Term Structure of Disagreement and Predictability over the Business Cycle*

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Abstract

Our objective is to identify the mechanism that causes return predictability to vary over the business cycle and the trading strategy that allows an investor to take advantage of it. We build an equilibrium model in which some investors believe the economy transits slowly from good to bad times, while others believe economic conditions can change precipitately. This particular type of heterogeneous beliefs leads to a time-varying term structure of disagreement whereby return predictability rises as economic conditions deteriorate. In bad times, disagreement spikes in the short term, generating short-term momentum followed by a long-term reversion to fundamentals. In good times, instead, disagreement is steady and therefore prices revert instantly to fundamentals. Investors implement momentum strategies to extract momentum in bad times and time the market to avoid momentum crashes when the market rebounds. We test these predictions and find that short-term momentum and momentum profits are larger during recessions, particularly during the crises of 1929 and 2008.

Keywords: Asset Pricing, Learning, Term Structure of Disagreement, Business Cycle, Predictability, Times-Series Momentum, Momentum Crashes

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1 Introduction

Recent evidence suggests that economic downturns are favorable times to invest in the stock market. First, stock return predictability rises during recessions—for instance, media pessimism (Garcia (2013)) and the price-dividend ratio (Henkel, Martin, and Nardari (2011)) better predict future returns in bad times.[1] Second, institutional investors perform better (Moskowitz (2000)) and are better able to time the market (Kacperczyk, van Nieuwerburgh, and Veldkamp (2013)) in recessions.[2] Although these facts are well-established, the mechanism that generates them remains unclear.

In this paper we ask what could cause return predictability to vary over the business cycle and examine which trading strategy would allow an investor to take advantage of it. We propose an explanation based on the empirical finding that disagreement is strongest in bad times (Patton and Timmermann (2010)).[3] We show that, when investors use different models to predict the state of the business cycle, large spikes in investors' disagreement are concentrated in bad times. Our main finding is that large disagreement in bad times causes asset prices to react to stale information. Stock return predictability therefore improves as economic conditions deteriorate. An investor whose model best predicts the state of the business cycle can exploit return predictability and make significantly larger profits in bad times.

We build a dynamic general equilibrium model in which two agents, A and B, trade one stock and one riskless bond. Agents do not observe the expected growth rate of dividends paid by the stock and have different views regarding the empirical process that governs it. Agent A shares the view that the expected growth rate of dividends is mean-reverting and smooth; she thinks that the economy transits from good to bad times according to an autoregressive model.[4] In contrast, Agent B believes that the economy may transit precipitately from good to bad times according to a Markov switching model; the Economics literature typically uses this type of model to forecast business cycle turning points and

[1] Rapach, Strauss, and Zhou (2010), Dangl and Halling (2012), and Gargano (2013) find similar patterns of predictability in recessions.

[2] Glode (2011) and Kosowski (2011) show that fund managers perform significantly better in recessions than they do in expansions. Lustig and Verdelhan (2010) show that the risk-return trade-off varies significantly over the business cycle.

[3] Carlin, Longstaff, and Matoba (2013) and Barinov (2014) provide empirical evidence that disagreement tends to be larger in downturns than in upturns. Veronesi (1999) also shows that the dispersion of analysts' forecasts is greater when the economy is contracting.

[4] This type of model is adopted by Detemple (1986, 1991), Wang (1993), Brennan and Xia (2001), Scheinkman and Xiong (2003), Bansal and Yaron (2004), Dumas, Kurshev, and Uppal (2009) among others.

recession probabilities.[5] Agents observe the same information (past dividends) but use their own model to infer the expected growth rate of dividends, which generates disagreement about the estimated growth rate.[6]

Agents agree to disagree on the model they use; which model is best is specific to an agent's need as both model have their own advantages and disadvantages. For instance, an autoregressive model better measures the level of the business cycle, while a Markov chain better detects sharp variations in the business cycle. Agent A's model therefore performs better in normal times, while Agent B's model performs better during turbulent periods in which the business cycle shifts quickly from a peak to a trough and vice-versa. We can, however, determine which model better fits historical data by calibrating each model to historical S&P500 dividends. We find that an information criterion favors Agent A's model and we choose to adopt the perspective of this agent when we analyze stock returns predictability and trading strategies.[7]

Heterogenous models give rise to a state-dependent term structure of disagreement, the shape of which varies significantly over the business cycle.[8] Such dynamics naturally arise as we let agents calibrate their own model to historical data. The data tell Agent A that the autoregressive model exhibits significant persistence, while the data tell Agent B that the Markov chain switches between a persistent expansionary regime and a non-persistent recessionary regime. This regime asymmetry together with the specific properties of each model affect disagreement in two different ways. First, they polarize agents' opinion, but only in bad times and in the short term. Namely, in bad times, Agent A expects the economy to recover, while Agent B expects economic conditions to further deteriorate in the short term. Second, they cause agents to adjust their expectations at different speeds. The persistence of Agent A's autoregressive model induces her to adjust her expectations slowly. Instead, Agent B adjusts her expectations only rarely; when these adjustments occur, they are fast and strong.

In bad times, polarization of opinions generates a spike in disagreement in the short term. In the long term, opinions align as Agent B eventually realizes that the economy is recovering. Since Agent B adjusts her views significantly faster than Agent A, disagreement decreases

[5]See Milas, Rothman, and van Dijk (2006) for a reference to the Economics literature, in particular Hamilton and Chauvet (2006). This type of model is adopted by David (1997, 2008), Veronesi (1999, 2000), Chen (2010), Bhamra, Kuehn, and Strebulaev (2010), and David and Veronesi (2013) among others.

[6]In contrast, disagreement arises in, for instance, Scheinkman and Xiong (2003), Dumas et al. (2009), and Xiong and Yan (2010) because some agents misinterpret public information.

[7]The Akaike Information Criterion shows that, between 1871 and 2013, the model of Agent A turns out to better fit S&P500 dividend data, than that of Agent B. Therefore, throughout the paper, we choose to work under the probabilistic views of Agent A and analyze the trading strategies of Agent A only.

[8]The term structure of disagreement plots the expected future disagreement among investors for different horizons.

towards zero before increasing again. In normal times, the high speed of adjustment of Agent B's expectations generates a spike in disagreement in the short term, which gradually shrinks as Agent A catches up with Agent B. In good times, agents' expectations are close to their long-term mean and therefore adjust at comparable speeds, implying little variation in disagreement. As a result, disagreement is strongest in bad times both in the short and the long term, consistent with the term structure of disagreement documented empirically by Patton and Timmermann (2010).

We show that this term-structure of disagreement leads to short-term downward pressure on prices followed by a reversion to fundamentals at all times. Consequently, disagreement generates low returns at short horizons, consistent with the empirical finding of Tetlock (2007). But, since the term structure of disagreement is time-varying, the magnitude of this effect varies over the business cycle. Short-term predictability concentrates in normal and bad times, and becomes negligible in good times, consistent with Garcia (2013).

In our model, short-term predictability in bad times results from a persistent and decreasing price reaction to a fundamental shock, along the lines of the *under-reaction* phenomenon of Jegadeesh and Titman (1993).[9] In bad times, polarization of opinions implies a spike in disagreement in the short term. Agents are slow to reach an agreement and therefore prices continue to react to the shock. On the contrary, short-term predictability in normal times results from a persistent and increasing price reaction to a fundamental shock, similar to the *over-reaction* phenomenon of De Bondt and Thaler (1985). In normal times, opinions align but Agent B revises her expectations faster than Agent A. This adjustment speed effect generates a spike in disagreement in the short term which causes prices to continue to react to the shock. Whether the price reaction is downward- or upward-slopping is determined by agents' relative optimism or pessimism. In bad times, Agent A is more optimistic than Agent B and the price reaction is downward-sloping. Instead, in normal times, Agent A is more pessimistic than Agent B and the price reaction is upward-slopping.

That prices continue to react to past information produces time-series momentum (Moskowitz, Ooi, and Pedersen (2012)) in excess returns. Returns keep moving in the same direction over short horizons and revert over long horizons. Importantly, our model implies that time-series momentum in the very short run—at a one-month lag—is stronger in bad times than it is in normal times. The reason is that, in the very short term, disagreement spikes significantly faster in bad times than in normal times. Moreover, in good times, the time-series properties of excess returns differ. In particular, excess returns exhibit an initial reversal spike at short

[9] In the behavioral finance literature, a price under/over-reaction to a shock is relative to the reaction that would have prevailed, had agents been rational (Daniel, Hirshleifer, and Subrahmanyam (1998), Barberis, Shleifer, and Vishny (1998)).

horizons, because there is little variation in disagreement in this state of the economy.

Agent A, whose model best fits historical data, exploits short-term predictability to time momentum. Both in normal and bad times, she is a short-term momentum trader. She recognizes that excess returns exhibit time-series momentum and takes advantage of this pattern by following the market. In the long run, she reverts her strategy and becomes a contrarian. She expects a long-term reversion to fundamentals and thus extracts stock return reversal by betting against the market. In good times, she exclusively implements a contrarian strategy. She anticipates that reversion to fundamentals will be fast and strong, and consistently reverts her strategy. Doing so, she avoids momentum crashes when the market rebounds (Daniel and Moskowitz (2013), Barroso and Santa-Clara (2014)) and exploits stock return reversal.

Agent A's profits are largest in bad times, consistent with the empirical finding of Moskowitz (2000).[10] The reason is as follows. In good times, prices accommodate shocks almost instantly, which makes it challenging for agents to reap large profits. In normal and bad times, however, stock prices exhibit short-term predictability. Agent A exploits this information to make larger profits. In particular, Agent A makes the largest profits in bad times, that is when short-term predictability is strongest. We view this result as a possible explanation for the finding that institutional investors focus on market-timing strategies during recessions (Kacperczyk et al. (2013)).[11] Moreover, our results are consistent with the finding of Moskowitz et al. (2012) that a time-series momentum strategy generates large profits in bad times and may incur large losses when the market rebounds.

We finally test an important implication of our model that, to the best of our knowledge, has not received attention in the empirical literature. Time-series momentum in the very short-term should be larger in downturns than in upturns, and therefore momentum strategies should generate larger profits in downturns than in upturns. We test this prediction using S&P500 returns recorded at a monthly frequency from January 1871 to November 2013. Looking back to the 19th century allows us to cover 29 NBER recessions, but also implies dramatic volatility changes over time. To make reasonable comparisons through time, we standardize returns by their ex-ante volatility. We then compute the serial correlation of returns at different lags. Consistent with the prediction of the model, we find that, over the 120 years of the sample, time-series momentum at a 1-month lag has been significantly stronger during recessions. We show that superior serial correlation at a 1-month lag is sufficient to make significantly larger profits in recessions than in expansions. In particular,

[10] See also Glode (2011), Kosowski (2011), and Lustig and Verdelhan (2010) for similar findings.

[11] Ferson and Schadt (1996) provide evidence that fund managers timing ability varies with economic conditions. Dangl and Halling (2012) show that market-timing strategies work best in recessions.

we implement a naive momentum strategy and find that it is, on average, twice as profitable in recessions as it is in expansions. Momentum profits are particularly impressive during the crises of 1929 and 2008.

Related Literature

In our framework, agents disagree because they use the *same* information to update *different* models—an Ornstein-Uhlenbeck process and a Markov chain. They do not have different interpretations of a public signal as in Scheinkman and Xiong (2003), Dumas et al. (2009), and Xiong and Yan (2010), nor do they have different initial priors as in Detemple and Murthy (1994), Zapatero (1998), and Basak (2000), nor do they have heterogeneous dogmatic beliefs as in Kogan, Ross, Wang, and Westerfield (2006), Borovicka (2011), Chen, Joslin, and Tran (2012), and Bhamra and Uppal (2013), nor do they disagree about the parameters of a same model as in David (2008) and Andrei, Carlin, and Hasler (2013)—a Markov chain in the former case and an Ornstein-Uhlenbeck in the latter case. Our model therefore has distinct implications for the dynamics of disagreement and, in particular, for the shape of the term structure of disagreement. Importantly, the large difference in the speed at which agents update their expectations, which is unique to our model, is needed to generate large spikes in disagreement in the short term and therefore momentum both in normal and bad times.

Our goal is to study patterns of stock return predictability. In that respect, our paper is related to Campbell and Cochrane (1999), Bansal and Yaron (2004), Menzly, Santos, and Veronesi (2004), and Constantinides and Gosh (2012) who rely on habit-formation, long-run risk, time-varying expected growth rates, and learning, respectively, to explain different types of stock-return predictability. Our emphasis, however, is on the mechanism through which investors' disagreement drives return predictability over the business cycle. Our result is also related to that of Veronesi (1999). While his analysis relies on a representative agent economy, ours relies on heterogenous models and therefore on disagreement.

In our model, momentum in bad times is similar to an under-reaction phenomenon, as in Hong and Stein (1999), while momentum in normal times resembles an over-reaction phenomenon, as in Daniel et al. (1998). Our emphasis is that these two effects may take place in different states of the business cycle and that momentum is strongest in bad times because disagreement spikes. Moreover, Andrei and Cujean (2014) show that momentum and reversal arise as a rational phenomenon when word-of-mouth communication accelerates the information flow through prices; Cujean (2013) analyzes how the serial correlation of returns induced by word-of-mouth communication affects investors' performance. In our model, the main channel driving momentum and investors' performance is the asymmetric shape of the term structure of disagreement, which is unique to our modeling assumptions.

Finally, in our model, prices continue to react to shocks both in normal and bad times and revert almost instantly to fundamentals in good times due to the asymmetric shape of the term structure of disagreement. Chalkley and Lee (1998), Veldkamp (2005), and Van Nieuwerburgh and Veldkamp (2006) obtain a similar result, but for a different reason. In their model, private information implies asymmetric learning over the business cycle, whereas, in our framework, public information serves to update heterogenous models. As mentioned by Patton and Timmermann (2010), the observed term structure of disagreement is more likely to reflect heterogenous models or priors than heterogenous information.

In contrast with the existing literature, our paper provides a model of heterogenous beliefs based on model heterogeneity that helps explain simultaneously the observed state-dependent term structure of disagreement and the patterns of both stock return predictability–momentum, reversal, continuing price reactions–and trading strategies over the business cycle. In addition, we test a novel implication of our model; short-term time-series momentum is stronger in recessions than in expansions.

The remainder of the paper is organized as follows. Section 2 presents and solves the model, Section 3 describes the estimation of its parameters, Sections 4 and 5 contain the results, Section 6 tests the main predictions of the model, and Section 7 concludes. Derivations and computational details are relegated to Appendix A.

2 The Model

We develop a dynamic general equilibrium in which investors use heterogeneous models to estimate the business cycle. We first describe the economy and the learning problem of investors. We then discuss the dynamics of investors' disagreement implied by their heterogeneous models and solve for the equilibrium stock price.

2.1 The Economy and Models of the Business Cycle

We consider an economy with an aggregate dividend that flows continuously over time. The market consists of two securities, a risky asset—the stock—in positive supply of one unit and a riskless asset—the bond—in zero net supply. The stock is a claim to the dividend process δ, which evolves according to

$$\mathrm{d}\delta_t = \delta_t f_t \mathrm{d}t + \delta_t \sigma_\delta \mathrm{d}W_t. \tag{1}$$

The random process $(W_t)_{t\geq 0}$ is a Brownian motion under the physical probability measure, which governs the empirical realizations of dividends. The expected dividend growth rate

f—henceforth the *fundamental*—is unobservable.

