feature of our model is that firms only follow a double-barrier policy when the cost of investment is low (i.e. $K < K^{**}$). To better understand this property of optimal policies, Figure 7, Panel A, plots the zero$-NPV$ threshold K^* and the threshold K^{**} that triggers a change in corporate policies, as functions of the arrival rate of investors λ, asset tangibility φ, cash flow volatility σ, and the carry cost of cash $\delta \equiv \rho - r$. Figure 7, Panel B, plots the internal rate of return of the project when $K = K^{**}$ (solid line) and $K = K^*$ (dashed line) defined as the solution to

$$\frac{\mu_1 - \mu_0}{R} = K + \left(1 - \frac{r}{R}\right)\left(C_1^*(R) - C_0^*(R)\right)$$

where $C_i^*(R)$ is the optimal cash buffer for a discount rate R. Firms with projects that fall below the solid line in panel B are firms for which $K > K^{**}$.

In our numerical analysis, we use the following parameter values: $\rho = 0.06$, $r = 0.02$, $\mu_0 = 0.10$, $\mu_1 = 0.125$, $\sigma = 0.10$, $\varphi = 0.75$, $\eta = 0.5$, $\lambda = 6$ and $N = 1$. These values imply an expected financing lag of $1/\lambda = 2$ months, a haircut of 35% of asset value in liquidation, and a 25% increase in the cash flow mean after investment.

Figure 7 shows that only projects with very high internal rates of return (or very low cost of investment) will lead to a double-barrier policy that mirrors those in prior contributions. In our base case environment in which the cost of capital is $\rho = 6\%$, only projects with an internal rate of return above 13.02% will induce the firm to follow barrier policies. As shown by the figure, the cutoff level for the internal rate of return increases with capital supply and the carry cost of cash and decreases with volatility and asset tangibility.

4.2.2 Determinants of cash holdings

Insert Figure 8 Here

Proposition 1 provides an analytical characterization of the properties of the cash buffer after investment, C_1^*. This section examines instead the determinants of target cash holdings before investment. In our base case, we have $K^{**} = 0.19$. Therefore, we consider two cases: one in which the cost of investment is low in that $K = 0.12$ and one in which it is high in that $K = 0.32$. Figure 8, Panel A, plots the target cash buffer C_U^* of the low investment cost case as a function of capital supply λ, asset tangibility φ, cash flow volatility σ, and agency costs δ. Panel B plots the target cash buffers C_W^* and C_H^* as well as the intermediate payout threshold C_L^* of the high investment cost case as functions of these same parameters.

The figure shows that the three target cash buffers C_U^*, C_W^*, and C_H^* decrease with the arrival rate of investors. That is, as λ increases, the likelihood of finding outside investors increases and the need to hoard cash within the firm decreases. This result is consistent with the evidence in OPSW (1999) and BKS (2009), who find that firms hold more cash when their access to external capital markets is more limited.

Another prediction of the model illustrated by the figure is that cash holdings should increase with cash flow volatility (since the risk of closure increases with σ), consistent with the evidence in Harford (1999) and BKS (2009). Interpreting $\delta \equiv \rho - r$ as an agency cost associated with free cash in the firm, the model also predicts that cash holdings decrease with the severity of agency conflicts, consistent with Harford, Mansi, and Maxwell (2008). Finally, the model predicts that target cash holdings decrease with asset tangibility φ, consistent with the evidence in Almeida and Campello (2007).

4.2.3 Financing investment

Insert Figure 9 Here

An important question is whether capital supply uncertainty affects investment and the source of funds used by firms when financing investment. To answer this question, we examine the determinants of the probabilities of investment with internal ($P_I(c)$) and external ($P_E(c)$). Appendix J shows how to compute these probabilities.

The top four panels of Figure 9 plot the average probability of investment with internal funds for a *cross-section* of firms with cash buffers uniformly distributed between 0 and C_U^* ($K < K^{**}$; dashed line) and between 0 and C_H^* ($K > K^{**}$; solid line) as a function of the arrival rate of investors λ, asset tangibility φ, cash flow volatility σ, and the carry cost of cash δ. The two lower panels plot the total probability of investing over a one-year (solid line) and over a three-year (dashed line) horizon as functions of the arrival rate of investors

λ for a firm with low investment cost (left) and high investment cost (right). In both cases, we assume that $c = K$ so that firms have enough cash to finance investment internally.

Consider the probability of investment with internal funds (top four panels). In our base case environment, the probability that the average firm invests with internal funds is 15.60% when the cost of investment is low and 2.87% when the cost of investment is high. This suggests that, in most environments, cash holdings will be used mostly to cover operating losses and that firms will wait until external financing arrives before investing, consistent with the large sample studies by OPSW (1999) and BKS (2009) and the survey of Lins, Servaes, and Tufano (2010). As expected, the probability of investment with internal funds is much smaller when the cost of investment is high because more firms find it optimal to invest exclusively with outside funds. Another property of the model illustrated by the figure is that the probability of financing investment with internal funds decreases with the arrival rate of investors and increases with asset tangibility. This last feature follows from the fact that C_U^* decreases with φ and implies that the investment-cash flow sensitivity increases with the tangibility of assets, consistent with Almeida and Campello (2007).

Lastly, the lower panels of Figure 9 show that the overall probability of investment decreases as λ decreases. Thus, our model predicts that a negative shock to the supply of capital may hamper investment even if firms have enough financial slack to fund all investment opportunities internally, consistent with the evidence in Gan (2007), Becker (2007), Lemmon and Roberts (2007), and Campello, Graham, and Harvey (2010).

5 Concluding remarks

We develop a model of investment, financing, and cash management decisions in which firms face capital supply uncertainty and have to search for investors when in need of capital. We characterize optimal policies explicitly and demonstrate that the smooth-pasting conditions used to characterize optimal polices in prior contributions are necessary, but may not be sufficient, for an optimum. We then show that a striking implication of this result is that it may not be optimal for firms to follow the standard Miller and Orr (1966) double-barrier policy. Lastly, we show that accounting for capital supply uncertainty and lumpy investment changes the predictions of models of financing constraints in ways that are consistent with stylized facts concerning firms' policy choices.

Appendix

A Bargaining with outside investors

See the supplementary appendix for a proof that introducing bargaining is equivalent to reducing the arrival rate of outside investors.

B Proofs of the results in Section 2.1

B.1 Intuition and road map

To facilitate the proofs, we start by introducing some notation that will be of repeated use throughout the appendix. Let $\mathcal{L}_i$ denote the differential operator defined by

$$\mathcal{L}_i\phi(c) := \phi'(c)(rc + \mu_i) + \frac{\sigma^2}{2}\phi''(c) - \rho\phi(c),$$

set

$$\mathcal{F}\phi(c) := \max_{f\geq 0} \lambda\{\phi(c+f) - \phi(c) - f\},$$

and denote by Θ the set of dividend and financing strategies such that

$$E_c\left[\int_0^{\tau_0} e^{-\rho s}(dD_s + f_s dN_s)\right] < \infty$$

for all $c \geq 0$ where τ_0 is the first time that the firm's cash holdings fall to zero and $E_c[\cdot]$ denotes an expectation conditional on the initial value $C_{0-} = c$.

Let $\hat{V}_i(c)$ denote the value of a firm with no growth option when the mean cash flow rate is μ_i. In order to apply dynamic programming techniques, we will proceed in four steps.

1. Derive the Hamilton-Jacobi-Bellmamn (HJB) equation.
2. Show that any smooth solution $\phi(c)$ to the HJB equation dominates the value function.
3. Conjecture an optimal policy and derive the corresponding firm value.
4. Show that the value of the firm associated with the conjectured optimal policy of Step 3 is indeed a smooth solution to the HJB equation.

In accordance with the theory of singular stochastic control (see Fleming and Soner (1993)), the HJB equation for the value of a firm with no option is given by

$$\max\{\mathcal{L}_i\phi(c) + \mathcal{F}\phi(c), 1 - \phi'(c), \ell_i(c) - \phi(c)\} = 0, \tag{32}$$

where $\ell_i(c) \equiv \ell_i + c$ denotes the liquidation value of the firm. Thus, we are done with Step 1. Lemma B.1 below accomplishes Step 2. In order to proceed to Step 3, we conjecture that the optimal policy is of a threshold form and show that the system defined by (1)-(4) and (8) has a unique smooth solution that is given by $(V_i(c), C_i^*) = (v_i(c; C_i^*), C_i^*)$ for some $C_i^* > 0$. Finally, to complete Step 4, we will show that this solution solves (32), i.e. that

(a) $V_i'(c) \geq 1$ for all $c \geq 0$,

(b) $\phi(c) = \ell_i(c)$ satisfies $\mathcal{L}_i\phi(c) + \mathcal{F}\phi(c) \leq 0$ for all $c \leq C_i^*$,

(c) $V_i(c) \geq \ell_i(c)$ for all $c \geq 0$.