The economy is populated by two agents, A and B, who consume the dividend and trade in the market. Being rational individuals, agents understand that the fundamental affects the dividend they consume and the price of the assets they trade. They cannot, however, observe the fundamental and therefore need to estimate it. To do so, they use the only source of information available to them—the empirical realizations of dividends. But this information is meaningless without a proper understanding of the data-generating process that governs dividends; agents need to have a model in mind.

Agent A assumes that the fundamental follows a mean-reverting process. In particular, she has the following model in mind

$$\begin{aligned} \mathrm{d}\delta_t &= f_t^A \delta_t dt + \sigma_\delta \delta_t \mathrm{d}W_t^A \\ \mathrm{d}f_t^A &= \kappa \left(\overline{f} - f_t^A\right) \mathrm{d}t + \sigma_f \mathrm{d}W_t^f, \end{aligned} \tag{2}$$

where W^A and W^f are two independent Brownian motions under Agent A's probability measure $\mathbb{P}^A$, which reflects her views about the data-generating process. Under Agent A's representation of the economy, the fundamental f^A transits smoothly from good to bad states, reverting to a long-term mean $\overline{f}$ at speed κ.

In contrast, Agent B believes that the fundamental follows a 2-state, continuous-time Markov chain and postulates the following model

$$\begin{aligned} &\mathrm{d}\delta_t = f_t^B \delta_t dt + \sigma_\delta \delta_t \mathrm{d}W_t^B \\ &f_t^B \in \{f^h, f^l\} \text{ with generator matrix } \Lambda = \begin{pmatrix} -\lambda & \lambda \\ \psi & -\psi \end{pmatrix}, \end{aligned} \tag{3}$$

where W^B is a Brownian motion under B's probability measure $\mathbb{P}^B$. Under Agent B's perception of the economy, the fundamental f^B is either high $f^h > 0$—the economy is in an expansionary state—or low $f^l < 0$—the economy is in a recessionary state. The economy transits from the high to the low state with intensity $\lambda > 0$ and from the low to the high state with intensity $\psi > 0$.

The Financial Economics literature has focussed, to a large extent, on the two types of model presented in Eqs. (2) and (3) to forecast the growth rate of aggregate output. In particular, the long-run risk literature (Bansal and Yaron (2004)) promotes the view that the growth rate of consumption is mean-reverting and persistent, and follows a Gaussian process; the model of Agent A reflects this perspective.[12] Another view, based on the work of David

[12]This type of model is adopted by Detemple (1986), Brennan and Xia (2001), Scheinkman and Xiong (2003), Dumas et al. (2009), and Xiong and Yan (2010) among others.

(1997) and Veronesi (1999) and extensions thereof, is that the growth rate of dividends evolves over a finite number of states and transits from one state to the other according to a Markov switching model.[13] This perspective, which is closer to models used in Economics to forecast business cycle turning points and recession probabilities, is reflected in the model of Agent B.[14]

To the best of our knowledge, these models have always been considered separately; our goal is to show that, when these models are considered within a unified framework, the resulting disagreement across agents can explain several empirical facts related to returns predictability. In the next section we discuss how these two models generate patterns of disagreement consistent with those observed among professional forecasters. In Section 3, we calibrate both models by maximum likelihood and discuss their respective ability to measure the business cycle.

2.2 Bayesian Learning and Disagreement

Agents learn about the fundamental by observing realizations of the dividend growth rate and, given the model they have in mind, update their expectations accordingly. Doing so, they come up with an estimate of the fundamental, which, from now on, we call the *filter*. We start by describing the dynamics of Agent A's filter, summarized in Proposition 1 below.

Proposition 1. *The filter of Agent A is defined by* $\widehat{f}_t^A = \mathbb{E}_t^{\mathbb{P}^A}\left[f_t^A\right]$ *and satisfies*

$$d\widehat{f}_t^A = \kappa\left(\overline{f} - \widehat{f}_t^A\right)dt + \frac{\gamma}{\sigma_\delta}d\widehat{W}_t^A, \tag{4}$$

where $\gamma = \sqrt{\sigma_\delta^2\left(\sigma_\delta^2\kappa^2 + \sigma_f^2\right)} - \kappa\sigma_\delta^2$ *denotes the steady-state posterior variance (or Bayesian uncertainty).* $\widehat{W}^A$ *is a Brownian motion under Agent A's probability measure* $\mathbb{P}^A$ *that satisfies*

$$\widehat{W}_t^A = \frac{1}{\sigma_\delta}\int_0^t\left(\frac{d\delta_s}{\delta_s} - \widehat{f}_s^A ds\right).$$

Proof. See Appendix A.1.

[13] This specification is also considered by Veronesi (2000), David (2008), and David and Veronesi (2013) among others.

[14] In the Economics literature, time-series models of the business cycles include Threshold AutoRegressive (TAR) models, Markov-Switching AutoRegressive (MSAR) models, and Smooth Transition AutoRegressive (STAR) models. See Hamilton (1994) and Milas et al. (2006) for further details.

The dynamics of the filter in Eq. (4) show that learning preserves the mean-reversion speed and the long-term mean of the initial model (in Eq. (3)), but decreases its volatility.[15] Moreover, the volatility of f_A and $\widehat{f}_A$ is constant, which implies that neither the perceived fundamental f_A nor its associated filter $\widehat{f}_A$ can change dramatically over a small period of time.

Agent B envisages only two states of the world—a high and a low state. She assigns a conditional probability π, defined by

$$\pi_t = \mathbb{P}_t^B \left(f_t^B = f^h \right),$$

to the high state, which she updates continuously. She then computes her filter by weighting each state accordingly:

$$\widehat{f}_t^B = \mathbb{E}_t^{\mathbb{P}^B} \left[f_t^B \right] = \pi_t f^h + (1 - \pi_t) f^l.$$

We highlight the dynamics of Agent B's filter in Proposition 2 below.

Proposition 2. *The dynamics of the filter computed by Agent B are given by*

$$d\widehat{f}_t^B = (\lambda + \psi) \left(f_\infty - \widehat{f}_t^B \right) dt + \frac{1}{\sigma_\delta} \left(\widehat{f}_t^B - f^l \right) \left(f^h - \widehat{f}_t^B \right) d\widehat{W}_t^B, \tag{5}$$

where

$$f_\infty = \lim_{t \to \infty} \mathbb{E} \left[f_t^B \right] = f^l + \frac{\psi}{\lambda + \psi} \left(f^h - f^l \right)$$

is the unconditional mean of the Markov chain. $\widehat{W}^B$ is a Brownian motion under Agent B's probability measure $\mathbb{P}^B$ that satisfies

$$\widehat{W}_t^B = \frac{1}{\sigma_\delta} \int_0^t \left(\frac{d\delta_s}{\delta_s} - \widehat{f}_s^B ds \right).$$

Proof. See Lipster and Shiryaev (2001).

Although Agent B only envisages two states of the world, the dynamics of her filter in Eq. (5) are continuous. The reason is that information flows continuously from dividends, causing Agent B to continuously reassess the likelihood of the high state. Moreover, just as Agent A's filter, her filter mean-reverts, yet at a different speed $\lambda + \psi \neq \kappa$ and to a different

[15] Straightforward computations show that $0 < \frac{\gamma}{\sigma_\delta} < \sigma_f$ as long as the parameters σ_δ, σ_f, and κ are positive.

long-term mean $f_\infty \neq \overline{f}$. Another difference is that the volatility of Agent B's filter is stochastic; her filter can therefore change precipitately. Such sudden changes typically occur when Agent B is the most uncertain about the current state of the economy, i.e., when she assigns equal probabilities to each state:

$$f_m \equiv \frac{1}{2} f^l + \frac{1}{2} f^h.$$

As the filter moves away from f_m, it gets rapidly absorbed towards the closest regime and stays there for some time until a regime shift occurs.

For reasons that will become apparent in Section 3, we choose to work under Agent A's probability measure. For the purpose of studying returns predictability, this choice is not without loss of generality; our results are, however, qualitatively equivalent under Agent B's measure. We convert Agent A's views into those of Agent B through the following change of measure:

$$\left.\frac{\mathrm{d}\mathbb{P}^B}{\mathrm{d}\mathbb{P}^A}\right|_{\mathscr{F}_t} \equiv \eta_t = \exp\left[-\frac{1}{2}\int_0^t \frac{g_u^2}{\sigma_\delta^2}\mathrm{d}u - \int_0^t \frac{g_u}{\sigma_\delta}\mathrm{d}\widehat{W}_u^A\right], \tag{6}$$

where the process $g \equiv \widehat{f}^A - \widehat{f}^B$ represents agents' disagreement about the estimated fundamental.

The change of measure in Eq. (6) implies that Agent A and B have *equivalent* perceptions of the world.[16] In particular, the Brownian motions $\widehat{W}^A$ and $\widehat{W}^B$ are related through

$$\mathrm{d}\widehat{W}_t^B = \mathrm{d}\widehat{W}_t^A + \frac{g_t}{\sigma_\delta}\mathrm{d}t.$$

This result may be surprising, if we consider that Agent B's filter is strictly confined within the interval $[f^l, f^h]$, while Agent A's filter can reach virtually any value. To see how both views are compatible, fix a value for Agent A's filter, say $\overline{f}^A$. This, in turn, imposes a restriction on the possible values of disagreement. Specifically, since Agent B assigns zero probabilities to the events $\{\widehat{f}^B > f^h\}$ and $\{\widehat{f}^B < f^l\}$, disagreement is at least equal to $\overline{f}^A - f^h$ and at most equal to $\overline{f}^A - f^l$. The dynamics of disagreement, g, are such that this

[16] Disagreement is the difference between an Ornstein-Uhlenbeck process, which is stationary, and a bounded process. Disagreement is therefore stationary, the Novikov condition is satisfied and Girsanov's Theorem applies.

property is always satisfied:

$$\mathrm{d}g_t = \left[\kappa\left(\bar{f} - \widehat{f}_t^A\right) - (\lambda + \psi)\left(f_\infty + g_t - \widehat{f}_t^A\right) - \frac{g_t}{\sigma_\delta^2}\left(\widehat{f}_t^A - g_t - f^l\right)\left(f^h + g_t - \widehat{f}_t^A\right)\right]dt \tag{7}$$
$$+ \frac{\gamma - \left(\widehat{f}_t^A - g_t - f^l\right)\left(f^h + g_t - \widehat{f}_t^A\right)}{\sigma_\delta}\mathrm{d}\widehat{W}_t^A.$$

In our model, agents disagree because they use identical information to update different models—they do not misinterpret information (Scheinkman and Xiong (2003), Dumas et al. (2009), and Xiong and Yan (2010)), nor do they use different estimation techniques, nor do they disagree about the parameters of the model (Andrei et al. (2013)). Our formulation has two main advantages. First, disagreement is not the result of an estimation error—agents just estimate different models, both of which are reasonable representations of the business cycle. Second, the dynamics of disagreement in Eq. (7) are consistent with observed facts, as we discuss now.

There exists a term structure of disagreement among professional forecasters. Patton and Timmermann (2010) find that forecasters' disagreement, as measured by the dispersion of their forecasts, is highest at long horizons and lower at short horizons. Their results also indicate that disagreement stems from *heterogeneous models*, as opposed to private information; our model is based on this fact.[17] Moreover, they document higher disagreement during recessions.[18] The dynamics of disagreement in Eq. (7) precisely follow this pattern—in recessions, disagreement is strong and persists over long horizons. We defer a detailed explanation of the mechanism through which disagreement operates to Section 4.1; we conclude this section with a discussion of the main differences with related models of disagreement.

Models in which agents either misinterpret information (e.g., Scheinkman and Xiong (2003)) or disagree about the parameters of their model (Andrei et al. (2013)) generate disagreement that disappears in the long-run. Instead, in our model, agents estimate different long-term means, as apparent in Eq. (7); disagreement may therefore persist over long horizons, consistent with empirical evidence. Moreover, when agents disagree exclusively about the length of the business cycle (Andrei et al. (2013)), disagreement persists in the short term, but symmetrically across expansions and recessions. In our model, agents disagree about three aspects of the fundamental—its persistence, its long-term mean, and its volatility. These three aspects together give rise to higher disagreement in recessions, an implication strongly supported by the data.

[17] See also Kandel and Pearson (1995) for similar evidence.

[18] Massa and Simonov (2005) show that forecasters strongly disagree about recession probabilities. Veronesi (1999) and Barinov (2014) show that the dispersion of analysts forecasts increases during recessions.

2.3 Optimization Problem

Agents choose their portfolios and consumption plans to maximize their expected lifetime utility of consumption. They have power utility preferences defined by

$$U(c,t) \equiv e^{-\rho t}\frac{c^{1-\alpha}}{1-\alpha},$$

where $\alpha > 0$ is the coefficient of relative risk aversion and $\rho > 0$ the subjective discount rate.

Since markets are complete, we solve the consumption-portfolio problem of both agents using the martingale approach of Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989). We write Agent A's problem as follows

$$\max_{c_A} \mathbb{E}^{\mathbb{P}^A}\left[\int_0^\infty e^{-\rho t}\frac{c_{At}^{1-\alpha}}{1-\alpha}\mathrm{d}t\right] + \phi_A\left(X_{A,0} - \mathbb{E}^{\mathbb{P}^A}\left[\int_0^\infty \xi_t c_{At}\mathrm{d}t\right]\right),$$

where ϕ_A denotes the Lagrange multiplier of Agent A's static budget constraint and ξ is the state-price density perceived by Agent A. Agent B solves an analogous problem but under her own probability measure $\mathbb{P}^B$. Rewriting Agent B's problem under Agent A's probability measure $\mathbb{P}^A$ yields

$$\max_{c_B} \mathbb{E}^{\mathbb{P}^A}\left[\int_0^\infty \eta_t e^{-\rho t}\frac{c_{Bt}^{1-\alpha}}{1-\alpha}\mathrm{d}t\right] + \phi_B\left(X_{B,0} - \mathbb{E}^{\mathbb{P}^A}\left[\int_0^\infty \xi_t c_{Bt}\mathrm{d}t\right]\right).$$

The first-order conditions lead to the following optimal consumption plans

$$c_{At} = \left(\phi_A e^{\rho t}\xi_t\right)^{-\frac{1}{\alpha}} \qquad\qquad c_{Bt} = \left(\frac{\phi_B e^{\rho t}\xi_t}{\eta_t}\right)^{-\frac{1}{\alpha}}. \tag{8}$$

Imposing the market-clearing condition finally gives the state-price density ξ, which satisfies:

$$\xi_t = e^{-\rho t}\delta_t^{-\alpha}\left[\left(\frac{\eta_t}{\phi_B}\right)^{\frac{1}{\alpha}} + \left(\frac{1}{\phi_A}\right)^{\frac{1}{\alpha}}\right]^{\alpha}. \tag{9}$$

Heterogeneous beliefs affect the state-price density through the variable η, the likelihood of Agent B's model relative to Agent A's model.[19] This relative likelihood is, in turn, determined by the disagreement process (through Eq. (6)). As a result, the disagreement process drives the dynamics of the state-price density and, hence, the dynamics of stock returns. In particular, Dumas et al. (2009) note that ξ is concave in η; Jensen's inequality

[19]By definition, the change of measure η represents the likelihood of the probability measure $\mathbb{P}^B$ relative to $\mathbb{P}^A$.

along with Eq. (6) together imply that an increase in disagreement reduces the expected values of future state-price densities under Agent A's measure, resulting in higher expected returns. Because disagreement is strongest and most persistent in recessions (see Section 4.1), this effect—disagreement predicts future returns—is concentrated in recessions.