As we show below, $V_i(c)$ is concave and therefore items (a) and (c) easily follow.[5] Item (b) follows by direct calculation. Finally, step 4 is proved in Lemma B.9. Thus, it remains to implement step 3.i and show that a solution exists and that it is concave. To this end, we will introduce another function $w(c; b)$ defined as the unique solution to (1) satisfying (4) and (8). As we note in the main text, (1) implies that (8) is equivalent to (9), that is

$$w_i(b; b) = (rb + \mu_i)/\rho.$$

Thus, for any fixed b, equation (1) turns into a standard ordinary differential equation that can be explicitly solved via special functions as we show below in Lemma B.2. Lemma B.4 proves the concavity of $w_i(c; b)$. The value function $v_i(c; b)$ satisfies (4)-(3). Thus, it order to determine the optimal threshold $b = C_i^*$, it remains to find the b such that $w_i(0; b) = \ell_i(0)$. This is done in Lemma B.8. Obviously, $w_i(c; C_i^*) = v_i(c; C_i^*)$ and the proof is complete.

B.2 Proofs

Lemma B.1 *If $\phi \in C^2((0, +\infty))$ is a solution to (32) then $\phi(c) \geq \hat{V}_i(c)$.*

Proof. Let ϕ be as in the statement, fix a strategy $(D, f) \in \Theta$ and consider the process

$$Y_t := e^{-\rho t}\phi(C_t) + \int_{0+}^{t} e^{-\rho s}(dD_s - f_s dN_s).$$

Using the assumption of the statement in conjunction with Itô's formula for semimartingales (see Dellacherie and Meyer (1980, Theorem VIII–25)), we get that $dY_t = dM_t - e^{-\rho t} dA_t$ where the process M is a local martingale and

$$\begin{aligned} dA_t &= (\phi(C_{t-} + f_t) - \phi(C_{t-}) - f_t - \mathcal{F}\phi(C_{t-}))dt \\ &\quad + (\Delta D_t - \phi(C_{t-} - \Delta D_t) + \phi(C_{t-})) + (\phi'(C_{t-}) - 1)dD_t^c. \end{aligned}$$

[5] Concavity is a rare and useful property for this class of models. As we show below, firm value is no longer globally concave when the firm has a growth option and, without this property, the verification of property (a) above become a lot more difficult.

where ΔD_t and D_t^c denote respectively the jump and the continuous components of the dividend policy under consideration, i.e.

$$\begin{aligned} \Delta D_t &= D_t - \lim_{s \uparrow t} D_s = D_t - D_{t-} \\ D_t^c &= D_t - \sum_{s \le t} \Delta D_s. \end{aligned}$$

The definition of $\mathcal{F}$ and the fact that $\phi' \ge 1$ then imply that A is nondecreasing and it follows that Y is a local supermartingale. The liquidation value being nonnegative, we have

$$Z_t := Y_{t \wedge \tau_0} \ge - \int_0^{\tau_0} e^{-\rho s} f_s dN_s$$

and since the random variable on the right hand side is integrable by definition of the set Θ, we conclude that Z is a supermartingale. In particular,

$$\begin{aligned} \phi(C_{0-}) = \phi(C_0) - \Delta\phi(C_0) = Z_0 - \Delta\phi(C_0) &\ge E_c[Z_{\tau_0}] - \Delta\phi(C_0) \\ &= E_c\left[e^{-\rho\tau_0}\phi(C_{\tau_0}) + \int_{0+}^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s)\right] - \Delta\phi(C_0) \\ &= E_c\left[e^{-\rho\tau_0}\ell_i(0) + \int_0^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s)\right] - \Delta D_0 - \Delta\phi(C_0) \\ &\ge E_c\left[e^{-\rho\tau_0}\ell_i(0) + \int_0^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s)\right] \end{aligned} \tag{33}$$

where the first inequality follows from the optional sampling theorem for supermartingales, the fifth equality follows from $C_{\tau_0} = 0$, and the last inequality follows from

$$\Delta D_0 + \Delta\phi(C_0) = \Delta D_0 + \phi(C_{0-} - \Delta D_0) - \phi(C_{0-}) = \int_{C_{0-}-\Delta D_0}^{C_{0-}} (1 - \phi'(c))dc \le 0$$

The desired result follows by taking supremum over $(D, f) \in \Theta$ on both sides of (33). ■

Lemma B.2 *Let $b \ge 0$ be fixed. Fix two constants $\phi(0)$ and $\phi(b)$. Then, the unique continuously differentiable solution to*

$$\mathcal{L}_i\phi(c) + \lambda(\phi(b) - b + c - \phi(c)) = 0, \qquad c \le b, \tag{34}$$

$$\phi(c) - \phi(b) + b - c = 0, \qquad c \ge b, \tag{35}$$

satisfies

$$\begin{aligned} \phi(c) = \phi(b)&H_i(c; b) + \phi(0)L_i(c; b) \\ &+ \Pi_i(c; b) - \Pi_i(b; b)H_i(c; b) - \Pi_i(0; b)L_i(c; b). \end{aligned}$$

In this equation, we have

$$\Pi_i(c;b) = \frac{\lambda}{\rho+\lambda}\left[\phi(b) - b + c + \frac{\mu_i + rc}{\rho+\lambda-r}\right]$$

and

$$L_i(c;b) = \frac{G_i(b)F_i(c) - F_i(b)G_i(c)}{G_i(b)F_i(0) - F_i(b)G_i(0)} \text{ and } H_i(c;b) = \frac{F_i(0)G_i(c) - G_i(0)F_i(c)}{G_i(b)F_i(0) - F_i(b)G_i(0)}$$

with

$$F_i(x) = M(-0.5\nu; 0.5; -(rx+\mu_i)^2/(\sigma^2 r)), \tag{36}$$

$$G_i(x) = \frac{rx+\mu_i}{\sigma\sqrt{r}} M(-0.5(\nu-1); 1.5; -(rx+\mu_i)^2/(\sigma^2 r)), \tag{37}$$

where $\nu = (\rho+\lambda)/r$ *and* M *is the confluent hypergeometric function defined by (see Dixit and Pindyck, 1994, pp.163):*

$$M(a,b;z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{z^3}{3!} + \cdots$$

The first claim of Lemma B.2 follows by standard arguments. The proof of the second claim is based on the following auxiliary result.

Lemma B.3 *The general solution to the homogenous equation*

$$\lambda\phi_i(c) = \mathcal{L}_i\phi_i(c) \tag{38}$$

is explicitly given by $\phi_i(c) = \gamma_1 F_i(c) + \gamma_2 G_i(c)$ *for some constants* γ_1, γ_2 *where the functions* F_i, G_i *are defined as in equations (36) and (37).*

Proof. The change of variable $\phi_i(c) = g_i(-(rc+\mu_i)^2/(r\sigma^2))$ transforms (38) for ϕ_i into Kummer's ODE for g_i and the conclusion now follows from standard results. ■

Let $w_i(c;b)$ be the unique solution to (34)-(35) with $w_i'(b;b) - 1 = w_i''(b;b) = 0$.

Lemma B.4 *The function* $w_i(c;b)$ *is increasing and concave with respect to* $c \leq b$, *and strictly monotone decreasing with respect to* b.

In order to prove Lemma B.4, we will rely on the following three useful results.

Lemma B.5 *Suppose that* k *is a solution to*

$$\mathcal{L}_i k(c) + \phi(c) = 0 \tag{39}$$

for some ϕ. *Then,* k *does not have negative local minima if* $\phi(c) \geq 0$ *and does not have positive local maxima if* $\phi(c) \leq 0$.

Proof. At a local minimum we have $k'(c) = 0$, $k''(c) \geq 0$ and the claim follows from (39) and the nonnegativity of ϕ. The case of a non-positive ϕ is analogous. ■

Lemma B.6 *Suppose that k is a solution to (39) for some $\phi(c) \leq 0$ and that $k'(c_0) \leq 0$, $k(c_0) \geq 0$ and $|k(c_0)| + |k'(c_0)| + |\phi(c_0)| > 0$. Then, $k(c) > 0$ and $k'(c) < 0$ for all $c < c_0$.*

Proof. Suppose on the contrary that $k'(c)$ is not always negative for $c < c_0$ and let z be the largest value of $c < c_0$ at which $k'(c)$ changes sign. Then, z is a positive local maximum and the claim follows from Lemma B.5. ■

Lemma B.7 *Suppose that k is a solution to (39) for some ϕ such that $\phi'(c) \leq 0$ and that $k'(c_0) \geq 0$, $k''(c_0) \leq 0$ and $|k'(c_0)| + |k''(c_0)| + |\phi'(c_0)| > 0$. Then, $k'(c) > 0$ and $k''(c) < 0$ for all $c < c_0$. Furthermore, if $k''(c_0) = 0$, then $k''(c) > 0$ for $c > c_0$ and $k'(c_0) = \min_{c \geq 0} k'(c)$.*

Proof. Differentiating (39) shows that $m = k'$ is a solution to $\mathcal{L}_i m(c) + rm(c) + \phi'(c) = 0$ and the conclusion follows from Lemma B.6. The case $c > c_0$ is analogous. ■