Substituting Eq. (9) into Eq. (8) gives the consumption share of Agent A, $\omega(\eta)$, which satisfies:

$$\omega\left(\eta_t\right) = \frac{\left(\frac{1}{\phi_A}\right)^{\frac{1}{\alpha}}}{\left(\frac{\eta_t}{\phi_B}\right)^{\frac{1}{\alpha}} + \left(\frac{1}{\phi_A}\right)^{\frac{1}{\alpha}}}.$$

The consumption share $\omega(\eta)$ of Agent A is also a function of the likelihood η. In particular, an increase in η raises the likelihood of Agent B's model relative to Agent A's model. The consumption share of Agent A is therefore decreasing in η—the more likely Agent B's model becomes, the less money Agent A makes and the less she consumes. This result applies symmetrically to the consumption share $1 - \omega(\eta)$ of Agent B.

Finally, an application of Itô's lemma to the state-price density ξ in Eq. (9) yields the risk-free rate, r^f, and the market price of risk, θ, of our model:

$$r_t^f = \rho + \alpha \widehat{f}_t^A - \frac{1}{2}\alpha(\alpha+1)\sigma_\delta^2 + \left(1 - \omega(\eta_t)\right) g_t \left(\frac{1}{2}\frac{\alpha - 1}{\alpha \sigma_\delta^2}\omega(\eta_t) g_t - \alpha\right) \tag{10}$$

$$\theta_t = \alpha \sigma_\delta + \frac{\left(1 - \omega(\eta_t)\right)}{\sigma_\delta} g_t. \tag{11}$$

If agents did not disagree ($g = 0$) or if only one agent populated the economy ($\omega = 0$ or $\omega = 1$), the risk-free rate in Eq. (10) and the market price of risk in Eq. (11) would be those of Lucas (1978). Inspecting the last term in Eq. (11), we conclude that heterogeneous beliefs cause the market price of risk to exceed that of a Lucas (1978) economy, but only when Agent A is more optimistic than Agent B ($g > 0$). In our model, this outcome, which plays an important role when we study agents' profits in Section 5, only arises in recessions.

2.4 Equilibrium Stock Price

Following Dumas et al. (2009), we assume that the coefficient of relative risk aversion α is an integer. This assumption allows us to obtain the following convenient expression for the equilibrium stock price:[20]

$$
\begin{aligned}
\frac{S_t}{\delta_t} &= \mathbb{E}_t^{\mathbb{P}^A}\left[\int_t^\infty \frac{\xi_u \delta_u}{\xi_t \delta_t} du\right] \\
&= \omega(\eta_t)^\alpha \sum_{j=0}^{\alpha} \begin{pmatrix} \alpha \\ j \end{pmatrix} \left(\frac{1-\omega(\eta_t)}{\omega(\eta_t)}\right)^j \mathbb{E}_t^{\mathbb{P}^A}\left[\int_t^\infty e^{-\rho(u-t)} \left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}} \left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha} du\right]. \quad (12)
\end{aligned}
$$

We start by computing the first and the last term of the sum in (12). These terms correspond to the prices of a Lucas (1978) economy in which the representative agent assumes that fundamental follows an Ornstein-Uhlenbeck process and a 2-state Markov chain, respectively. These prices have (semi) closed-form solutions, which we present in Proposition 3.

Proposition 3. *Suppose the economy is populated by a single agent.*

1. *If the agent's filter follows the Ornstein-Uhlenbeck process described in Eq. (4), the equilibrium price-dividend ratio satisfies*

$$
\left.\frac{S_t}{\delta_t}\right|_{O.U.} = \int_0^\infty e^{-\rho\tau + \alpha(\tau) + \beta_2(\tau)\widehat{f}_t^A} d\tau,
$$

where the functions $\alpha(\tau)$ and $\beta_2(\tau)$ are the solutions to a set of Ricatti equations.

2. *If the agent's filter follows the filtered 2-state Markov chain process described in Eq. (5), the equilibrium price-dividend ratio satisfies*

$$
\begin{aligned}
\left.\frac{S_t}{\delta_t}\right|_{M.C.} &= \pi_t H_1 + (1-\pi_t) H_2 = \frac{\widehat{f}_t^B - f^l}{f^h - f^l} H_1 + \left(1 - \frac{\widehat{f}_t^B - f^l}{f^h - f^l}\right) H_2 \\
&= \frac{\widehat{f}_t^A - g_t - f^l}{f^h - f^l} H_1 + \left(1 - \frac{\widehat{f}_t^A - g_t - f^l}{f^h - f^l}\right) H_2,
\end{aligned}
$$

where

$$
\begin{aligned}
H &= \begin{pmatrix} H_1 & H_2 \end{pmatrix}^\top = A^{-1} \mathbf{1}_2 \\
A &= -\Omega - (1-\alpha) \begin{pmatrix} f^h & 0 \\ 0 & f^l \end{pmatrix} + (\rho + \frac{1}{2}\alpha(1-\alpha)\sigma_\delta^2) \mathbf{Id}_2.
\end{aligned}
$$

$\mathbf{Id}_2$ *is a 2-by-2 identity matrix and* $\mathbf{1}_2$ *is a 2-dimensional vector of ones.*

[20] We refer the reader to Dumas et al. (2009) for the details of the derivation.

Proof.

1. See Appendix A.2 for the solutions of $\alpha(\tau)$ and $\beta_2(\tau)$.

2. See Veronesi (2000).

We now rewrite the price in (12) as

$$\frac{S_t}{\delta_t} = \omega(\eta_t)^\alpha \left.\frac{S_t}{\delta_t}\right|_{O.U.} + \omega(\eta_t)^\alpha \sum_{j=1}^{\alpha-1} \begin{pmatrix} \alpha \\ j \end{pmatrix} \left(\frac{1-\omega(\eta_t)}{\omega(\eta_t)}\right)^j F^j\left(\widehat{f}_t^A, g_t\right) + (1-\omega(\eta_t))^\alpha \left.\frac{S_t}{\delta_t}\right|_{M.C.}. \tag{13}$$

The last step consists in computing the intermediate terms F^j in (13), which relate to heterogeneous beliefs. Each term solves a differential equation, which we present in Proposition 4.

Proposition 4. *The function F^j, defined as*

$$F^j\left(\widehat{f}_t^A, g_t\right) \equiv \mathbb{E}_t^{\mathbb{P}^A}\left[\int_t^\infty e^{-\rho(u-t)}\left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha} du\right], \tag{14}$$

solves the following partial differential equation

$$\widetilde{\mathscr{L}}^{\widehat{f}^A,g} F^j + X^j F^j + 1 = 0, \tag{15}$$

where $\widetilde{\mathscr{L}}$ denotes the infinitesimal generator of $\left(\widehat{f}^A, g\right)$ under the probability measure $\widetilde{\mathbb{P}}^A$.

Proof. See Appendix A.3.

We finally solve the equation in (15) for each term j numerically through Chebyshev collocation. For convenience, we discuss the details in Appendix A.4.

3 Calibration and Model Fit to the U.S. Economy

In this section we calibrate the models of Agent A and B and discuss their relative fit to the U.S. business cycle. In our model, agents use a single source of information for the purpose of updating their expectations—the time-series of dividends δ. As a proxy for the dividend stream, we use the S&P500 dividend time-series recorded at a monthly frequency from

January 1871 to November 2013, which we obtain from Robert Shiller's website. Looking back to the 19th century allows us to cover a large number of business cycle turning points, but obviously adds strong seasonality effects (Bollerslev and Hodrick (1992)). To reduce these effects, we apply the filter developed by Hodrick and Prescott (1997) to the time-series of dividends.[21]

We assume that both agents observe the S&P500 dividend time-series, after seasonalities have been smoothed out. Because this data is available monthly, agents need to first estimate a discretized version of their model and then map the parameters they estimated into their continuous-time model. In particular, Agent A estimates the following discrete-time model

$$\log\left(\frac{\delta_{t+1}}{\delta_t}\right) = f_t^A + \sqrt{v^\delta}\epsilon_{1,t+1} \tag{16}$$

$$f_{t+1}^A = m^{f^A} + a^{f^A} f_t^A + \sqrt{v^{f^A}}\epsilon_{2,t+1}, \tag{17}$$

while Agent B estimates the following discrete-time model

$$\log\left(\frac{\delta_{t+1}}{\delta_t}\right) = f_t^B + \sqrt{v^\delta}\epsilon_{3,t+1} \tag{18}$$

$$f_t^B \in \{s^h, s^l\} \quad \text{with transition matrix} \quad P = \begin{pmatrix} p^{hh} & 1-p^{hh} \\ 1-p^{ll} & p^{ll} \end{pmatrix}$$

where ϵ_1, ϵ_2, and ϵ_3 are normally distributed with zero-mean and unit-variance; ϵ_1 and ϵ_2 are independent. The transition matrix P contains the probabilities of staying in the high and the low state over the following month.

We estimate the parameters of the models in Eqs. (17) and (18) by Maximum Likelihood.[22] After mapping the resulting estimates into the continuous-time model of each agent, we obtain the parameter values presented in Table 1. For convenience, we discuss the methodological details in Appendix A.5. All discussions and results that follow are based on the calibrated parameters of Table 1.

The parameters of Table 1 show that both agents interpret the data unambiguously. First, Agent A finds a low reversion speed κ and therefore concludes that the fundamental is highly persistent. The long-run risk literature largely supports this view (see, e.g., Bansal and Yaron (2004), Bansal and Shaliastovich (2011), Colacito and Croce (2013), Bansal, Kiku, Shaliastovich, and Yaron (2013)). Second, Agent B finds that the high state f^h and the low state f^l unmistakably identify an expansionary and a recessionary phase, respectively.

[21] The smoothing parameter is set to 120.

[22] See Hamilton (1994) for the likelihood function of each model. We present the estimated parameters, their standard errors, and their statistical significance in Table 3 in Appendix A.5.

Parameter	Symbol	Value
Volatility Dividend Growth	σ_δ	0.0225
Mean-Reversion Speed f^A	κ	0.1911
Long-Term Mean f^A	$\bar{f}$	0.0630
Volatility f^A	σ_f	0.0056
High State f^B	f^h	0.0794
Low State f^B	f^l	-0.0711
Intensity High to Low	λ	0.3022
Intensity Low to High	ψ	0.3951
Relative Risk Aversion	α	2
Subjective Discount Rate	ρ	0.01

Table 1: Parameter calibration

This table reports the estimated parameters of the continuous-time model of Agent A and Agent B along with the relative risk aversion α and the subjective discount rate ρ.

Perhaps unsurprisingly, the Economics literature widely resorts to Markov switching models to identify business cycle turning points (Milas et al. (2006)).

Both models produce distinct interpretations of the data and therefore also have distinct implications for agents' respective filter. First, the parameter values for λ and ψ imply that Agent B's filter reverts about 3.5 times faster than Agent A's filter.[23] Second, the unconditional mean $f_\infty \approx 0.014$ of Agent B's filter is significantly lower than the long-term mean $\overline{f} = 0.063$ of Agent A's filter. To illustrate how these two aspects—difference in adjustment speed and difference in mean—affect agents' ability to forecast the business cycle, we plot in Figure 1 agents' filters against NBER recession dates over the sample period.

Both models do a reasonable job at identifying business cycle turning points. In particular, sharp declines in the filter of Agent A (the blue line) and Agent B (the red line) often coincide with an NBER recession (the shaded areas). Agents' filters, however, react with some delay relative to NBER announcements. The reason is that NBER turning points are ex-post indicators, while agents perform their estimation in real time.

Figure 1 also illustrates that each model measures, respectively, a different aspect of the business cycle. By construction, a mean-reverting process better captures level effects, while a Markov chain better captures sharp variations. Differences in level are clearly apparent during expansions and recessions—a mean-reverting model is more versatile and allows Agent A's filter to take values largely beyond those of Agent B's filter.[24] Differences in adjustment speed, in turn, clearly appear at business cycle turning points—a Markov chain better detects sudden changes and allows Agent B's filter to adapt faster than Agent A's filter.

[23] Agent B's expectations, under her own probability measure, revert at speed $\psi + \lambda \approx 0.7$.

[24] An Ornstein-Uhlenbeck process can be viewed as a limit ($n \to \infty$) of an n-state Markov chain.

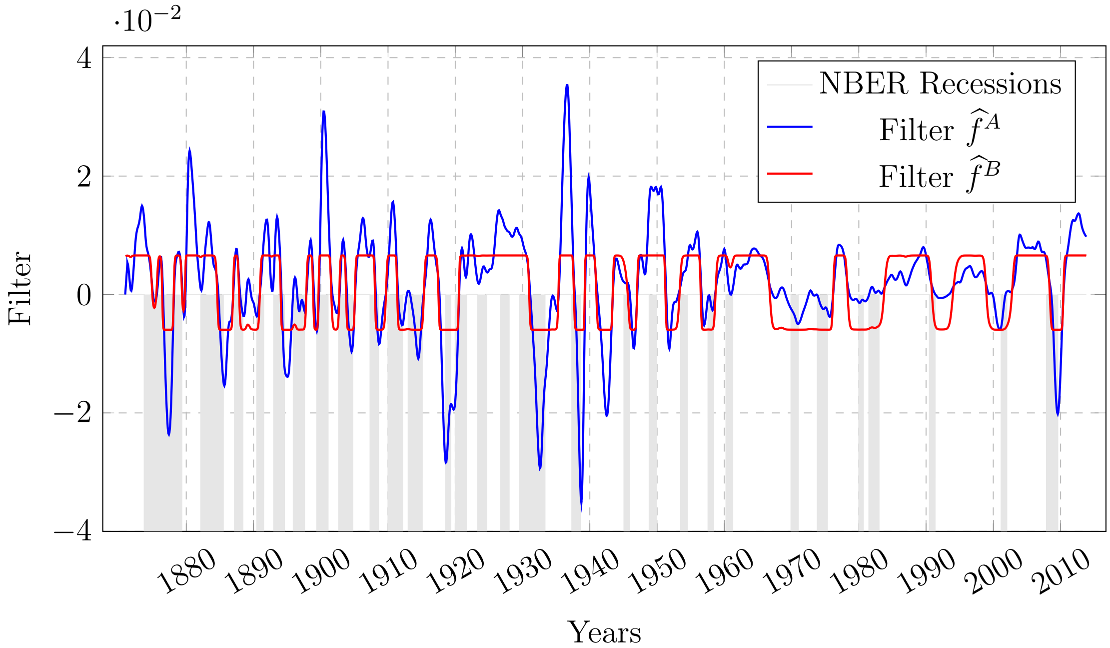

Figure 1: Empirical filters.

The blue and red curves depict the filters estimated by Agent A and Agent B, respectively. The estimation is performed by using monthly data from 01/1871 to 11/2013.

Which model is best is obviously specific to an agent's needs; we can, however, determine which model better fits historical data by applying a model selection method such as the *Akaike* Information Criterion (AIC).[25] Because an Ornstein-Uhlenbeck process is more versatile and relies on fewer parameters than a 2-state Markov chain, the information criterion favors Agent A's model.[26] This fact ultimately justifies our choice of computing and analyzing the equilibrium under Agent A's probability measure $\mathbb{P}^A$.

4 Predictability and Disagreement

Recent evidence shows that stock return predictability is concentrated in in bad times.[27] In this section, our objective is to determine the mechanism that causes return predictability to vary over the business cycle. We first show that disagreement among investors is strongest and most persistent in in bad times (Subsection 4.1). We then demonstrate that

[25]The Akaike Information Criterion is defined as follows: $AIC = 2K - 2\log(L)$, where K is the number of parameters estimated and L is the Likelihood function. Therefore, the smaller the criterion, the better is the model.

[26]The AIC for Agent A and B's model are $AIC_{\text{Agent A}} = -1.7229 \times 10^4$ and $AIC_{\text{Agent B}} = -1.2154 \times 10^4$.