Proof of Lemma B.4. As is easily seen, the function

$$k(c) = w_i(c; b) - \frac{\lambda}{\lambda + \rho}\left(w_i(b; b) - b + \frac{(\rho + \lambda)c + \mu_i}{\lambda + \rho - r}\right)$$

is a solution to (38) and satisfies $k'(b) = 1 > 0$ as well as $k''(b) = 0$. Together with Lemma B.7 this implies that $k(c)$, and hence also $w_i(b; b)$, is increasing and concave for $c \leq b$.To establish the required monotonicity, let $b_1 < b_2$ be fixed and consider the function $m(c) = w_i(c; b_1) - w_i(c; b_2)$. Using the first part of the proof it is easily seen that m solves

$$\mathcal{L}_i m(c) - \lambda m(c) - \lambda(1 - r/\rho)(b_1 - b_2) = 0$$

with the boundary conditions $m'(b_1) = 1 - w_i'(b_1; b_2) < 0$, $m''(b_1) = -w_i''(b_1; b_2) \geq 0$. Thus, it follows by a straightforward modification of Lemma B.7 that m is monotone decreasing and it only remains to show that $m(b_1) > 0$. To this end, observe that

$$\begin{aligned} m(b_1) &= w_i(b_1; b_1) - w_i(b_1; b_2) \\ &= w_i(b_1; b_1) - w_i(b_2; b_2) + \int_{b_2}^{b_1} w_i'(c; b_2)dc \\ &\geq w_i(b_1; b_1) - w_i(b_2; b_2) + b_2 - b_1 = (r/\rho - 1)(b_1 - b_2) > 0 \end{aligned}$$

where the first inequality follows from $w_i'(b; b) = 1$ and the first part of the proof, and the last inequality follows from the fact that by assumption $\rho > r$. ■

Lemma B.8 *There exists a unique solution C_i^* to $w_i(0; C_i^*) = 0$ and the function*

$$V_i(c) = w_i(c \wedge C_i^*; C_i^*) + (c - C_i^*)^+$$

is a twice continuously differentiable solution to (32).

Proof. By Lemma B.2 we have that $V_i(c)$ is twice continuously differentiable, solves (1) subject to (4), (8) and (3) so we only need to show that

$$w_i(0; C_i^*) = \ell_i(0) \tag{40}$$

does have a unique solution C_i^*. By Lemma B.4, we have that $w_i(0; b)$ is monotone decreasing in b. On the other hand, a direct calculation shows that $w_i(0; 0) = \mu_i/\rho > \ell_i(0)$, $w_i(0; \infty) < 0$ and it follows that (40) has a unique solution. To complete the proof, it remains to show that V_i is a solution to the HJB equation. Using the concavity of $V_i(c)$ in conjunction with the smooth pasting condition we obtain that $1 - V_i'(c)$ is negative below the threshold C_i^* and zero otherwise so that

$$\ell_i(c) - V_i(c) = \int_0^c (1 - V_i'(x))dx \leq 0.$$

On the other hand, using the concavity of $V_i(c)$ in conjunction with Lemma B.2 and the smooth pasting condition we obtain

$$\begin{aligned}(\mathcal{L}_i + \mathcal{F})V_i(c) &= \mathcal{L}_i V_i(c) + 1_{\{c < C_i^*\}}\lambda(V_i(C_i^*) - b + c - V_i(c)) = 1_{\{c \geq C_i^*\}}\mathcal{L}_i V_i(c) \\ &= (r - \rho)(c - C_i^*)^+ \leq 0\end{aligned}$$

Combining the above results shows that V_i is a solution to (32). ∎

Lemma B.9 *We have $\hat{V}_i(c) \geq V_i(c)$ for all $c \geq 0$.*

Proof of Lemma B.9. Combining the results of Lemmas B.1 and B.8 shows that $V_i \geq \hat{V}_i$. In order to establish the reverse inequality, consider the dividend and financing strategy defined by $D_t^* = L_t$ and $f_t^* = (C_i^* - C_{t-})^+$ where the process C evolves according to

$$dC_t = (rC_{t-} + \mu_i)dt + \sigma dB_t - dD_t^* + f_{t-}^* dN_t$$

with initial condition $C_{0-} = c \geq 0$ and $L_t = \sup_{s \leq t}(b_t - C_i^*)^+$ where

$$db_t = (rb_{t-} + \mu_i)dt + \sigma dB_t + (C_i^* - b_{t-})^+ dN_t.$$

In order to show that the strategy (D^*, f^*) is admissible, we start by observing that

$$E_c\left[\int_0^\infty e^{-\rho t} f_{t-}^* dN_t\right] \leq E_c\left[\int_0^\infty e^{-\rho t} C_i^* dN_t\right] = \frac{\lambda C_i^*}{\rho}$$

where the inequality follows from the definition of f^*. Using this bound in conjunction with Itô's lemma and the assumption that $r < \rho$ we then obtain

$$E_c\left[\int_0^t e^{-\rho s} dD_s^*\right] = C_0 + E_c\left[\int_0^t e^{-\rho s}((r-\rho)C_{s-} + \mu_i)ds + \int_0^t e^{-\rho s} f_{s-}^* dN_s\right]$$
$$\leq C_0 + E_c\left[\int_0^\infty e^{-\rho s}\mu_i ds + \int_0^\infty e^{-\rho s} f_{s-}^* dN_s\right] \leq C_0 + \frac{1}{\rho}(\mu_i + \lambda C_i^*)$$

for any $t < \infty$ and it now follows from Fatou's lemma that $(D^*, f^*) \in \Theta$. Applying Itô's formula for semimartingales to the process

$$Y_t = e^{-\rho(t\wedge\tau_0)} V_i(C_{t\wedge\tau_0}) + \int_{0+}^{t\wedge\tau_0} e^{-\rho s}(dD_s^* - f_{s-}^* dN_s)$$

and using the definition of (D^*, f^*) in conjunction with the fact that the function V_i solves the HJB equation we obtain that the process Y is a local martingale. Now, using the fact that $C_t \in [0, C_i^*]$ for all $t \geq 0$ together with the increase of V_i we deduce that

$$|Y_\theta| \leq |V_i(C_i^*)| + \int_0^\infty e^{-\rho t}(dD_t^* + f_{t-}^* dN_t)$$

for any stopping time θ and, since the right hand side is integrable, we conclude that the process Y is a uniformly integrable martingale. In particular, we have

$$V_i(c) = Y_{0-} = Y_0 - \Delta Y_0 = Y_0 + \Delta D_0^* = E_c[Y_{\tau_0}] + \Delta D_0^*$$
$$= E_c\left[e^{-\rho\tau_0} V_i(C_{\tau_0}) + \int_{0+}^{\tau_0} e^{-\rho s}(dD_s^* - f_{s-}^* dN_s)\right] + \Delta D_0^*$$
$$= E_c\left[e^{-\rho\tau_0}\ell_i(0) + \int_0^{\tau_0} e^{-\rho s}(dD_s^* - f_{s-}^* dN_s)\right]$$

where the third equality follows from the definition of V_i and the fourth from the martingale property of Y. This shows that $V_i \geq \hat{V}_i$ and establishes the desired result. ■

Lemma B.10 *The level of cash holdings C_i^* that is optimal for a firm with no growth option is monotone decreasing in λ and φ and increasing in σ^2.*

Proof. Monotonicity in φ follows from the definition of C_i^* and the monotonicity of ℓ_i. To establish the required monotonicity in λ, it suffices to show that $w_i(0; b, \lambda)$ is monotone decreasing in λ. Indeed, in this case we have

$$\ell_i(0) = w_i(0; C_i^*(\lambda_1), \lambda_1) \leq w_i(0; C_i^*(\lambda_1); \lambda_2)$$

for all $\lambda_1 < \lambda_2$ and therefore $C_i^*(\lambda_2) \leq C_i^*(\lambda_1)$ due to the fact that $w_i(0; b, \lambda)$ is decreasing in b. To establish the required monotonicity observe that $w_i(b; b, \lambda) = \frac{rb+\mu_i}{\rho}$ does not depend on λ. As a result, it follows from Lemma B.2 that the function defined by

$$k(c) = w_i(c; b, \lambda_1) - w_i(c; b, \lambda_2)$$

for some $\lambda_1 < \lambda_2$ satisfies

$$k(b) = k'(b) = k''(b) = k^{(3)}(b) = k^{(4)}(b) = 0$$

and solves the ODE

$$\mathcal{L}_i k(c) - \lambda_1 k(c) = (\lambda_2 - \lambda_1)(w_i(b; b, \lambda_2) - w_i(c; b, \lambda_2) - (b - c)). \tag{41}$$

Since, by Lemma B.4, $w_i(c; b, \lambda_2)$ is concave in c and $w_i'(b; b, \lambda_2) = 1$, the right hand side of (41) is nonnegative for all $c \le b$. It follows by a slight modification of Lemma B.5 that $k(c)$ cannot have a positive local maximum. Since

$$k^{(5)}(b) = \frac{2}{\sigma^2}(\lambda_1 - \lambda_2) w_i^{(3)}(b; b, \lambda_2) = \frac{2}{\sigma^2}(\lambda_1 - \lambda_2)(\rho - r) < 0,$$

we conclude that k is decreasing in a small neighborhood of b. Therefore, $k(c)$ is decreasing for all $c \le b$ and hence $k(c) > k(b) = 0$ for all $c \le b$. Similarly, if $\sigma_1^2 > \sigma_2^2$, then $k(c) = w_i(c; b; \sigma_1^2) - w_i(c; b; \sigma_2^2)$ satisfies

$$\mathcal{L}_i(\sigma_1^2) k(c) - \lambda k(c) = 0.5(\sigma_2^2 - \sigma_1^2) w_i''(c; b; \sigma_2^2) \ > \ 0$$

for $c \le b$ and the required monotonicity follows by the same arguments as above. ■

C Proof of Proposition 2

The proof of Proposition 2 will be based on a series of lemmas. To facilitate the presentation, let $\hat{V}$ denote the value of the firm and Π denote the set of triples $\pi = (\tau, D, f)$ where τ is a stopping time and $(D, f) \in \Theta$ is an admissible dividend and financing strategy.

Our first result shows that for any fixed policy π, the value of the firm is equal to the present value of all dividends net of issuing costs, up to the time τ of investment, plus the present value of the value of the firm at the time of investment.

Lemma C.1 *The value of the firm satisfies*

$$\hat{V}(c) = \sup_{\pi \in \Pi} E_c \left[\int_0^{\tau \wedge \tau_0} e^{-\rho t} (dD_t - f_t dN_t) + 1_{\{\tau \ge \tau_0\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau < \tau_0\}} e^{-\rho \tau} V_1(C_\tau) \right].$$

If $V_0(c) \ge V_1(c - K)$ for all $c \ge K$ then it is optimal to abandon the growth option.

Proof. The proof of the first part follows from standard dynamic programming arguments and therefore is omitted. To establish the second part assume that $V_0(c) \ge V_1(c - K)$ for all $c \ge K$ and observe that $\Delta C_\tau = -K + 1_{\{\tau \in \mathcal{N}\}} f_\tau$ where $\mathcal{N}$ denotes the set of jump times of the Poisson process. Using this identity in conjunction with the first part, we obtain

$$V(c) \leq \sup_{\pi\in\Pi} E_c\bigg[\int_0^{\tau\wedge\tau_0} e^{-\rho t}(dD_t - f_t dN_t) \\ + 1_{\{\tau\geq\tau_0\}}e^{-\rho\tau_0}\ell_0(0) + 1_{\{\tau<\tau_0\}}e^{-\rho\tau}V_0(C_{\tau-} + 1_{\{\tau\in\mathcal{N}\}}f_{\tau-})\bigg].$$

and the desired result follows since the right hand side of this inequality is equal to $V_0(c)$ by standard dynamic programming arguments. ■

By Lemma C.1, the option has a non-positive net present value if and only if $V_0(c) \geq V_1(c-K)$ for all $c \geq 0$. Thus, in order to establish Proposition 2 it now suffices to show that this condition is equivalent to the inequality $K \geq K^*$. This is the objective of the following:

Lemma C.2 *The constant K^* is nonnegative and the following are equivalent:*

(1) $K \geq K^*$

(2) $K \geq V_1(C_1^*) - V_0(C_0^*) - (C_1^* - C_0^*)$

(3) $V_0(c) \geq V_1(c-K)$ *for all* $c \geq K$.

Proof. The equivalence of (1) and (2) follows from the definition of K^* and the fact that

$$V_i(C_i^*) = (rC_i^* + \mu_i)/\rho.$$

In order to show that the constant K^* is nonnegative we argue as follows: Since $\mu_0 < \mu_1$, the set of feasible strategies for V_0 is included in the set of feasible strategies for V_1. It follows that $V_0 \leq V_1$ and combining this with the definition of C_i^* shows that

$$K^* = V_1(C_1^*) - V_0(C_0^*) - (C_1^* - C_0^*) = \max_{C\geq 0}\{V_1(C) - C\} - \max_{C\geq 0}\{V_0(C) - C\} \geq 0.$$

To establish the implication (1) ⇒ (3) it suffices to show that under (1) we have $V_1(c-K^*) \leq V_0(c)$ for all $c \geq K^*$. Indeed, if that is the case then (3) also holds since

$$V_1(c-K) \leq V_1(c-K^*), \qquad c \geq K \geq K^*$$

due to the increase of the function V_1. For $c \geq K^* \vee C_0^*$ the concavity of V_1 and the fact that V_0 is linear with slope one above the level C_0^* imply that

$$V_1(c-K^*) \leq V_1(C_1^*) + (c - K^* - C_1^*) = V_0(c) + C_0^* - V_0(C_0^*) - K^* - C_1^* = V_0(c)$$

and it remains to prove the result for $c \in [K^*, C_0^*]$. Consider the function $k(c) = V_0(c) - V_1(c-K^*)$. Using Lemma B.2 in conjunction with the fact that $C_0^* < C_1^* + K^*$ by Lemma C.3 below we have that the function k is a solution to

$$\mathcal{L}_0 k(c) - \lambda k(c) + (-\mu_1 + \mu_0 + rK^*)V_1'(c-K^*) = 0$$

on the interval $[K^*, C_0^*]$. Combining Lemma C.3 below with the increase of V_1 shows that the last term on the left hand side is positive and since

$$k(C_0^*) = V_0(C_0^*) - V_1(C_0^* - K^*) \geq V_0(C_0^*) - V_1(C^*) - (C_0^* - K^* - C_1^*) = 0,$$
$$k'(C_0^*) = V_0'(C_0^*) - V_1'(C_0^* - K^*) = 1 - V_1'(C_0^* - K^*) \leq 0$$

by the concavity of V_1, we can apply Lemma B.6 to conclude that $k(c) \geq 0$ for all $c \leq C_0^*$. Finally, the implication (3) $\Rightarrow$ (2) follows by taking $c > C_0^* \vee (C_1^* + K)$. ■

The optimal cash levels before and after investment are given by C_0^* and $C_1^* + K$ respectively. By Lemma **??**, $V_0(c) \geq V_1(c - K^*)$. Therefore, it is natural to expect that the better-off firm $V_0(c)$ will have a lower optimal cash level, that is $C_0^* \leq C_1^* + K^*$. Furthermore, it is also natural to expect that the increase in the mean cash flow rate net of cost is nonnegative for $K \leq K^*$, that is $(\mu_1 - \mu_0)/r \leq K$. It turns out that both of these intuitive results are indeed true, as is shown by the following lemma.

Lemma C.3 *We have $C_0^* < C_1^* + K^*$ and $\mu_1 - \mu_0 - rK^* > 0$.*

Proof. The definition of K^* implies that the first inequality is equivalent to the second which is in turn equivalent to $C_1^* - C_0^* > (\mu_0 - \mu_1)/r$. Denote by $C^*(\mu)$ the optimal cash holdings of the firm with mean cash flow rate μ and without a growth option. To prove the inequality $C_1^* - C_0^* > (\mu_0 - \mu_1)/r$, it suffices to show that $C^*(\mu)$

$$-\partial C^*(\mu)/\partial\mu < 1/r. \tag{42}$$

Let $w(c; b, \mu)$ be the function $w_0(c; b)$ with $\mu_0 = \mu$. By Lemma B.2, using the identity $w(b; b, \mu) = \frac{\mu + br}{\rho}$, we get that

$$w(c; b, \mu) \;=\; \tilde{V}(c; b, \mu) \;+\; \frac{\lambda}{\rho + \lambda}\left[\frac{\mu + br}{\rho} - b + c + \frac{\mu_i + rc}{\rho + \lambda - r}\right]$$

where the function $\tilde{V}$ is defined by

$$\tilde{V}(c; b, \mu) = \alpha(b; \mu) F(c; \mu) - \beta(b; \mu) G(c; \mu)$$

for α and β are defined as

$$\alpha_i(c) = \frac{-G_i''(c)(\rho - r)}{2\sigma^{-3}\sqrt{r}\,(\rho + \lambda - r)(\rho + \lambda)\, e^{-(\sigma^2 r)^{-1}(rC + \mu)^2}},$$
$$\beta_i(c) = \frac{-F_i''(c)(\rho - r)}{2\sigma^{-3}\sqrt{r}\,(\rho + \lambda - r)(\rho + \lambda)\, e^{-(\sigma^2 r)^{-1}(rC + \mu)^2}}$$

Therefore, equation $w(0; C^*(\mu), \mu) = \frac{\varphi}{\rho}\mu$ takes the form

$$\tilde{V}(0; C^*(\mu), \mu) + \frac{\lambda^*}{\lambda^* + \rho}\left(\frac{\mu + (r-\rho)C^*(\mu)}{\rho} + \frac{\mu}{\lambda^* + \rho - r}\right) = \varphi\frac{\mu}{\rho}\,.$$