[27]Garcia (2013) shows that sentiment, as proxied by media pessimism, better predicts stock returns in recessions. Rapach et al. (2010), Henkel et al. (2011), and Dangl and Halling (2012) find that price-dividend ratios better predict stock returns in recessions.

spikes in disagreement lead to continuing price reaction to news shocks in both normal and bad times (Subsection 4.2). In good times, disagreement exhibit little variation and prices therefore revert almost instantly to fundamentals. Finally, we show that these phenomena produce time-series momentum (Moskowitz et al. (2012)) and momentum crashes (Daniel and Moskowitz (2013)). We argue that short-term time-series momentum is strongest in bad times and that momentum crashes occur when the market sharply rebounds (Subsection 4.3).

4.1 Term Structure of Disagreement

In this section, we show how agents disagree, both in the short- and long-term, in different states of the economy. In our model, there is little disagreement in good times; disagreement is especially large and volatile in bad times. In bad times, agents disagree about the short-term direction of the economy. In the long-run, agents' expectations align, yet disagreement persists due to agents' expectations adjusting at different speed. Proposition 5 describes mathematically how agents' expectations behave.

Proposition 5. *Agent A's filter is (conditionally) normally distributed with mean $m_t = \widehat{f}_0^A e^{-\kappa t} + \bar{f}(1 - e^{-\kappa t})$ and variance $\frac{\gamma^2}{\sigma_\delta^2}\frac{1}{2\kappa}(1 - e^{-2\kappa t})$. Under Agent A's probability measure and conditional on Agent A's filter being equal to its conditional mean m_t, the conditional distribution $\mathbb{P}^A(\widehat{f}_t^B | \widehat{f}_t^A = m_t, \widehat{f}_0^B) = p(\widehat{f}^B, t; \widehat{f}_0^B, \widehat{f}_0^B)$ of Agent B's filter satisfies the Fokker-Planck equation*

$$\frac{\partial}{\partial t} p = -\frac{\partial}{\partial \widehat{f}^B}\left(p \frac{1}{dt}\mathbb{E}_t^{\mathbb{P}^A}[d\widehat{f}_t^B | \widehat{f}_t^A = m_t]\right) + \frac{1}{2}\frac{\partial^2}{\partial (\widehat{f}^B)^2}\left(p\frac{1}{dt}(d\widehat{f}_t^B)^2\right) \tag{19}$$

with boundary conditions $\lim_{\widehat{f}^B \nearrow f^h} p(\widehat{f}^B, t; \widehat{f}_0^B, \widehat{f}_0^B) = \lim_{\widehat{f}^B \searrow f^l} p(\widehat{f}^B, t; \widehat{f}_0^B, \widehat{f}_0^B) = 0$ and initial condition $p(\widehat{f}^B, 0; \widehat{f}_0^B, \widehat{f}_0^B) = \widehat{\delta}_{\widehat{f}^B = \widehat{f}_0^B}$, where $\widehat{\delta}$ is the Dirac delta function.

Proof. The first part of the proposition follows from standard computations. Boundary conditions are derived in David (1997).

Agent A updates her views in a rather simple manner. Her views revert towards their long-term mean $\bar{f}$ at speed κ, as shown in Proposition 5. That is, when her filter is below $\bar{f}$ her expectations revert upwards; when her filter is above $\bar{f}$ her expectations revert downwards. Due to the high persistence of her filter—the estimated reversion speed κ is low—Agent A typically adjusts her views upwards or downwards slowly.

The mechanism through which Agent B updates her views differs significantly from Agent A's. When Agent B's filter sits exactly at f_m, she assigns equal probabilities to the up and the down state. As her expectations drop below f_m, her filter is rapidly attracted towards the

Good Times	Normal Times	Bad Times
$(\widehat{f}^A, \widehat{f}^B) \in [\bar{f}, f^h], \quad g = 0$	$(\widehat{f}^A, \widehat{f}^B) \in [f_m, \bar{f}], \quad g = 0$	$(\widehat{f}^A, \widehat{f}^B) \in [f^l, f_m], \quad g = 0$

Table 2: Definition of regimes

This table describes the 3 different regimes of the economy—good, normal, and bad times.

down state; when her expectations rises above f_m, her filter is rapidly attracted towards the up state. Now, because the up state is more persistent than the down state—the estimated intensity ψ is larger than λ—her filter tends to stick around the up state over a longer time period. This asymmetry therefore explains why her filter reverts towards a long-run mean f_∞ that is above f_m, i.e., if, instead, the down state were more likely, f_∞ would lie below f_m.

Based on these learning patterns, we focus our study on three different regimes. We say that the economy is in *good times* when agents' expectations are above $\bar{f}$ and in *bad times* when agents' expectations are below f_m. Finally, we say the economy is in *normal times* when filters lie between f_m and $\bar{f}$. To emphasize that future disagreement—as opposed to current disagreement—drives our result, we set current disagreement to zero in each case. Table 2 summarizes these 3 cases.

To illustrate how agents update their expectations in each regime, we first compute Agent A and B's average filter path. Then, conditional on Agent A's average filter path, we compute the probability that Agent B assigns to a given outcome over time, which solves Eq. (19) in Proposition 5. We plot the result in Figure 2. For the sake of brevity and for the rest of the analysis, we assume that filters start in the middle of each interval, as defined in Table 2.

In normal times (the middle panel), average filter paths move in the same direction, but at different speeds—Agent B quickly adjusts her expectations towards the up state, while Agent A's expectations slowly revert upwards. Therefore, disagreement spikes in the short term and then shrinks slowly over time. In good times (the left panel), both agents adjust their views downwards at comparable speeds and disagreement therefore exhibits little variations. Agent B is somewhat slower to adjust her views, which tend to stick around the up state. For this reason, Agent B is more optimistic than Agent A, both in normal and good times.

Learning differs significantly in bad times, as apparent from the right panel of Figure 2. First, agents disagree about the direction in which the economy is heading in the very short term. While Agent A expects a steady recovery, Agent B expects the economy to go further south—average filter paths diverge. Hence, in the short-term, Agent B is pessimistic, while

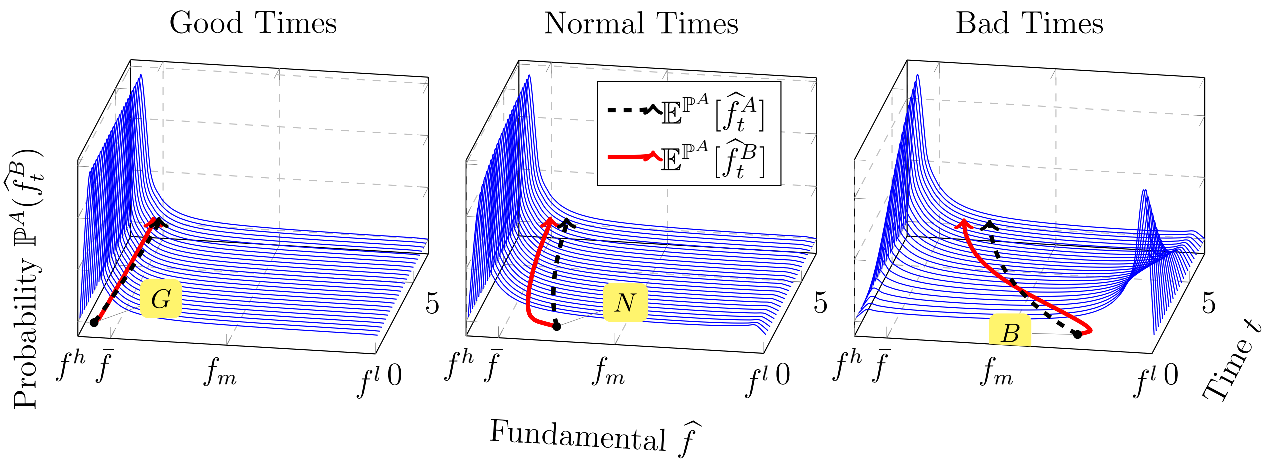

Figure 2: Filtered dynamics in 3 states of the economy.

The solid red and dashed black lines represent Agent A and B's average filters ($\mathbb{E}^{\mathbb{P}^A}[\widehat{f}_t^A]$ and $\mathbb{E}^{\mathbb{P}^A}[\widehat{f}_t^B]$), respectively. Both lines are plotted as functions of time and the probability that Agent B assigns to a given outcome (the blue lines) under Agent A's probability measure ($\mathbb{P}^A(\widehat{f}_t^B)$). Each Panel corresponds to a specific state of the economy—good (G), normal (N), and bad (B) times.

Agent A is optimistic. Second, to Agent B's mind, bad times do not last as long as good times and, therefore, she expects a long-run recovery. As a result, disagreement gradually decreases and bounces back in the long-run as Agent B adjusts her views significantly faster than Agent A. Hence, in the long-run, Agent B is more optimistic than Agent A. Overall, in our model, bad times are characterized by large disagreement driven by agents' divergence of views in the short term and large disagreement driven by agents' different updating speeds in the long term.

This pattern of disagreement is consistent with that documented empirically among forecasters. In particular, Patton and Timmermann (2010), Carlin et al. (2013), and Barinov (2014) provide empirical evidence that disagreement tends to be larger in downturns than in upturns. Veronesi (1999) also shows that the dispersion of analysts' forecasts is greater when the economy is contracting.

4.2 Disagreement driving Stock Return Predictability

The goal of this section is to link disagreement to return predictability. In Subsection 4.2.1, we show that disagreement predicts short-term downward pressure on prices followed by a reversion to fundamentals. This pattern is weakest in good times, strongest in normal times, while downward price pressure is largest and most persistent in bad times. The upshot of this is that prices continue to react to news shocks in bad and normal times, but not in good

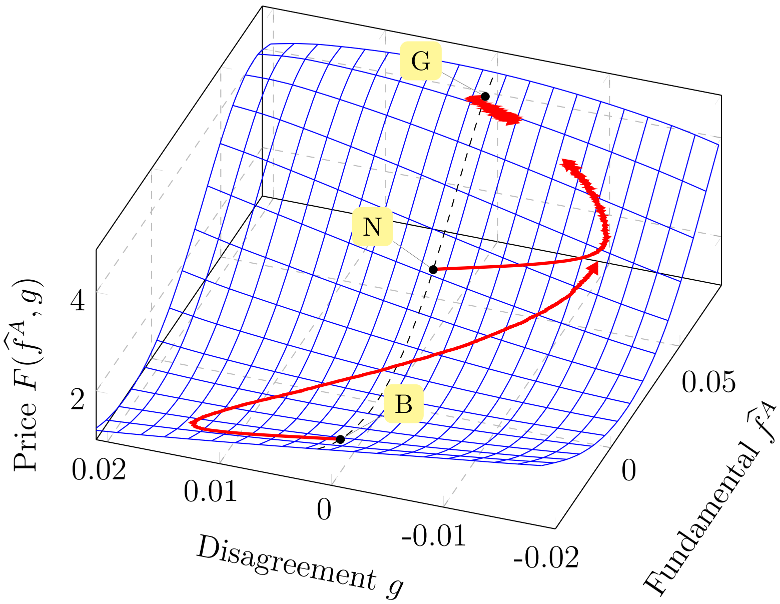

Figure 3: Disagreement, fundamental, and expected prices.
The blue surface represents the middle price component in Eq. (13), plotted as a function of disagreement and fundamentals. The red arrows describe the average price paths over the next 7 years, each corresponding to a state of the economy—good (G), normal (N), and bad (B) times.

times during which reversion to fundamentals prevails (Subsection 4.2.2).

4.2.1 Disagreement and Downward Price Pressure

In an influential paper, Tetlock (2007) provides evidence that media pessimism predicts low future returns at short horizons and high future returns at longer horizons. In a subsequent paper, Garcia (2013) demonstrates that the relation between media pessimism and return predictability is concentrated in recessions. Since disagreement among investors is a measure of relative pessimism, we show that our model offers an explanation for this fact.

In our model, predictability is driven by the interaction between disagreement and fundamentals. To show this, we first isolate the component of prices that exclusively captures this interaction—the function F in Eq. (13). We plot this price component (the blue surface) in Figure 3, both as a function of disagreement and the fundamental. This surface provides a static view on prices, disagreement, and fundamental. To see how these three dimensions interact dynamically, we draw their average path over time (the red arrows).

The joint effect of disagreement and fundamentals on prices operates through the competing implications of the precautionary savings and substitution effects. In bad times (B), Agent A thinks the fundamental will increase slowly, while Agent B thinks it will continue to decrease in the short term. Therefore, disagreement increases sharply in the short term (the red arrow moves towards the left), which causes a strong downward pressure on prices because the precautionary savings effect implied by both the sharp increase in disagreement and the slow increase in the fundamental dominates the substitution effect. Agents' views then align but Agent B is faster to update her expectations upwards (the red arrow changes direction). This decrease in disagreement towards zero dampens the substitution effect implied by the increase in the fundamental, and therefore causes prices to be steady in the medium term. In the long term, the substitution effect implied by the increase in the fundamental dominates and pulls prices up.

In normal times (N), disagreement increases quickly and significantly in the short term (the red arrow moves towards the right), because Agent A adjusts her views slowly and Agent B jumps to conclusion. Since Agent A is more pessimistic than Agent B, she disinvests as she anticipated a downward pressure on prices. But since disagreement shrinks over time (the red arrow reverts back towards zero) and the fundamental increases, Agent A starts reinvesting which brings prices back up.

In good times (G), both disagreement and fundamentals tend to stick around the same level (the red arrow is confined to a small area). In that case, prices exhibit little variations, as they would in the representative agent model of Lucas (1978).

We now consider the entire price function in Eq. (13) and plot its average path over time in Figure 4. As anticipated by Tetlock (2007), disagreement tends to predict low returns at short horizons and high returns at longer horizons. This conclusion is clear-cut in normal times—the solid red line is strongly U-shaped. Moreover, consistent with Garcia (2013), short-term predictability is concentrated in bad times—the dashed line exhibits the strongest downward pressure. In good times, return predictability vanishes as disagreement and fundamentals exhibit little variations.

Overall, the speed at which disagreement changes in the short term determines the slope of the downward price pressure. In bad times, disagreement spikes, which generates a steep downward price pressure. In normal times, disagreement increases at a lower speed and the downward price pressure is therefore less pronounced. In good times, disagreement is almost constant and price pressures are weak.

4.2.2 Price Reaction to News

In our model, short-term predictability in bad times results from a persistent and decreasing price reaction to news shocks, along the lines of Jegadeesh and Titman (1993). In bad times, opinions regarding future economic conditions diverge in the short term, causing a spike in disagreement. The market is slow to reach an agreement and the price continues to react to

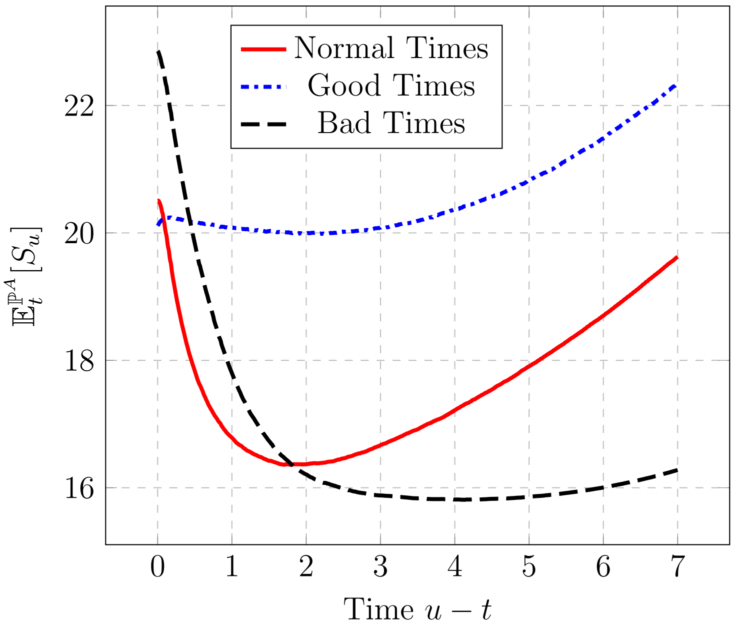

Figure 4: Average expected stock price over time.
This figure plots the average price trajectory over time (7 years). Each line corresponds to a state of the economy—good (dashed black), normal (solid red), and bad (dash-dotted blue) times.

stale information.[28] On the contrary, short-term predictability in normal times results from a persistent and increasing price reaction to news shocks, consistent with De Bondt and Thaler (1985). In normal times, opinions align but Agent B revises her expectations precipitately—she jumps to conclusions—which causes a similar price reaction, but for a different reason. In good times, there is little variation in disagreement and prices immediately revert to fundamentals.