Using equations (36) and (37) in conjunction with the definition of the functions α and β, we obtain

$$\tilde{V}_\mu(0; b, \mu) = (\tilde{V}_c(0; b, \mu) + \tilde{V}_b(0; b, \mu))/r,$$

where a subscript denotes a partial derivative. It follows that

$$\frac{\partial C^*(\mu)}{\partial \mu} = \frac{-\tilde{V}_b(0; C^*(\mu), \mu)/r - \tilde{V}_c(0; C^*(\mu); \mu)/r + \varphi/\rho - B}{\tilde{V}_b(0; C^*(\mu), \mu) - A}$$

where we have set

$$A = \frac{\lambda}{\lambda + \rho}\left(1 - \frac{r}{\rho}\right), \quad B = \frac{\lambda}{\lambda + \rho}\left(\frac{1}{\rho} + \frac{1}{\lambda + \rho - r}\right).$$

By Lemma B.4 we have that $\tilde{V}$ is decreasing in b and since $A > 0$ it follows that the validity of equation (42) is equivalent to

$$-\tilde{V}_b(0; C^*(\mu), \mu) - \tilde{V}_c(0; b; \mu) - r\,(B - \varphi/\rho) < A - \tilde{V}_b(0; C^*(\mu), \mu),$$

which in turn follows from

$$\tilde{V}_c(0; b; \mu) + r(B - 1/\rho) > 0. \tag{43}$$

Since the difference $\tilde{V} - V$ is a linear function of c we have from Lemma B.4 that the function $\tilde{V}$ is concave. Thus, it follows from the smooth pasting condition that

$$\begin{aligned}\tilde{V}_c(0; C^*(\mu), \mu) &= V_c(0; C^*(\mu), \mu) - \frac{\lambda}{\lambda + \rho - r} \\ &\geq V_c(C^*(\mu); C^*(\mu), \mu) - \frac{\lambda}{\lambda + \rho - r} = \frac{\rho - r}{\lambda + \rho - r}\end{aligned}$$

and combining this with a straightforward calculation shows that (43) holds. ■

D Additional proofs

These proofs are relegated to the Supplementary Appendix.

References

Abel, A. and J. Eberly, 1996, "Optimal Investment with Costly Reversibility,", *Review Economic Studies* 63, 581-593.

Abel, A. and J. Eberly, 2011, "How Q and Cash Flow Affect Investment without Frictions: An Analytic Explanation,", *Review Economic Studies* 78, 1179-1200.

Almeida, H., and M. Campello, 2007, "Financial Constraints, Asset Tangibility, and Corporate Investment," *Review of Financial Studies* 20, 1429-1460.

Alvarez, F., and F. Lippi, 2012, "The Demand of Liquid Assets with Uncertain Lumpy Expenditures,"Working paper, University of Chicago.

Alvarez, F., and F. Lippi, 2009, "Financial Innovation and the Transactions Demand for Cash," *Econometrica* 77, 363-402.

Bar-Ilan, A., 1990, "Overdrafts and the Demand for Money," *American Economic Review* 80, 1201-1216.

Bar-Ilan, A., D. Perry, and W. Stadje, 2004, "A Generalized Impulse Control Model of Cash Management," *Journal of Economic Dynamics and Control* 28, 1013-1033.

Bar-Ilan, A., and W. Strange, 1996, "Investment Lags," *American Economic Review* 86, 610-622.

Bates, T., K. Kahle, and R. Stulz, 2009, "Why Do U.S. Firms Hold so Much More Cash Than They Used To?" *Journal of Finance,* 64, 1985-2021.

Baumol, W., 1952, "The Transactions Demand for Cash: An Inventory Theoretic Model," *Quarterly Journal of Economics* 66, 545-556.

Becker, B., 2007, "Geographical Segmentation of US Capital Markets," *Journal of Financial Economics* 85, 151-178.

Bolton, P., H. Chen, and N. Wang, 2011, "A Unified Theory of Tobin's q, Corporate Investment, Financing, and Risk Management," *Journal of Finance,* 66, 1545-1578.

Bolton, P., H. Chen, and N. Wang, 2012, "Market Timing, Investment, and Risk Management," *Journal of Financial Economics* Forthcoming.

Boyarchenko, S., 2004, "Irreversible Decisions and Record-Setting News Principles," *American Economic Review* 94, 557-568.

Busaba, W., L. Benveniste and R.-J. Guo, 2001, "The Option to Withdraw IPOs during the Premarket: Empirical Analysis," *Journal of Financial Economics* 60, 73-102.

Caballero, R., and E. Engel, 1999, "Explaining Investment Dynamics in the U.S. Maufacturing: A Generalized (S,s) Approach," *Econometrica* 67, 783-826.

Campello, M., J. Graham, and C. Harvey, 2010, "The Real Effects of Financial Constraints: Evidence from a Financial Crisis," *Journal of Financial Economics* 97, 470-487.

Carlson, M., A. Fisher, and R. Giammarino, 2010, "SEO Risk Dynamics," *Review of Financial Studies* 23, 3966-3995.

Choi, D., M. Getmansky, B. Henderson, and H. Tookes, 2010, "Convertible Bond Arbitrageurs as Suppliers of Capital," *Review of Financial Studies* 23, 2492-2522.

Cooper, R., J. Haltiwanger, and L. Power, 1999, "Machine Replacement and the Business Cycle: Lumps and Bumps," *American Economic Review* 89, 921-946.

Dayanik, S., and I. Karatzas, 2003, "On the Optimal Stopping Problem for One-Dimensional Diffusions," *Stochastic Processes and Their Applications* 107, 173-212.

Décamps, J.P., T. Mariotti, J.C. Rochet, and S. Villeneuve, 2011, "Free Cash Flow, Issuance Costs, and Stock Price Volatility," *Journal of Finance,* 66, 1501-1544.

Décamps, J.P., and S. Villeneuve, 2007, "Optimal Dividend Policy and Growth Option," *Finance and Stochastics* 11, 3-27.

Dellacherie, C., and P.A. Meyer, 1980, *Probabilité et Potentiel: Théorie des Martingales*, Hermann.

Dixit, A., 1989, "Entry and Exit Decisions under Uncertainty," *Journal of Political Economy* 97, 620-638.

Doms, M., and T. Dunne, 1998, "Capital Adjustment Patterns in Manufacturing Plants," *Review of Economic Dynamics* 1, 409-429.

Duchin, R., O. Ozbas, and B. Sensoy, 2010, "Costly External Finance, Corporate Investment, and the Mortgage Credit Crisis," *Journal of Financial Economics* 97, 418-435.

Duffie, D., N. Garleanu, and L. Pedersen, 2005, "Over-the-Counter Markets," *Econometrica* 73, 1815-1847.

Dumas, B., 1991, "Super Contact and Related Optimality Conditions,"*Journal of Economic Dynamics and Control* 15, 675-685.

Dunbar, C., and S. Foerster, 2008, "Second Time Lucky? Withdrawn IPOs That Return to the Market," *Journal of Financial Economics* 87, 610-635.

Fleming, W., and M. Soner, 1993, *Controlled Markov Processes and Viscosity Solution*, Springer Verlag, New-York.

Gan, J., 2007, "The Real Effects of Asset Market Bubbles: Loan- and Firm-Level Evidence of a Lending Channel," *Review of Financial Studies* 20, 1941-1973.

Gomes, J., 2001, "Financing Investment," *American Economic Review* 91, 1263-1285.

Guo, X., J. Miao, and E. Morellec, 2005, "Irreversible Investment with Regime Shifts," *Journal of Economic Theory* 122, 37-59.

Harford, J., 1999, "Corporate Cash Reserves and Acquisitions", *Journal of Finance* 54, 1969-1997.

Harford, J., S. Mansi, and W. Maxwell, 2008, "Corporate Governance and a Firm's Cash Holdings," *Journal of Financial Economics* 87, 535–555.

Hartman, P. 1982. Ordinary Differential Equations. 2nd ed. Boston: Birkhaüser.

Karatzas, I., and S. Shreve, 1991, *Brownian Motion and Stochastic Calculus*, Springer Verlag, New York.

Kashyap, A., O. Lamont, and J. Stein, 1994, "Credit Conditions and the Cyclical Behavior of Inventories," *Quarterly Journal of Economics* 109, 565-592.

Kashyap, A., J. Stein, and D. Wilcox, 1993, "Monetary Policy and Credit Conditions: Evidence from the Composition of External Finance," *American Economic Review* 83, 78-98.

Kim, W., and M. Weisbach, 2008, "Motivations for Public Equity Offers: An International Perspective," *Journal of Financial Economics* 87, 281-307.

Lagos, R., and G. Rocheteau, 2009, "Liquidity in Asset Markets with Search Frictions," *Econometrica* 77, 403-426.

Lemmon, M., and M. Roberts, 2010, "The Response of Corporate Financing and Investment to Changes in the Supply of Credit," *Journal of Financial and Quantitative Analysis* 45, 555-587.

Lins, K., H. Servaes, and P. Tufano, 2010, "What Drives Corporate Liquidity? An International Survey of Cash Holdings and Lines of Credit," *Journal of Financial Economics* 98, 160-176.

Manso, G., 2008, "Investment Reversibility and the Agency Cost of Debt," *Econometrica* 76, 437-442.

Massa, M., A. Yasuda, and L. Zhang, 2010, "Supply Investment Horizon of the Bond Investor Base and the Leverage of the Firm," Working Paper, INSEAD.

McDonald, R., Siegel, J., 1986, "The value of waiting to invest," *Quarterly Journal of Economics* 101, 707-728.

McLean, D., 2011, "Share Issuance and Cash Savings," *Journal of Financial Economics* 99, 693-715.

Miller, M. and D. Orr, 1966, "A Model of Demand for Money by Firms," *Quarterly Journal of Economics* 80, 413-435.

Modigliani, F. and M. Miller, 1958, "The Cost of Capital, Corporation Finance and the Theory of Investment," *American Economic Review* 48, 261-297.

Morellec, E., B. Nikolov, and F. Zucchi, 2013, "Cash Holdings and Competition," Working Paper, University of Rochester.

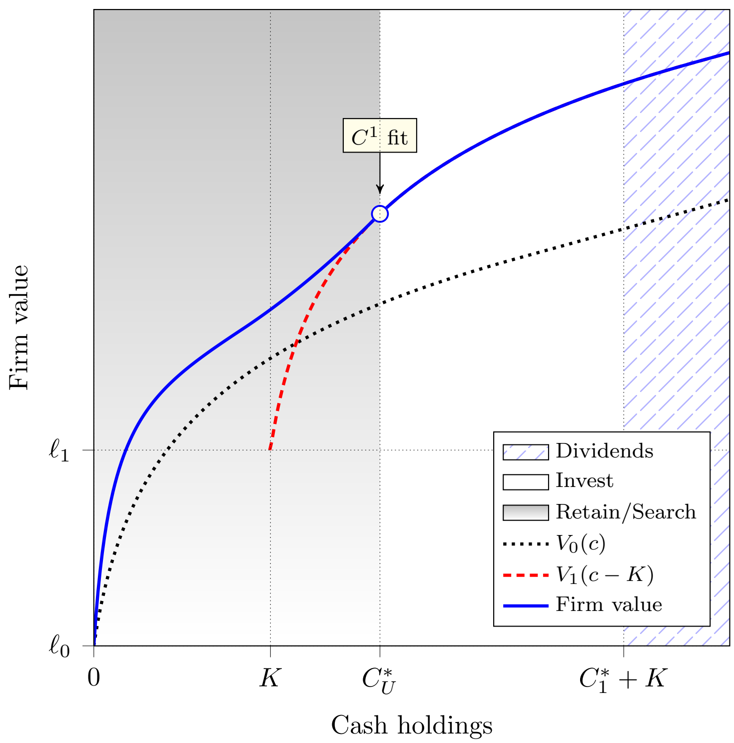
Opler, T., L. Pinkowitz, R. Stulz, and R. Williamson, 1999, "The determinants and implications of corporate cash holdings," *Journal of Financial Economics* 51, 219–244.

Radner, R., and L. Shepp, 1996, "Risk vs. Profit Potential: A Model for Corporate Strategy", *Journal of Economic Dynamics and Control* 20, 1373-1393.

Tobin, J., 1968, "The Interest Elasticity of Transactions Demand for Money," *Review of Economics and Statistics* 38, 241-247.

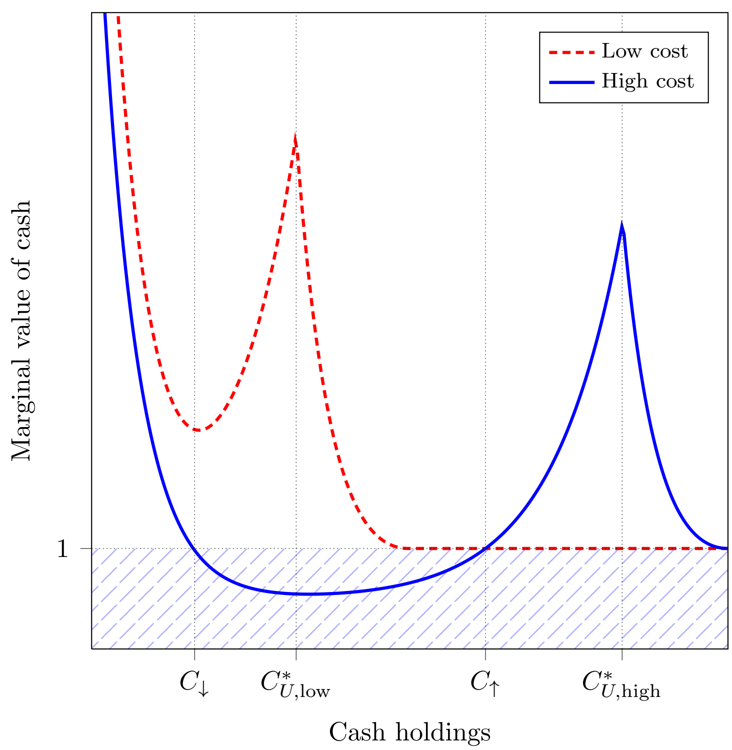
Vayanos, D., and P. O. Weill, 2008, "A Search-Based Theory of the On-the-Run Phenomenon," *Journal of Finance* 63, 1351-1389.

Figure 1: Value of a firm with a growth option when $\underline{K} < K \leq K^{**}$



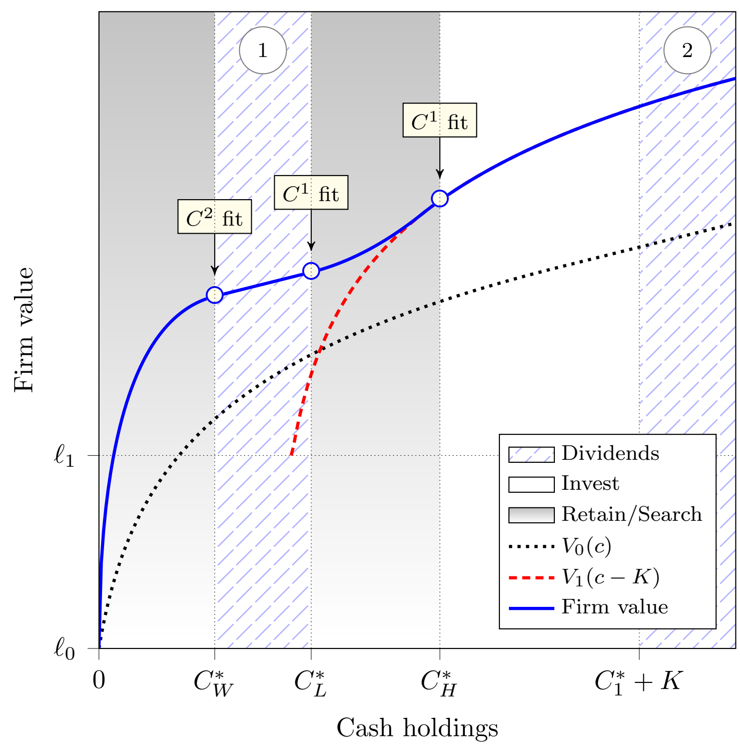
This figure represents the value of a firm with a growth option as a function of its cash holdings when $\underline{K} < K \leq K^{**}$. In the shaded area below the investment trigger C_U^*, the optimal policy is to retain earnings and search for investors. In the unshaded area between C_U^* and $C_1^* + K$, the optimal policy is to invest in the growth option from internal funds. In the hatched area above $C_1^* + K$, the optimal policy is to invest with internal funds and distribute dividends to decrease cash holdings to the target level C_1^* after investment.

Figure 2: Marginal value of cash under the optimal barrier strategy



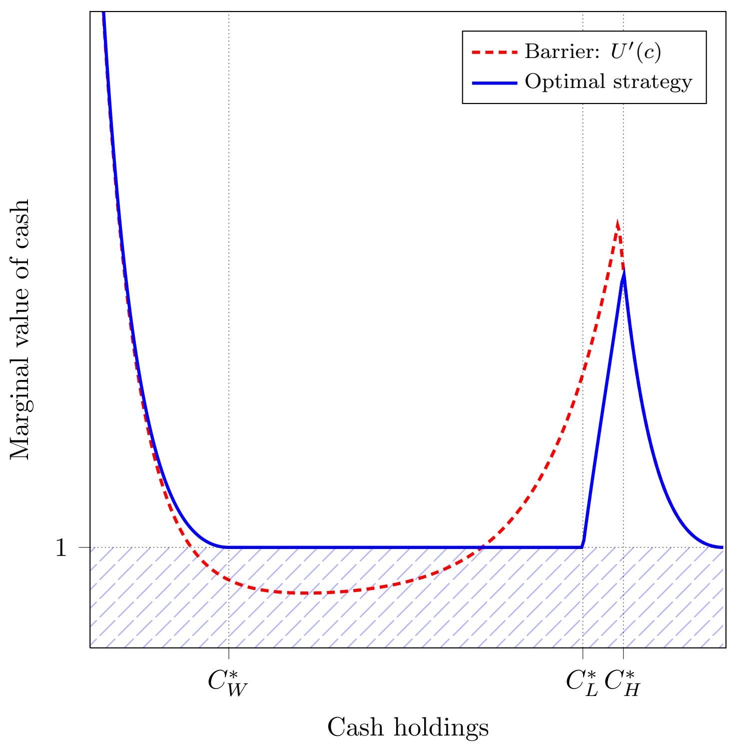
This figure plots the marginal value of cash $U'(c)$ under the optimal barrier strategy in an environment where the investment cost is low (dashed line) and in an environment where the costs of investment is high (solid line). In the latter case the marginal value of the cash drops to one at the point $C_\downarrow$ and remains below one over the interval $(C_\downarrow, C_\uparrow)$ indicating that shareholders would rather abandon the option of investing from internal funds and receive dividends than continue hoarding cash in side the firm.