To show how prices react to news in our model, we perform the following experiment. We consider a trajectory of news—in our model, the only source of news is the Brownian innovation $\widehat{W}^A$. We then introduce a news *surprise* today—an initial perturbation of the news trajectory—and keep the news trajectory otherwise unchanged, as illustrated in the upper panel of Figure 5.

To each news trajectory—both perturbed and unperturbed—corresponds a price trajectory, as illustrated in the lower panel of Figure 5. The difference $\mathcal{D}S$ between these price trajectories (the shaded area) precisely captures the price reaction to the news surprise *relative* to the price that would have prevailed, had there been no news surprise. Finally, because we are interested in *all* possible trajectories, we compute an average price reaction $E[\mathcal{D}S]$ (the solid black line).

[28] Garcia (2013) documents a similar phenomenon.

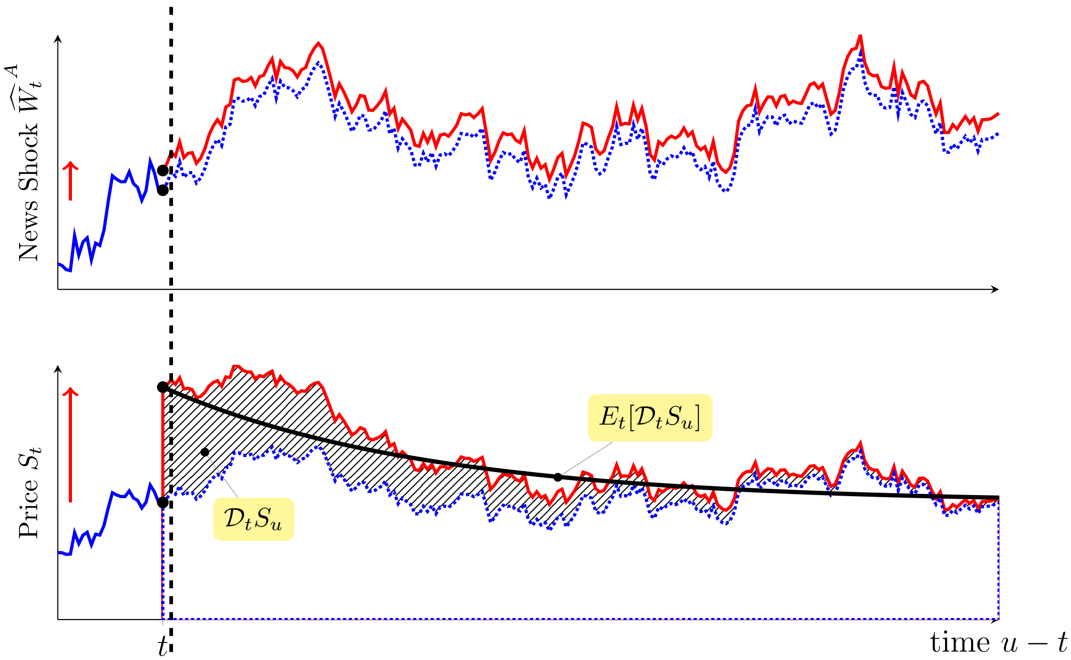

Figure 5: Illustration of a price impulse response to a news shock. The upper panel shows a trajectory of news before (dashed blue line) and after (solid red line) a news surprise. The lower panel shows the associated price trajectory before (dashed blue line) and after (solid red line) a news surprise. The shaded area represents the price reaction $\mathcal{D}S$ to a news shock and the solid black line represents the average price reaction $E[\mathcal{D}S]$.

When the news surprise is "small", the price reaction $\mathcal{D}S$ becomes a well-defined mathematical object known as a Malliavin derivative.[29] To borrow the insightful analogy of Dumas et al. (2009), "Just as the standard derivative measures the local change of a function to a small change in an underlying variable, the Malliavin derivative measures the change in a path-dependent function implied by a small change in the initial value of the underlying Brownian motions." Its economic meaning is that of an impulse-response function following a shock in initial values—in our case, a news shock. However, unlike a standard impulse-response function, a Malliavin derivative takes future uncertainty into account—it does not assume that the world becomes deterministic after the shock has occurred. For that reason, we compute an *average* price response, as described in Proposition 6.

[29] See, e.g., Detemple and Zapatero (1991), Detemple, Garcia, and Rindisbacher (2003, 2005), Berrada (2006), and Dumas et al. (2009) for applications of Malliavin calculus in Financial Economics.

Proposition 6. *The expected average price response to a news surprise satisfies*

$$\mathbb{E}_t^{\mathbb{P}^A}[\mathcal{D}_t S_u] = \sigma_\delta \mathbb{E}_t^{\mathbb{P}^A}[S_u] + \underbrace{\frac{\gamma}{\kappa\sigma_\delta}\left(\left(1-\alpha e^{-\kappa(u-t)}\right)\mathbb{E}_t^{\mathbb{P}^A}[S_u] + (\alpha-1)\mathbb{E}_t^{\mathbb{P}^A}\left[\int_u^\infty \frac{\xi_s}{\xi_u}\delta_s e^{-\kappa(s-t)}\,ds\right]\right)}_{\textit{fundamental effect}}$$

$$+\underbrace{\frac{1}{\sigma_\delta^2}\mathbb{E}_t^{\mathbb{P}^A}\left[\int_u^\infty \frac{\xi_s}{\xi_u}\delta_s\left((\omega(\eta_s)-\omega(\eta_u))\int_t^u g_v\mathcal{D}_t g_v\,dv - (1-\omega(\eta_s))\int_u^s g_v\mathcal{D}_t g_v\,dv\right)ds\right]}_{\textit{disagreement effect}}, \tag{20}$$

where the disagreement response $\mathcal{D}g$ to a news surprise solves

$$d\mathcal{D}_t g_v = \nabla\mu_g(\widehat{f}_v^A, g_v)^\top \begin{pmatrix} \frac{\gamma}{\sigma_\delta}e^{-\kappa(v-t)} \\ \mathcal{D}_t g_v \end{pmatrix} dv + \nabla\sigma_g(\widehat{f}_v^A, g_v)^\top \begin{pmatrix} \frac{\gamma}{\sigma_\delta}e^{-\kappa(v-t)} \\ \mathcal{D}_t g_v \end{pmatrix} d\widehat{W}_v^A, \tag{21}$$

with initial condition $\mathcal{D}_t g_t = \sigma_g(\widehat{f}_t^A, g_t)$ and where μ_g and σ_g represent the drift and the diffusion of disagreement in Eq. (7), respectively. The operator ∇ stands for the gradient.

Proof. See Appendix A.6.

The news surprise affects prices through three different channels. The first term in Eq. (20) represents the dividend channel; it is just a scaled average price, which we discussed in the previous section. The second term captures the filter channel. This term is initially negative; it becomes gradually positive and brings prices closer to their fundamental levels. The last term accounts for the disagreement channel.

Our calibration indicates that the first two effects will be of small short-term magnitude as compared to the disagreement channel. Disagreement is therefore the main channel through which the price reacts to the news surprise in the short term. In particular, the news surprise moves prices through the term $g \times \mathcal{D}g$, disagreement multiplied by the reaction of disagreement to the news surprise (Eq. (21)). We plot its average path $E_t[g_u\mathcal{D}_t g_u]$ in the right panel of Figure 6.

The news shock exacerbates disagreement in the short term, both in normal and bad times, but not for the same reason. In the former case, Agent B is faster to adjust her views to the news shock, which sharpens the difference in adjustment speed across agents. In the latter case, the shock polarizes agents' opinions and expectations move further apart. In good times, the reaction of disagreement is similar to that in normal times; it is, however, significantly less pronounced and persistent.

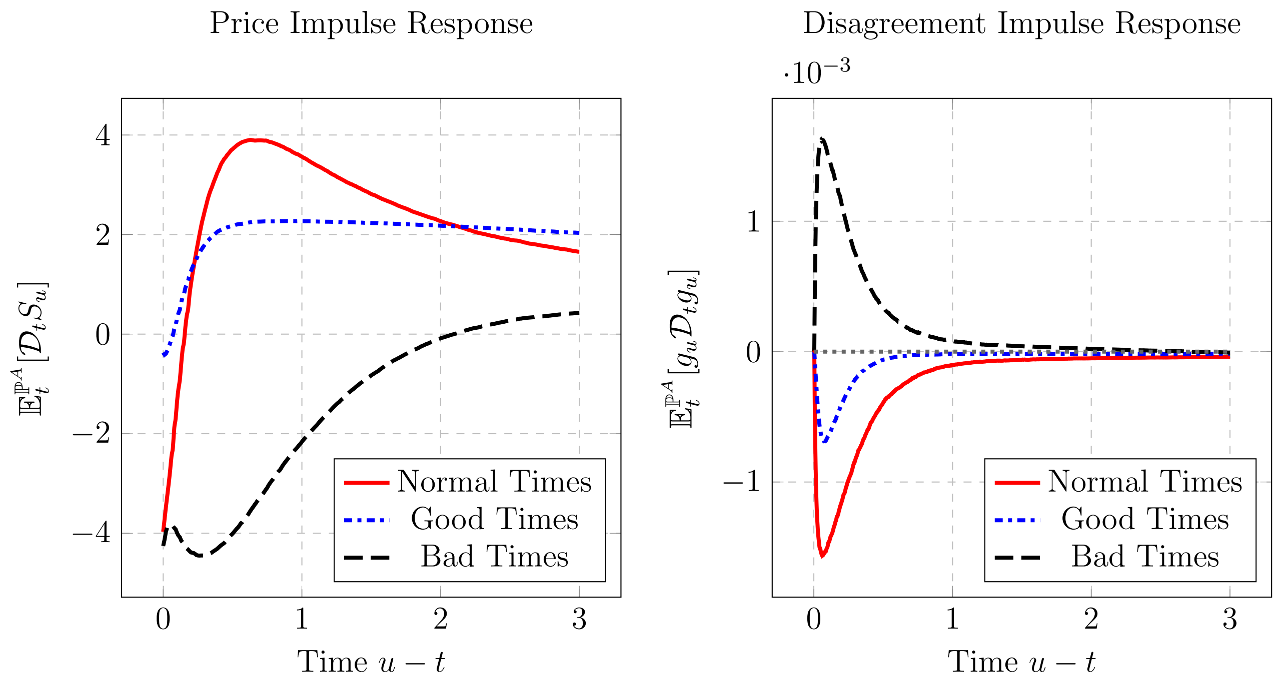

Figure 6: Impulse response to a news shock.

The left-hand panel shows the average price response $\mathbb{E}_t^{\mathbb{P}^A}[\mathcal{D}_t S_u]$ to a news shock. The right-hand panel shows the average price response of disagreement to a news shock multiplied by disagreement $\mathbb{E}_t^{\mathbb{P}^A}[g_u \mathcal{D}_t g_u]$. Each is plotted as a function of time. Each line corresponds to a different regime.

We now want to understand how the reaction of disagreement drives the reaction of prices. We plot the 3-year average price response to a news shock today in the left panel of Figure 6. As the news surprise hits, the price experiences an initial drop. In bad times (the black dashed line), agents interpret the news surprise in opposite ways and their divergence of opinions slows down the price adjustment to the news shock. The price continues to react to the shock, for as long as their dispute induces sufficient disagreement, and then slowly reverts as opinions gradually align. While the news shock generates no debate in normal times (the solid red line), Agent B updates her expectations significantly faster than Agent A, inducing a spike in disagreement and continuing price reaction. The price then slowly corrects as Agent A catches up with Agent B. In good times (the dash-dotted blue line), there is little variations in disagreement and prices therefore revert immediately. Overall, prices continue to react to news shocks only if disagreement spikes in the short term. This phenomenon, unique to our model, arises because Agent B adjusts her expectations significantly faster than Agent A.

The sign of the change in disagreement drives the direction of the price reaction in normal and bad times. In bad times, Agent A is more optimistic than Agent B and the price reaction is decreasing. In contrast, in normal times, Agent A is more pessimistic than Agent B and the price reaction is increasing. The former result is similar to the continuing under-reaction of Hong and Stein (1999), while the latter result resembles the continuing over-reaction of Daniel et al. (1998). Our emphasis is that these effects may take place at different points over the business cycle. Our result is also related to that of Veronesi (1999). While his analysis relies on a representative agent economy, ours relies on disagreement.

We now show that the speed at which disagreement changes in the short term leads to short-term momentum in stock returns.

4.3 Implications for Time-Series Momentum and Momentum Crashes

Stock return momentum is one of the most pervasive facts in Financial Economics. Returns trend over short horizons (Jegadeesh and Titman (1993)), typically 6 months to 1 year, and revert over longer horizons (De Bondt and Thaler (1985)). In a recent study, Moskowitz et al. (2012) uncovers a similar pattern in the time-series of returns, which they coin "time-series momentum". Time-series momentum is a natural implication of our model, on which we focus in this section. We show both theoretically (Subsection 4.3.1) and empirically (Section 6) that time-series momentum is strongest in bad times, particularly at a one-month lag. We further show that our model gives rise to momentum *crashes* following sharp market rebounds, consistent with the novel evidence of Daniel and Moskowitz (2013) and Barroso and Santa-Clara (2014).

4.3.1 Time-Series Momentum and Momentum Crashes

Our goal is to study the implications of short-term predictability for the serial correlation of returns. To do so, we focus on *excess* stock returns R, the dynamics of which satisfy

$$\mathrm{d}R_t = \frac{\mathrm{d}S_t + \delta_t \mathrm{d}t}{S_t} - r_t^f \mathrm{d}t = \sigma_t(\theta_t \mathrm{d}t + \mathrm{d}\widehat{W}_t^A).$$

Following Banerjee, Kaniel, and Kremer (2009), we then compute the serial correlation of returns at different lags h according to

$$\mathbb{E}^{\mathbb{P}^A}[R_{t+h} - R_t | R_t - R_{t-h}] = \frac{\mathrm{cov}^{\mathbb{P}^A}\left(R_{t+h} - R_t, R_t - R_{t-h}\right)}{\mathrm{var}^{\mathbb{P}^A}\left(R_t - R_{t-h}\right)} \left(R_t - R_{t-h}\right) = \rho_t(h)\left(R_t - R_{t-h}\right).$$

The coefficient ρ determines whether returns exhibit momentum ($\rho > 0$) or reversal ($\rho < 0$). We plot this coefficient for different lags, ranging from 1 month to 3 years, in Figure 7.