Figure 3: Value of a firm with a growth option when $K^{**} < K \leq K^{*}$



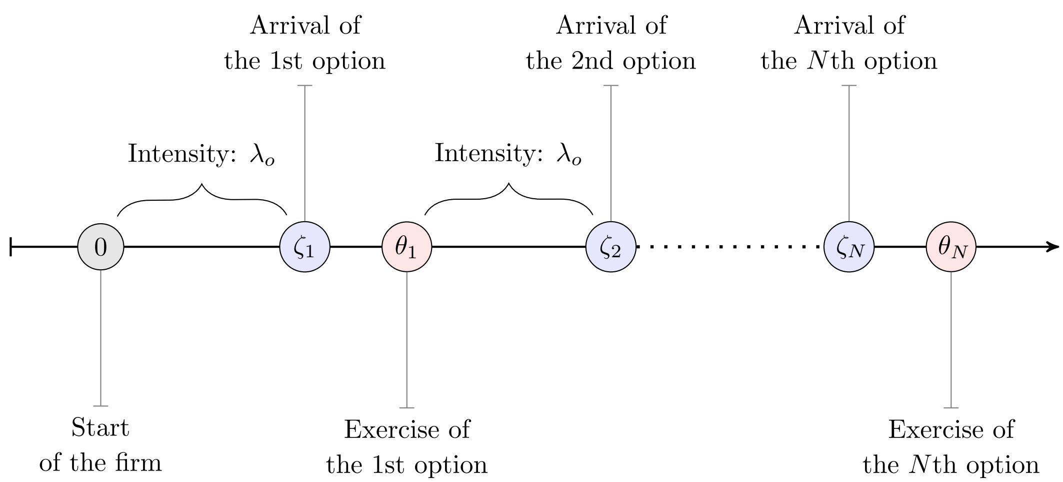
This figure represents the value of a firm with a growth option as a function of its cash holdings when $K^{**} \leq K < K^{*}$. In the shaded areas the optimal policy is to retain earnings and search for investors. In the first hatched area, the optimal policy is to distribute dividends to decrease the level of cash holdings to C_W^*. In the unshaded area between C_H^* and $C_1^* + K$ the optimal policy is to invest in the growth option with internal funds. In the second hatched area, the optimal policy is to invest with internal funds and distribute dividends in order to decrease cash holdings to the target level C_1^* after investment.

Figure 4: Marginal value of cash for a firm with a growth option



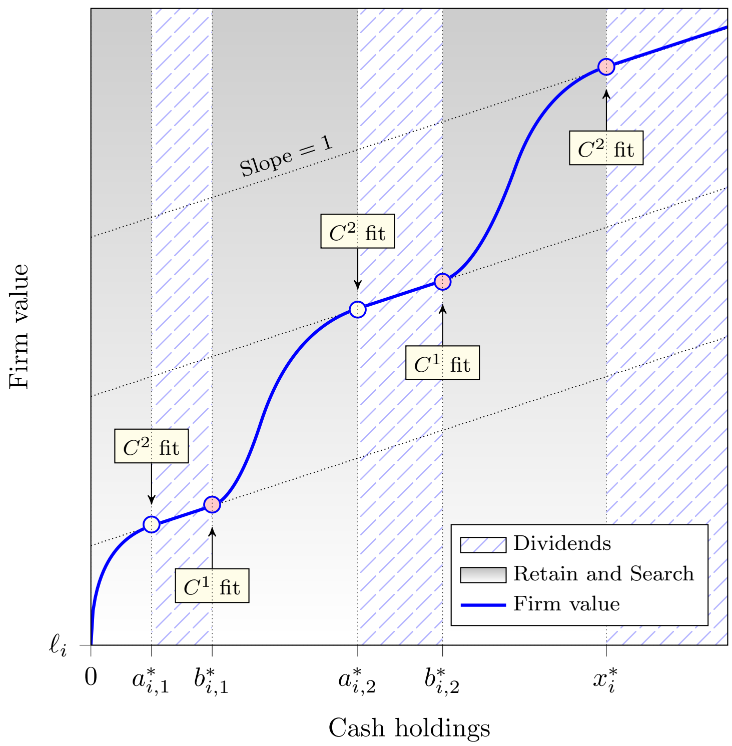
This figure plots the marginal value of cash $V'(c)$ under the globally optimal strategy (dashed line) and the marginal value of cash $U'(c)$ under the optimal barrier strategy (solid line) in an environment where the investment cost is high. In the latter case the failure of global optimality is due to the fact that the marginal value of the cash drops below one indicating that shareholders would rather abandon the option of investing from internal funds and receive dividends than continue hoarding cash in side the firm.

Figure 5: The model with multiple growth options



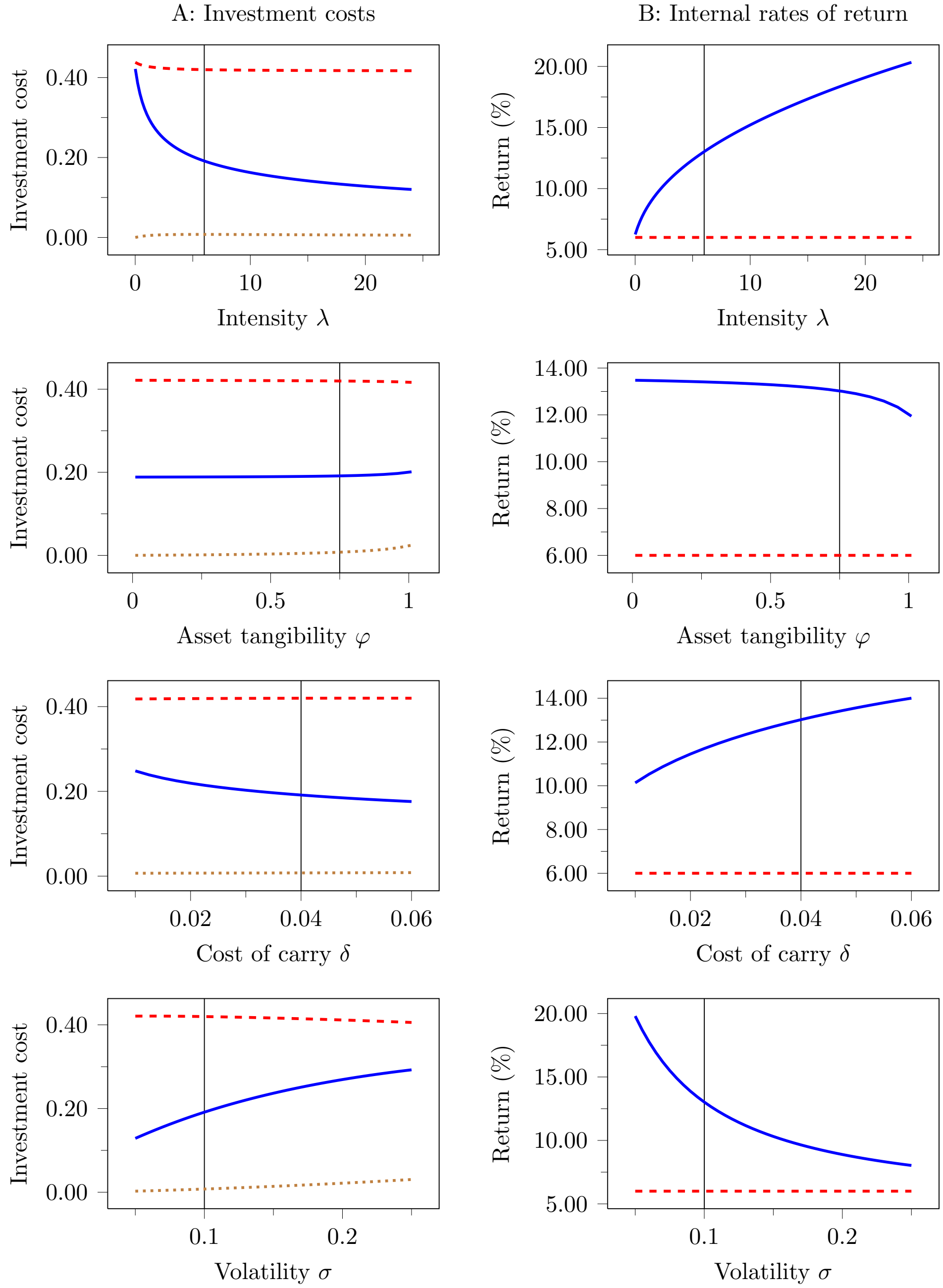
This figure illustrates the model with multiple options: The firm initially has mean cash flow rate μ_0 and does not any growth option. At the exponentially distributed time ζ_1 the firm receives its first growth option and exercises optimally at the stopping time θ_1. The second growth option then arrives after the exponentially distributed time $\zeta_2 - \theta_1$ has elapsed and is optimally exercised at the stopping time θ_2. This goes on until the optimal exercise of the last growth option at the stopping time θ_N. After that time the mean cash flow rate of the firm remains constant.