In our model, returns exhibit short-term momentum both in normal (the center panel)

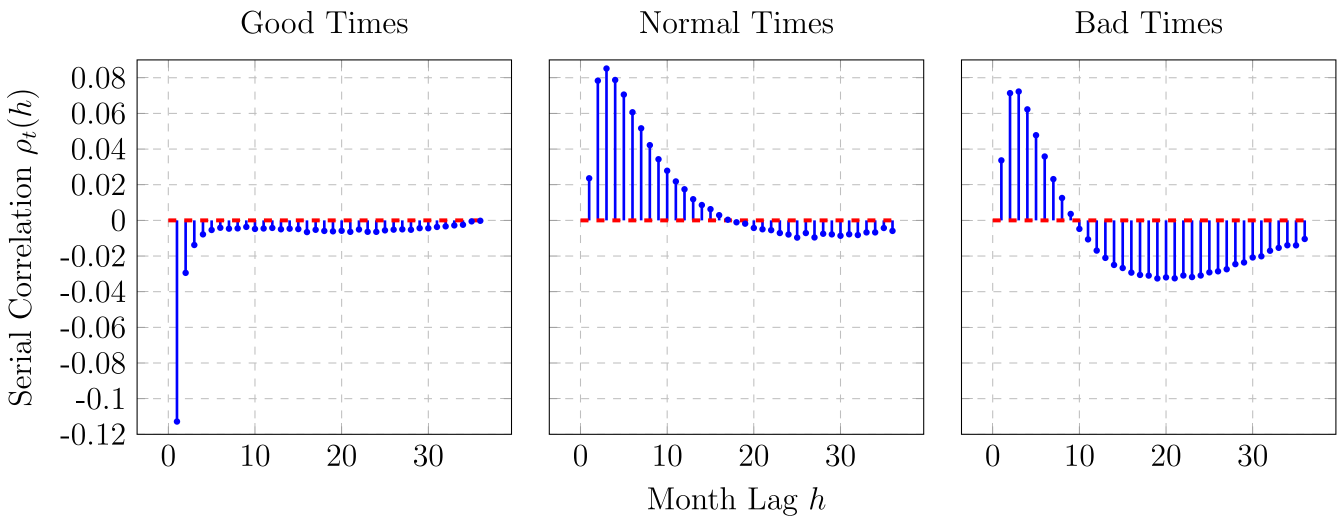

Figure 7: Serial correlation of excess returns.
This figure plots the serial correlation $\rho_t(h)$ of excess returns for different lags h, ranging from 1 month to 3 years. Each panel corresponds to a different state of the economy.

and bad times (the right panel)—returns keep moving in the same direction over an horizon of 10 to 17 months and then revert over subsequent periods, all the more so in bad times. Both the magnitude and the time horizon of these effects are in line with empirical evidence. Specifically, Moskowitz et al. (2012) document significant time-series momentum in equity index returns over roughly 12 months, systematically followed by long-term reversal. In addition, the model implies that the 1-month serial correlation in returns is largest in bad times—it is about twice as large in bad times as it is in normal times, and it is negative in good times.

Based on our previous analysis, momentum in our model results from continuing price reaction to news shocks in both normal and bad times. In that respect, Figure 4 suggests that momentum should persist over a longer time period in normal times, which is confirmed by Figure 7. Time-series momentum in the very short term (at a one-month lag), however, is largest in bad times due to the spike in disagreement. Returns have different time-series properties in good times (the left panel). In particular, returns exhibit strong initial reversal and weak serial correlation over subsequent months. The following mechanism operates. In good times, the main force driving prices is the filter and not disagreement[30]. Since the filter reverts downwards towards its long-term mean, a positive contemporaneous price shock predicts a drop in future prices and therefore a price reversal.

An important consequence of the initial reversal spike in good times is that a momentum

[30]Figure 3 shows that, in good times, the price is significantly more sensitive to a change in the filter than it is to a change in disagreement.

strategy may crash if the market rises sharply. Suppose we start implementing a momentum strategy in normal times, or in bad times for that matter. The center and right panels then indicate that this strategy would be profitable as long as the market does not rise suddenly. If, instead, the market sharply rebounds—as was the case in January 2001, for instance—the momentum strategy would crash, consistent with the evidence of Daniel and Moskowitz (2013) and Barroso and Santa-Clara (2014).

5 Trading Strategies and Profits

Return predictability varies over the business cycle. Evidence suggests that institutional investors exploit this fact to time the market.[31] In this section, our objective is to identify the trading strategy that allows an investor to take advantage of short-term predictability over the business cycle. We show that Agent A implements a momentum strategy in bad and normal times, whereas she follows a contrarian strategy in good times. Because predictability increases as economic conditions deteriorate, her strategy makes significantly more money in bad times.

To identify Agent A's trading strategy, we compute her wealth and deduce from it the two components of her strategy. We highlight Agent A's wealth in Proposition 7 below.

Proposition 7. *Agent A's wealth, V, satisfies*

$$\begin{aligned} V_t &= \mathbb{E}_t^{\mathbb{P}^A}\left[\int_t^\infty \frac{\xi_u}{\xi_t} c_{Au}\, du\right] \\ &= \delta_t \omega\left(\eta_t\right)^\alpha \sum_{j=0}^{\alpha-1} \begin{pmatrix} \alpha-1 \\ j \end{pmatrix} \left(\frac{1-\omega\left(\eta_t\right)}{\omega\left(\eta_t\right)}\right)^j \mathbb{E}_t^{\mathbb{P}^A}\left[\int_t^\infty e^{-\rho(u-t)} \left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}} \left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha} du\right] \\ &\overset{(\alpha=2)}{=} \delta_t \omega\left(\eta_t\right)^2 \left.\frac{S_t}{\delta_t}\right|_{O.U.} + \delta_t \omega\left(\eta_t\right)\left(1-\omega\left(\eta_t\right)\right) F\left(\widehat{f}_t^A, g_t\right). \end{aligned} \tag{22}$$

Proof. See Dumas et al. (2009).

Our calibration ($\alpha = 2$) implies that Agent A's wealth is the sum of two terms, both of which have an intuitive interpretation. If Agent A were the only agent populating the economy, she would hold one unit of the stock over time, otherwise markets would not clear. For that reason, her wealth would be exactly equal to the price that prevails in a

[31] Moskowitz (2000), Glode (2011), and Kosowski (2011) show that fund managers perform significantly better in recessions than in expansions. Kacperczyk et al. (2013) provide evidence that fund managers time the market in recessions and pick stocks in expansions.

representative agent economy, hence the first term in Eq. (22). But, because she can trade with Agent B, her portfolio fluctuates over time; it fluctuates in a way that reflects both her expectations of the fundamental and how they differ from Agent B's. The second term in Eq. (22) precisely accounts for that. In particular, the function F is the price component we plotted in Figure 3, which represents how the interaction between fundamentals and disagreement operates on prices.

From Agent A's wealth, we now deduce the strategy she implements. We present her strategy in Proposition 8.

Proposition 8. *The number Q of shares held by Agent A satisfies*

$$Q = \frac{1}{\sigma S}\left(\frac{\partial V}{\partial \delta}\sigma_\delta \delta + \frac{\partial V}{\partial \widehat{f}^A}\frac{\gamma}{\sigma_\delta} + \frac{\partial V}{\partial g}\frac{\gamma - \left(\widehat{f}^A - g - f^l\right)\left(f^h + g - \widehat{f}^A\right)}{\sigma_\delta} - \frac{\partial V}{\partial \eta}\frac{g\eta}{\sigma_\delta}\right), \quad (23)$$

where σ denotes the diffusion of stock returns, which satisfies

$$\sigma = \sigma_\delta + \frac{1}{S}\left(\frac{\partial S}{\partial \widehat{f}^A}\frac{\gamma}{\sigma_\delta} + \frac{\partial S}{\partial g}\frac{\gamma - \left(\widehat{f}^A - g - f^l\right)\left(f^h + g - \widehat{f}^A\right)}{\sigma_\delta} - \frac{\partial S}{\partial \eta}\frac{g\eta}{\sigma_\delta}\right). \quad (24)$$

The number of shares $Q \equiv M + H$ can be further decomposed into a myopic portfolio $M = \frac{\mu - r^f}{\alpha\sigma^2}\frac{V}{S}$ and a hedging portfolio

$$H = Q - M = \frac{\alpha - 1}{\alpha \sigma_t S_t} E_t^{\mathbb{P}^A}\left[\int_t^\infty \frac{\xi_s}{\xi_t} c_{As}\left(\frac{\mathcal{D}_t \xi_s}{\xi_s} - \frac{\mathcal{D}_t \xi_t}{\xi_t}\right) ds\right]. \quad (25)$$

Proof. See Appendix A.7.

Agent A's portfolio in Equation (23) has two parts. The first part, M, is a myopic demand through which Agent A seeks to extract the immediate Sharpe ratio. The second term, H, is an hedging demand through which Agent A seeks to exploit returns predictability. In particular, the term $\mathcal{D}_t \xi_s$ in Eq. (25) indicates that Agent A performs the experiment of Section 4.2.2—she attempts to forecast how a news shock today affects future returns. Because the hedging demand tells us how Agent A trades on returns predictability, we focus the analysis on this term.

To demonstrate how Agent A exploits returns predictability, we follow (Wang, 1993; Brennan and Cao, 1996) and compute the correlation, $\mathrm{Corr}_t(\mathrm{d}H_t, \mathrm{d}P_t)$, between changes in Agent A's hedging portfolio and changes in stock prices. This correlation measures Agent A's trading behavior. A positive correlation means that Agent A tends to buy after price

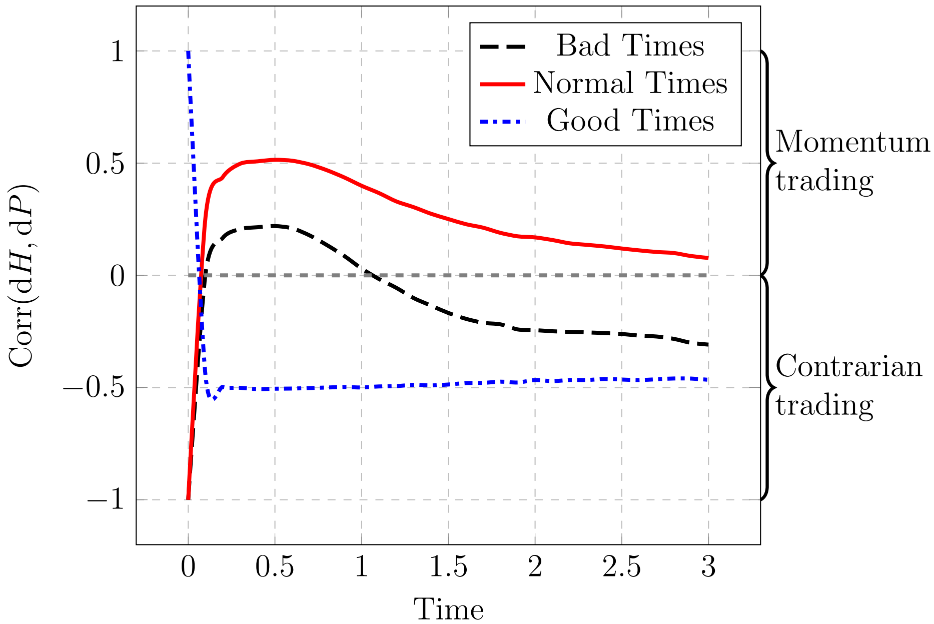

Figure 8: Model-implied trading behavior.
The solid red line, the dash-dotted blue line, and the dashed black line depict the average correlation between the hedging component $\mathrm{d}H$ and the stock price $\mathrm{d}S$ in normal, good, and bad times, respectively. The correlation reported above is an average computed over 100,000 simulations.

increases and sell after price decreases—she is a momentum trader. Instead, a negative correlation means that she is a contrarian. We plot this trading measure in Figure 8.

Agent A is a successful market timer. In the short-term (over the first year), she implements a momentum strategy both in normal times (the solid red line) and in bad times (the dashed black line). In particular, Agent A anticipates that prices will continue to react to news shocks in both normal and bad times. Continuing price reaction generates time-series momentum, which she extracts by implementing a momentum strategy. Moreover, Agent A recognizes that time-series momentum is stronger in normal times on average and therefore trades more aggressively in that regime. In the long term, Agent A reverts her strategy and becomes a contrarian, but only in bad times; she remains a trend-follower in normal times. The reason is that reversal is strong and arises earlier in bad times, while reversal is insignificant and only arises later in normal times. In good times, Agent A is a contrarian (the dash-dotted blue line). Agent A anticipates that reversion to fundamentals will be fast and strong, and consistently reverts her strategy. Doing so, she avoids momentum crashes and exploits stock return reversal.

Agent A consistently times the market; we now want to understand when her strategy is most profitable over the business cycle. To do so, we compute the mean cumulative profit

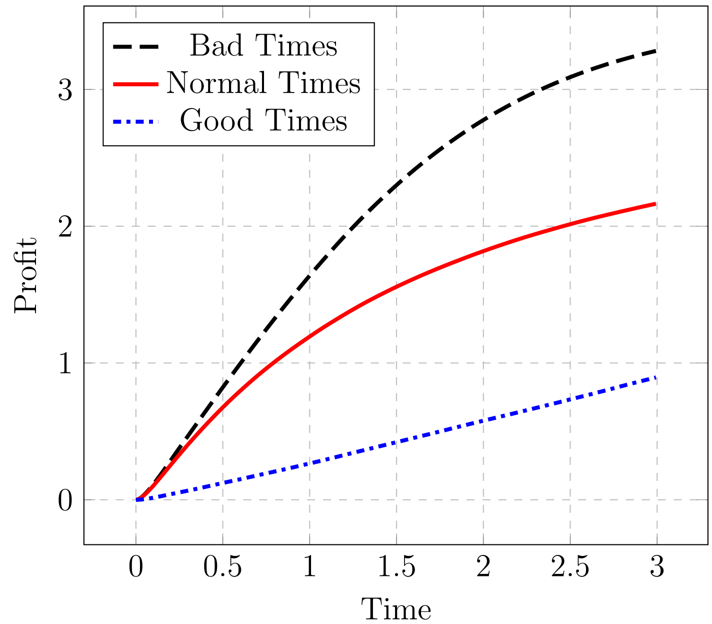

Figure 9: Model-implied profits.

The solid red line, the dash-dotted blue line, and the dashed black line depict the average of the mean cumulative profit made by Agent A's hedging strategy in normal, good, and bad times, respectively. The profit reported above is an average computed over 100,000 simulations.

P that her strategy generates. It satisfies

$$P_t = \int_0^t \left(Q_u S_u - \frac{\mu_u - r_u^f}{\alpha \sigma_u^2} V_u \right) \left(\mu_u - r_u^f \right) \mathrm{d}u = \int_0^t H_u S_u \left(\mu_u - r_u^f \right) \mathrm{d}u. \tag{26}$$

The profits in Eq. (26) represents the cumulative expected dollar amount made by following a fully leveraged dynamic strategy that consists in borrowing HS dollars and investing this amount in the stock. We plot the average cumulative profit P in Figure 9.

Consistent with the empirical findings of Moskowitz (2000), Glode (2011), Kosowski (2011), and Lustig and Verdelhan (2010), the profits that Agent A's strategy generates are smallest in good times and largest in bad times. The reason is as follows. In good times, prices accommodate news shocks almost instantly, which makes it challenging for Agent A to reap large profits. In normal and bad times, however, stock prices exhibit short-term predictability (continuing price reaction). Agent A exploits this information to make larger profits. In particular, Agent A makes the largest profits in bad times, that is when short-term predictability is strongest.