Figure 6: Value of the firm in the waiting period between growth options



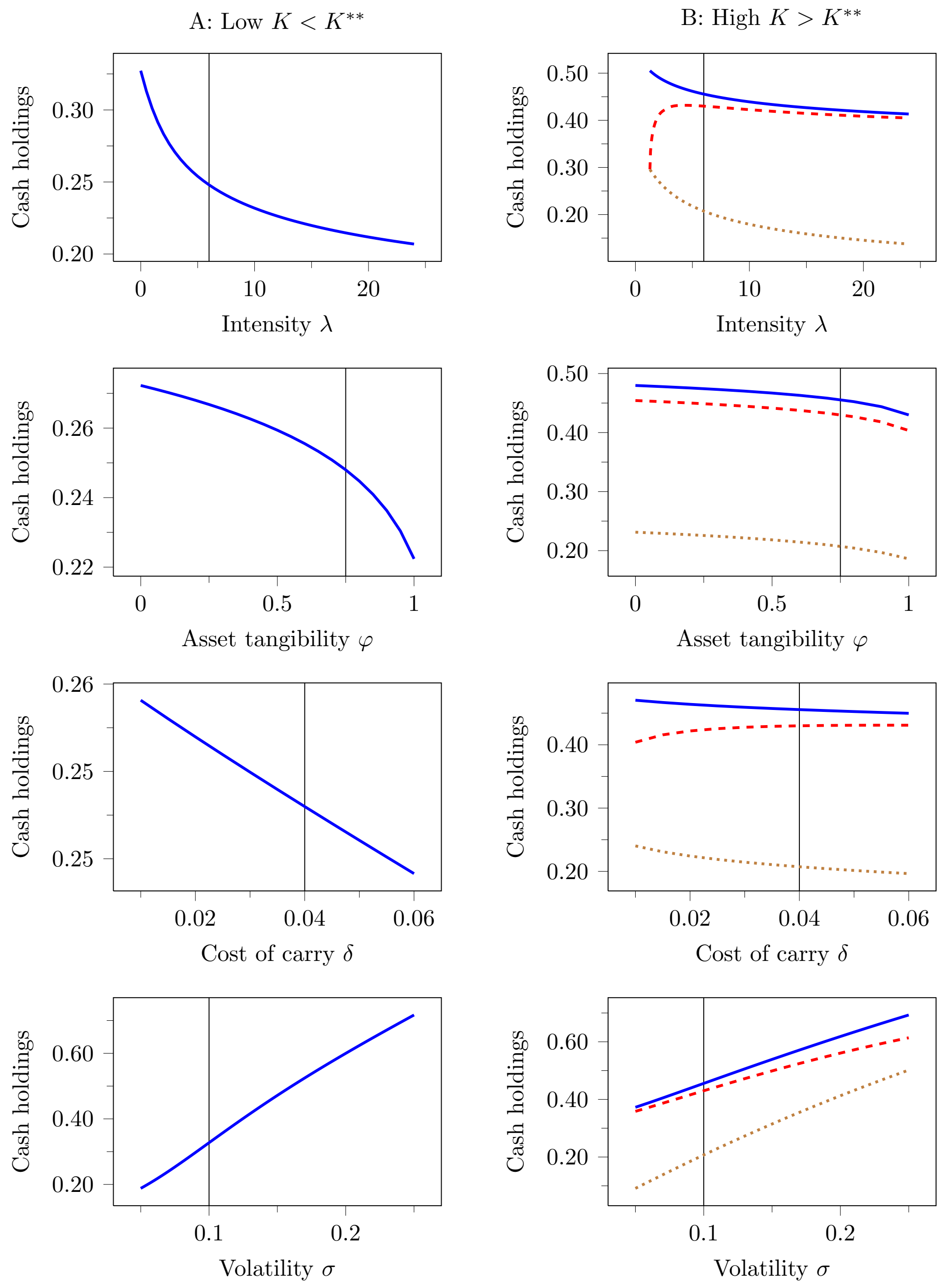
This figure represents the value of a firm as a function of its cash holdings in the waiting period between the exercise of the i'th growth option and the arrival of the next one. In this picture the optimal strategy includes two intermediate dividend distribution intervals and three earnings retention intervals whose location are specified by the vectors (a^*_i, b^*_i) and the target x^*_i.

Figure 7: Critical investment costs and internal rates of return



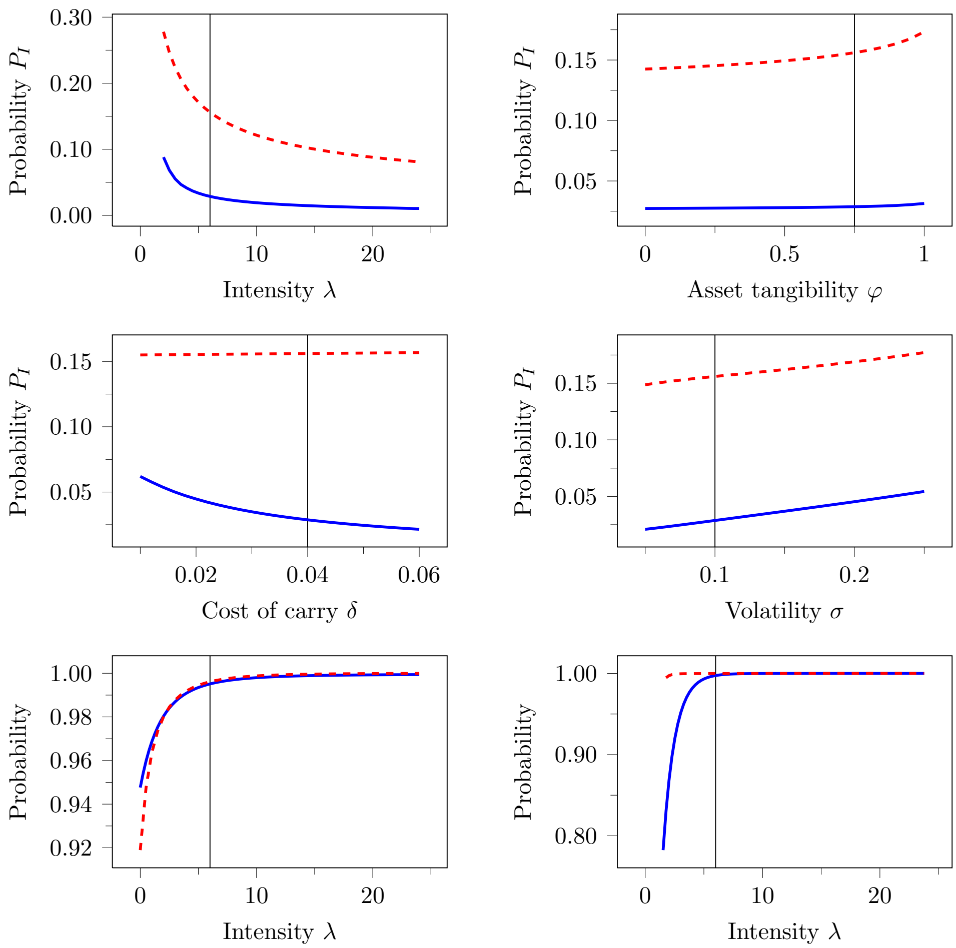
Panel A plots the critical investment costs $\underline{K}$ (dotted), K^{**} (solid) and K^{*} (dashed) as functions of the arrival rate of investors λ, the tangibility of assets φ, the carry cost of cash δ and the volatility of cash flows σ. Panel B plots the internal rate of returns associated with K^{**} (solid) and K^{*} (dashed) as functions of the same parameters. In each plot the vertical line indicates the base value of the parameter.

Figure 8: Optimal cash holdings for a firm with a growth option



Panel A plots the investment threshold C_U^* for a firm with a low investment cost as a function of the arrival rate of investors λ, the tangibility of assets φ, the carry cost of cash δ and the volatility of cash flows σ. Panel B plots the investment threshold C_H^* (solid) and the payout thresholds C_L^* (dashed) and C_W^* (dotted) for a firm with a high investment cost as functions of the same parameters. In each plot the vertical line indicates the base value of the parameter.

Figure 9: Probabilities of investment



The top four panels plots the average probability of investment with internal funds for a firm with a low investment cost (dashed line) and a firm with a high investment cost (solid line) as functions of the arrival rate of investors λ, the tangibility of assets φ, the carry cost of cash δ and the volatility of cash flows σ. The two lower panels plot the total probability of investment at an horizon of one year (solid line) and three years (dashed line) for a firm with cash holdings $C = K < K^{**}$ (left) and $C = K > K^{**}$ (right) as functions of the arrival rate of investors λ. In each plot the vertical line indicates the base value of the parameter.