Our results show that a market-timing strategy is most profitable in bad times. In that respect, our model offers an explanation for the finding that institutional investors focus on market-timing strategies during recessions (Kacperczyk et al. (2013)).[32] In addition, Moskowitz et al. (2012) show that a time-series momentum strategy generates large profits at the beginning of a crisis and incurs large losses at the end of the crisis when the market rebounds. Our model is consistent with this fact—a momentum strategy makes large profits in bad times and incurs losses after sharp economic recoveries.

6 Empirical Evidence

Our model predicts that time-series momentum at a one-month lag is stronger in bad times and that profits of momentum strategies are consequently largest in bad times. In this section, we provide empirical support for these predictions. First, we show that over the last 120 years, time-series momentum at a one-month lag has been significantly stronger during recessions. Second, we show that momentum profits are, on average, twice as large in recessions as they are in expansions.

6.1 Time-Series Momentum across the Business Cycle

We consider the monthly excess returns r^e on the S&P500 from January 1871 to November 2013, which we obtain from Robert Shiller's website. Looking back to the 19th century allows us to account for a large number of recessions—NBER reports 29 recessions since the beginning of our sample. An undesirable result, however, is that the volatility of returns varies dramatically over the sample—returns have been remarkably volatile over the period surrounding the 1929 market crash, for instance. To make meaningful comparisons through time, we run a GARCH(1,1) and scale excess returns r^e_t at time t by their ex-ante volatility σ_{t-1} at time $t-1$. We then estimate the following regression

$$r^e_t/\sigma_{t-1} = \alpha + \beta_{1,h} r^e_{t-h}/\sigma_{t-h-1} + \beta_{2,h} D_{t-h} r^e_{t-h}/\sigma_{t-h-1} + \epsilon_t, \tag{27}$$

where D_t is a dummy variable that takes value 1 if t belongs to an NBER recession.

The coefficient $\beta_{1,h}$ in Eq. (27) captures the time-series momentum effect of Moskowitz et al. (2012) at a $h-$month lag. The second coefficient β_2, absent in Moskowitz et al. (2012), measures times-series momentum in recessions exclusively, beyond that measured by β_1. If $\beta_{2,h}$ is significantly positive, there is additional time-series momentum in recessions at a $h-$month lag; if not, then time-series momentum is either identical or lower in recessions.

[32]Ferson and Schadt (1996) provide evidence that fund managers timing ability varies with economic conditions. Dangl and Halling (2012) show that market-timing strategies work best in recessions.

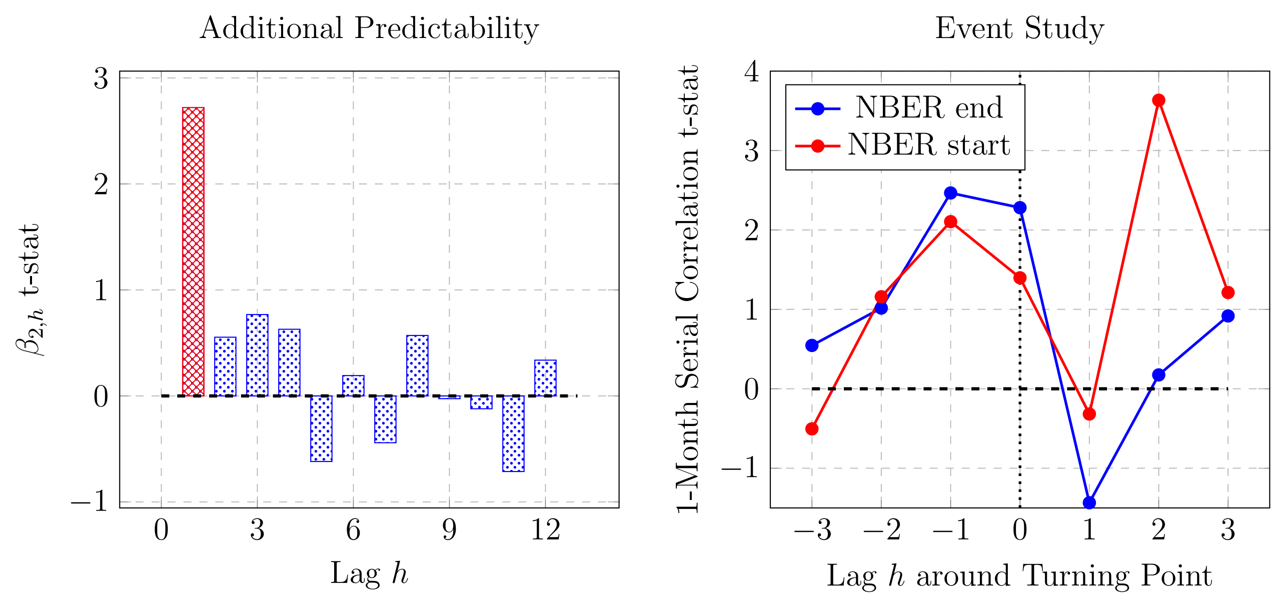

Figure 10: Time-series momentum t-statistics during recessions.
The left panel plots the t-statistics of the serial correlation in recessions at different lags, ranging from 1 month to 1 year. Red bars represent significance at the 99% level. The right panel plots the t-statistics of the 1-month serial correlation 3 months around an NBER turning point. The blue line corresponds to troughs and the red line corresponds to peaks.

We plot the t-statistics of $\beta_{2,h}$ for different lags h, ranging from 1 month to 1 year, in the left panel of Figure 10.

Time-series momentum is significantly larger in recessions, mostly at a 1-month lag (the red bar). There is additional, although statistically insignificant, momentum in recessions up to the 4-month lag. The momentum difference between recessions and expansions then vanishes over subsequent lags. In Section 6.2, we show that superior predictability at a 1-month lag is sufficient to make significantly higher profits in recessions than in expansions.

Focusing on momentum at a 1-month lag—precisely where additional predictability is concentrated—we conduct the following event study. We look at the 1-month serial correlation over a 3-month window around an NBER turning point and plot the associated t-statistics in the right panel of Figure 10. T-statistics before a trough (the blue line) are positive and of the same order of magnitude as those before a peak (the red line). Predictability then vanishes right after the turning point. Momentum subsequently reappears, but only after a peak and not after a trough. This is another indication that the magnitude of momentum differs across recessions and expansions.

Our finding—time-series momentum is stronger in recessions—contrasts with evidence reported in the *cross-section* of returns. In particular, Chordia and Shivakumar (2002) and Cooper, Gutierrez, and Hameed (2004) find that cross-sectional momentum is weaker in down-markets than in up-markets. While Moskowitz et al. (2012) show that time-series and cross-sectional momentum are strongly related, our results suggest that their relation varies over the business cycle.

6.2 Momentum Trading and Profits

To show that momentum profits are largest in recessions, we build the following naive momentum strategy: at time t, we go long when $r^e_{t-1} > 0$ and short when $r^e_{t-1} < 0$, where r^e denotes the excess return on the S&P500. We then compute the cumulative profits, P^e, of this strategy according to

$$P^e_t = \sum_{u=1}^{t-1} Y_u r^e_{u+1}$$
$$Y_u = \begin{cases} 1 \text{ if } r^e_u > 0 \\ -1 \text{ otherwise} \end{cases}$$

The profit P^e is recorded at a monthly frequency from January 1871 to November 2013, a time period that covers 29 NBER recessions. Figure 11 shows the cumulative profit made in expansions (the blue line) and in recessions (the black line), along with total profits (the red line). The profit made in expansions is zero if the economy is in a recession, and vice versa.

The conclusion we draw from Figure 11 is clear—even though NBER recessions only cover a total of 500 months out of the 1715 months of the sample, the cumulative profit made in recessions is overall larger than that made in expansions. In particular, the average monthly profit made in NBER recessions is 0.0132, whereas that made in expansions is only 0.0059. That is, consistent with the prediction of the model, the average monthly profit made in recession is roughly twice larger than that made in expansions. Moreover, focusing on the crises of 1929 and 2008, we observe that a momentum strategy yields impressive profits over a very short time period—the cumulative profit steeply increases during these two major crises.

7 Conclusion

We have presented a dynamic general equilibrium model in which two agents use different models to update their forecast of future economic conditions. One agent believes economic conditions change slowly from good to bad times, while the other believes they can change precipitately.

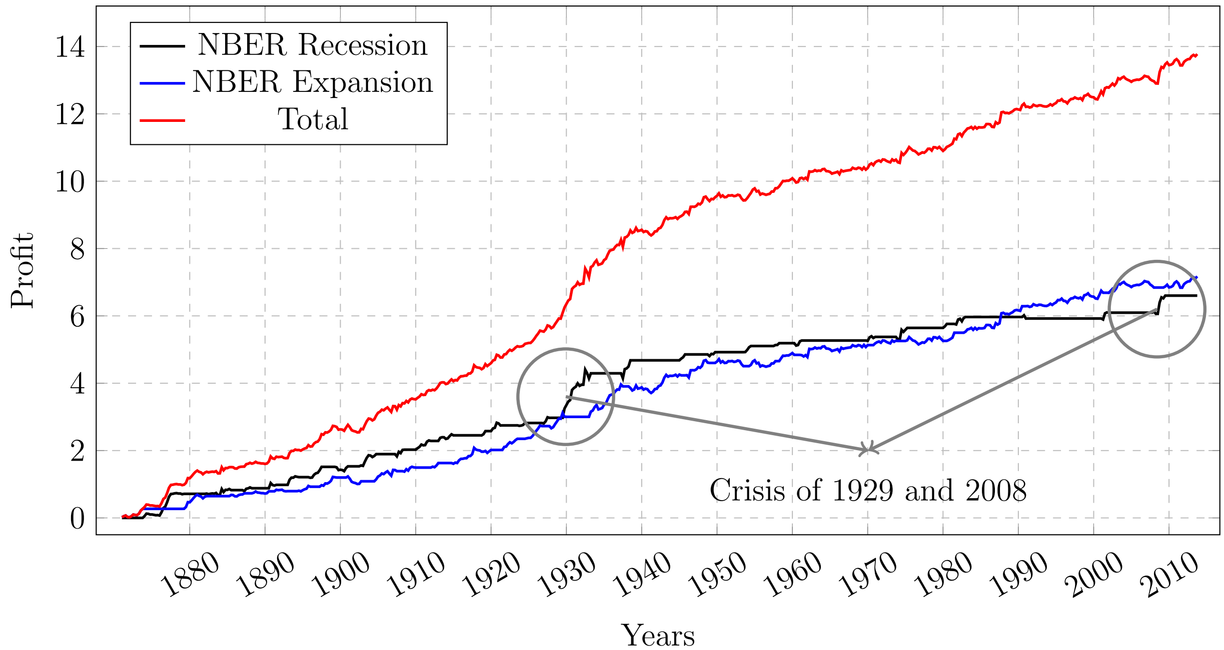

Figure 11: Profit of momentum strategy in NBER recessions and expansions. The blue and black curves depict the cumulative profit made by the naive momentum strategy in NBER expansions and recessions, respectively. The red curve shows the total profit. Monthly data from 01/1871 to 11/2013 are considered.

Heterogeneous beliefs lead to an asymmetric term structure of disagreement—disagreement is large and persistent in bad times, whereas it is small and short-lived in good times. This asymmetry causes stock return predictability to be concentrated in bad times. In bad times, prices continue to react to news shocks and then slowly revert to fundamentals. By contrast, in good times, prices revert almost instantly to fundamentals. The upshot of this is that stock returns exhibit short-term momentum in bad times and strong immediate reversal in good times. An investor can use this information to make large profits in bad times—she follows the market in bad times and revert her strategy to avoid momentum crashes when the market rebounds. We test these predictions and find that momentum profits are about twice as large in recessions as they are in expansions.

This paper suggests at least two interesting avenues for future research. First, our analysis describes how a single stock—an index—reacts to news shocks, but individual stocks composing the index may react differently. In particular, the performance of one stock relative to another may vary over the business cycle—some may be "losers", others may be "winners", and this relation may persist or revert depending on economic conditions. Extending our framework to an economy with two trees would allow to study both this cross-sectional relation and how investors pick stocks over the business cycle. Second, in our framework, investors estimate heterogeneous models, both of which measures a different aspect of the business cycle. In a representative-agent economy, we would like to study how the agent picks dynamically one model over the other depending on economic conditions. We believe that such endogenous "paradigm shifts" can explain several empirical facts regarding the dynamics of stock returns volatility.

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A Appendix

A.1 Proof of Proposition 1

We follow the notations in Lipster and Shiryaev (2001) and write the observable process as

$$\frac{\mathrm{d}\delta_t}{\delta_t} = \left(A_0 + A_1 f_t^A\right)\mathrm{d}t + A_1 \mathrm{d}W_t^f + A_2 \mathrm{d}W_t^A$$

and the unobservable process as

$$\mathrm{d}f_t^A = \left(a_0 + a_1 f_t^A\right)\mathrm{d}t + b_1 \mathrm{d}W_t^f + b_2 \mathrm{d}W_t^A.$$

Using the SDEs in (1) and (2), we write $A \circ A = \sigma_f^\delta$, $b \circ b = \sigma_f^2$ and $b \circ A = 0$. Applying Theorem 12.7 in Lipster and Shiryaev (2001), the dynamics of the filter satisfy

$$\mathrm{d}\widehat{f}_t^A = \left(a_0 + a_1 \widehat{f}_t^A\right)\mathrm{d}t + \left(A \circ A + \gamma A_1^\top\right)(A \circ A)^{-1}\left(\frac{\mathrm{d}\delta_t}{\delta_t} - \left(A_0 + A_1 \widehat{f}_t^A\right)\mathrm{d}t\right)$$

where the steady-state variance γ solves the algebraic equation

$$a_1\gamma + \gamma a_1^\top + b \circ b - \left(b \circ A + \gamma A_1^\top\right)(A \circ A)^{-1}\left(b \circ A + \gamma A_1^\top\right)^\top = 0.$$

Substituting the coefficients, we obtain (4), the steady-state variance and

$$\mathrm{d}\widehat{W}_t^A = \frac{1}{\sigma_\delta}\left(\frac{\mathrm{d}\delta_t}{\delta_t} - \widehat{f}_t^A \mathrm{d}t\right).$$

□

A.2 Proof of Proposition 3

Following Duffie (2010), the functions $\alpha(\tau)$ and $\beta(\tau) \equiv (\beta_1(\tau), \beta_2(\tau))$ solve the following system of Ricatti equations

$$\beta'(\tau) = K_1^\top \beta(\tau) + \frac{1}{2}\beta(\tau)^\top H_1 \beta(\tau)$$
$$\alpha'(\tau) = K_0^\top \beta(\tau) + \frac{1}{2}\beta(\tau)^\top H_0 \beta(\tau)$$

with boundary conditions $\beta(0) = (1-\alpha, 0)$ and $\alpha(0) = 0$. The H and K matrices satisfy

$$K_0 = \begin{pmatrix} -\frac{1}{2}\sigma_\delta^2 \\ \kappa\bar{f} \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 0 & 1 \\ 0 & -\kappa \end{pmatrix}$$

$$H_0 = \begin{pmatrix} \sigma_\delta^2 & \gamma \\ \gamma & \left(\frac{\gamma}{\sigma_\delta}\right)^2 \end{pmatrix}, \qquad H_1 = 0_2 \otimes 0_2.$$

The solutions to this system are

$$\begin{aligned}
\beta_1(\tau) =& 1-\alpha \\
\beta_2(\tau) =& -\frac{(\alpha-1)e^{-\kappa\tau}\left(e^{\kappa\tau}-1\right)}{\kappa} \\
\alpha(\tau) =& \frac{(\alpha-1)^2\gamma^2 e^{-2\kappa\tau}\left(e^{2\kappa\tau}(2\kappa\tau-3)+4e^{\kappa\tau}-1\right)}{4\kappa^3\sigma_\delta^2} \\
& -\frac{(\alpha-1)e^{-2\kappa\tau}\left(4\kappa\sigma_\delta^2 e^{\kappa\tau}\left(e^{\kappa\tau}(\kappa\tau-1)+1\right)\left(\kappa\bar{f}-\alpha\gamma+\gamma\right)+2\alpha\kappa^3\tau\sigma_\delta^4 e^{2\kappa\tau}\right)}{4\kappa^3\sigma_\delta^2}.
\end{aligned}$$

□

A.3 Derivation of the PDE Satisfied by the Interior Expectation Terms

We introduce two sequential changes of probability measure, one from $\mathbb{P}^A$ to a probability measure $\overline{\mathbb{P}}$ according to

$$\left.\frac{\mathrm{d}\overline{\mathbb{P}}}{\mathrm{d}\mathbb{P}^A}\right|_{\mathscr{F}_t} \equiv e^{-\frac{1}{2}\int_0^t\left(\frac{j}{\alpha}\frac{g_s}{\sigma_\delta}\right)^2\mathrm{d}s-\int_0^t\frac{j}{\alpha}\frac{g_s}{\sigma_\delta}\mathrm{d}\widehat{W}_s^A}$$

and one from $\overline{\mathbb{P}}$ to a probability measure $\widetilde{\mathbb{P}}$ according to

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\overline{\mathbb{P}}}\right|_{\mathscr{F}_t} \equiv e^{-\frac{1}{2}\int_0^t(1-\alpha)^2\sigma_\delta^2\mathrm{d}s+\int_0^t(1-\alpha)\sigma_\delta\mathrm{d}\overline{W}(s)}$$

where, by Girsanov's Theorem, $\overline{W}$ is a $\overline{\mathbb{P}}-$Brownian motion satisfying

$$\overline{W}_t = \widehat{W}_t^A + \int_0^t \frac{j}{\alpha}\frac{g_s}{\sigma_\delta}\mathrm{d}s \tag{28}$$

and $\widetilde{W}$ is a $\widetilde{\mathbb{P}}-$Brownian motion satisfying

$$\widetilde{W}_t = \overline{W}_t - \int_0^t (1-\alpha)\,\sigma_\delta\mathrm{d}t. \tag{29}$$

Now, implementing sequentially the changes of probability measures in Eqs. (28) and (29) allows us to rewrite the interior expectation in (14) as

$$F^j\left(\widehat{f}_t, g_t\right) = \mathbb{E}^{\widetilde{P}}\left[\left.\int_t^\infty e^{\int_t^u X_s^j\mathrm{d}s}\mathrm{d}u\right|\mathscr{F}_t\right], \tag{30}$$

where

$$X_t^j = -\left(\rho+\frac{1}{2}(1-\alpha)\,\alpha\sigma_\delta^2\right)+\frac{1}{2}\frac{j}{\alpha}\left(\frac{j}{\alpha}-1\right)\frac{g_t^2}{\sigma_\delta^2}-(1-\alpha)\,\frac{j}{\alpha}g_t+(1-\alpha)\,\widehat{f}_t^A.$$

To obtain the partial differential equation that the function F^j has to satisfy, we use the fact that Eq. (30) can be rewritten as follows

$$\begin{aligned} F^j\left(\widehat{f}_t, g_t\right) &= e^{-\int_0^t X_s^j \mathrm{d}s} \mathbb{E}^{\widetilde{P}}\left[\int_t^\infty e^{\int_0^u X_s^j \mathrm{d}s} \mathrm{d}u \middle| \mathscr{F}_t\right] \\ &= e^{-\int_0^t X_s^j \mathrm{d}s}\left(-\int_0^t e^{\int_0^u X_s^j \mathrm{d}s} \mathrm{d}u + \mathbb{E}^{\widetilde{P}}\left[\int_0^\infty e^{\int_0^u X_s^j \mathrm{d}s} \mathrm{d}u \middle| \mathscr{F}_t\right]\right) \\ &\equiv e^{-\int_0^t X_s^j \mathrm{d}s}\left(-\int_0^t e^{\int_0^u X_s^j \mathrm{d}s} \mathrm{d}u + \widetilde{M}_t\right), \end{aligned}$$

where $\widetilde{M}$ is a $\widetilde{\mathbb{P}}-$Martingale. An application of Itô's lemma along with the Martingale Representation Theorem then gives the partial differential equation in (15).[33]

□

A.4 Proof of Proposition 4

We solve the partial differential equation in (15) using a collocation method. In particular, we approximate the functions $F^j(\widehat{f}^A, g)$ for $j = 1, \ldots, \alpha - 1$ as follows:

$$P^j\left(\widehat{f}^A, g\right) = \sum_{i=0}^{n}\sum_{k=0}^{m} a_{i,k}^j T_i\left(\widehat{f}^A\right) T_k\left(g\right) \approx F^j(\widehat{f}^A, g)$$

where T_i is the Chebyshev polynomial of order i. Following Judd (1998), we mesh the roots of the Chebyshev polynomial of order n with those of the Chebyshev polynomial of order m to obtain the interpolation nodes. We then substitute $P^j(\widehat{f}^A, g)$ and its derivatives in Eq. (15), and we evaluate this expression at the interpolation nodes. Since all the boundary conditions are absorbing, this approach directly produces a system of $(n+1)\times(m+1)$ equations with $(n+1)\times(m+1)$ unknowns that we solve numerically. □

A.5 Derivation of the Parameter Values Provided in Table 1

We estimate the discrete-time models in Eqs. (16) and (18) by Maximum Likelihood. We report the estimated parameters, their standard errors, and their statistical significance in Table 3.[34]

We then map the parameters of Table 3 into the associated continuous-time models. Straightforward applications of Itô's lemma show that the dividend stream, δ, and the fundamental perceived

[33] As proved in David (2008), the boundary conditions are absorbing in both the $\widehat{f}^A-$ and the $g-$dimension.
[34] Parameters are all significant at a 99% confidence level.

Parameter	Symbol	Estimate
Variance Dividend Growth	v^{δ}	$4.23 \times 10^{-5***}$
		(1.48×10^{-6})
Persistence Growth Rate f^A	a^{f^A}	0.9842^{***}
		(9.39×10^{-5})
Mean Growth Rate f^A	m^{f^A}	$9.96 \times 10^{-4***}$
		(0.0022)
Variance Growth Rate f^A	v^{f^A}	$2.53 \times 10^{-6***}$
		(2.12×10^{-7})
High State of f^B	s^h	0.0066^{***}
		(0.0003)
Low State of f^B	s^l	-0.0059^{***}
		(0.0003)
Prob. of Staying in High State	p^{hh}	0.9755^{***}
		(0.0912)
Prob. of Staying in Low State	p^{ll}	0.9680^{***}
		(0.0940)

Table 3: Output of the maximum-likelihood estimation

Parameter values resulting from a discrete-time Bayesian-Learning Maximum-Likelihood estimation. The estimation is performed on monthly data from 01/1871 to 11/2013. Standard errors are reported in brackets and statistical significance at the 10%, 5%, and 1% levels is labeled with */**/***, respectively.

by Agent A, f^A, satisfy

$$\begin{aligned} \log\left(\frac{\delta_{t+\Delta}}{\delta_t}\right) &= \int_t^{t+\Delta}\left(f_u^A - \frac{1}{2}\sigma_\delta^2\right)\mathrm{d}u + \sigma_\delta\left(W_{t+\Delta}^A - W_t^A\right) \\ &= \int_t^{t+\Delta}\left(f_u^B - \frac{1}{2}\sigma_\delta^2\right)\mathrm{d}u + \sigma_\delta\left(W_{t+\Delta}^B - W_t^B\right) \\ &\approx \left(f_t^A - \frac{1}{2}\sigma_\delta^2\right)\Delta + \sigma_\delta\left(W_{t+\Delta}^A - W_t^A\right) && (31) \\ &\approx \left(f_u^B - \frac{1}{2}\sigma_\delta^2\right)\Delta + \sigma_\delta\left(W_{t+\Delta}^B - W_t^B\right) && (32) \end{aligned}$$

$$f_{t+\Delta}^A = e^{-\kappa\Delta} f_t^A + \bar{f}\left(1 - e^{-\kappa\Delta}\right) + \sigma_f \int_t^{t+\Delta} e^{-\kappa(t+\Delta-u)}\mathrm{d}W_u^f. \qquad (33)$$

The relationship between the transition matrix P and the generator matrix Λ is written as follows

$$\begin{aligned} P &= \begin{pmatrix} p^{hh} & 1-p^{hh} \\ 1-p^{ll} & p^{ll} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\psi}{\lambda+\psi} + \frac{\lambda}{\lambda+\psi} e^{-(\lambda+\psi)\Delta} & \frac{\lambda}{\lambda+\psi} - \frac{\lambda}{\lambda+\psi} e^{-(\lambda+\psi)\Delta} \\ \frac{\psi}{\lambda+\psi} - \frac{\psi}{\lambda+\psi} e^{-(\lambda+\psi)\Delta} & \frac{\lambda}{\lambda+\psi} + \frac{\psi}{\lambda+\psi} e^{-(\lambda+\psi)\Delta} \end{pmatrix}. \end{aligned} \tag{34}$$

We perform the Maximum-Likelihood estimation on monthly data and accordingly set $\Delta = \frac{1}{12} =$ 1 month.

Matching Eq. (16) to Eq. (31) and Eq. (17) to Eq. (33) yields the following system of equation for κ, $\bar{f}$, σ_δ, and σ_f

$$\begin{aligned} a^{f^A} &= e^{-\kappa\Delta} \\ m^{f^A} &= \bar{f}\left(1-e^{-\kappa\Delta}\right) - \frac{1}{2}\sigma_\delta^2\Delta \\ v^\delta &= \sigma_\delta^2\Delta \\ v^{f^A} &= \frac{\sigma_f^2}{2\kappa}\left(1-e^{-2\kappa\Delta}\right), \end{aligned} \tag{35}$$

where the last equation relates the variance of the Ornstein-Uhlenbeck process to its empirical counterpart. Matching Eq. (18) to Eq. (32) yields the following system of equations for f^l and f^h

$$\begin{aligned} s^h &= \left(f^h - \frac{1}{2}\sigma_\delta^2\right)\Delta \\ s^l &= \left(f^l - \frac{1}{2}\sigma_\delta^2\right)\Delta. \end{aligned} \tag{36}$$

Solving the system comprised of Eqs. (34), (35), and (36) yields the parameters presented in Table 1.

□

A.6 Proof of Proposition 6

The Malliavin derivative of the stock price satisfies

$$\mathcal{D}_t S_u = \mathbb{E}_u^{\mathbb{P}^A}\left(\int_u^\infty \left[-\frac{\xi_s}{\xi_u^2}\delta_s \mathcal{D}_t\xi_u + \frac{1}{\xi_u}\delta_s\mathcal{D}_t\xi_s + \frac{\xi_s}{\xi_u}\mathcal{D}_t\delta_s\right] \mathrm{d}s\right). \tag{37}$$

The definitions of the state-price density ξ, the dividend δ, the change of measure η, and the filter $\widehat{f}^A$ imply that

$$\mathcal{D}_t\xi_s = -\alpha\frac{\xi_s}{\delta_s}\mathcal{D}_t\delta_s + \frac{\xi_s(1-\omega(\eta_s))}{\eta_s}\mathcal{D}_t\eta_s \tag{38}$$

$$\mathcal{D}_t\delta_s = \delta_s\left(\sigma_\delta + \int_t^s \mathcal{D}_t\widehat{f}_v^A\mathrm{d}v\right) \tag{39}$$

$$\mathcal{D}_t\eta_s = -\frac{\eta_s}{\sigma_\delta}\left(g_t + \int_t^s \mathcal{D}_t g_v \mathrm{d}\widehat{W}_v^A + \frac{1}{\sigma_\delta}\int_t^s g_v\mathcal{D}_t g_v \mathrm{d}v\right) \tag{40}$$

$$\mathcal{D}_t\widehat{f}_s^A = \frac{\gamma}{\sigma_\delta}e^{-\kappa(s-t)}. \tag{41}$$

Substituting Eq. (41) in Eq. (39) yields

$$\frac{\mathcal{D}_t\delta_s}{\delta_s} = \sigma_\delta + \frac{\gamma}{\kappa\sigma_\delta}\left(1-e^{-\kappa(s-t)}\right), \tag{42}$$

while substituting Eqs. (39), (40), and (41) in Eq. (38) yields

$$\frac{\mathcal{D}_t\xi_s}{\xi_s} = -\alpha\left(\sigma_\delta + \frac{\gamma}{\kappa\sigma_\delta}\left(1-e^{-\kappa(s-t)}\right)\right) - \frac{1-\omega(\eta_s)}{\sigma_\delta}\left(g_t + \int_t^s \mathcal{D}_t g_v\mathrm{d}\widehat{W}_v^A + \frac{1}{\sigma_\delta}\int_t^s g_v\mathcal{D}_t g_v\mathrm{d}v\right) \tag{43}$$

Then, substituting Eqs. (42) and (43) in Eq. (37) yields

$$\begin{aligned}
\mathcal{D}_t S_u = \sigma_\delta S_u &+ \frac{\gamma}{\kappa\sigma_\delta}\left(\left(1-\alpha e^{-\kappa(u-t)}\right)S_u + (\alpha-1)\mathbb{E}_u^{\mathbb{P}^A}\left[\int_u^\infty \frac{\xi_s}{\xi_u}\delta_s e^{-\kappa(s-t)}\mathrm{d}s\right]\right) \\
&+ \frac{1}{\sigma_\delta^2}\left(S_u(1-\omega(\eta_u))\int_t^u g_v\mathcal{D}_t g_v\mathrm{d}v - \mathbb{E}_u^{\mathbb{P}^A}\left(\int_u^\infty \frac{\xi_s}{\xi_u}\delta_s(1-\omega(\eta_s))\int_t^s g_v\mathcal{D}_t g_v\mathrm{d}v\mathrm{d}s\right)\right) \\
&+ \frac{1-\omega(\eta_u)}{\sigma_\delta}S_u\left(g_t + \int_t^u \mathcal{D}_t g_v\mathrm{d}\widehat{W}_v^A\right) - \frac{1}{\sigma_\delta}\mathbb{E}_u^{\mathbb{P}^A}\left(\int_u^\infty \frac{\xi_s}{\xi_u}\delta_s(1-\omega(\eta_s))\left(g_t + \int_t^s \mathcal{D}_t g_v\mathrm{d}\widehat{W}_v^A\right)ds\right).
\end{aligned} \tag{44}$$

Taking conditional expectation at time t and setting $g_t = 0$ implies that the last line of Eq. (44) vanishes. That is,

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{P}^A}\left(\mathcal{D}_t S_u\right) = \sigma_\delta\mathbb{E}_t^{\mathbb{P}^A}\left(S_u\right) &+ \frac{\gamma}{\kappa\sigma_\delta}\left(\left(1-\alpha e^{-\kappa(u-t)}\right)\mathbb{E}_t^{\mathbb{P}^A}\left(S_u\right) + (\alpha-1)\mathbb{E}_t^{\mathbb{P}^A}\left[\int_u^\infty \frac{\xi_s}{\xi_u}\delta_s e^{-\kappa(s-t)}\mathrm{d}s\right]\right) \\
&+ \frac{1}{\sigma_\delta^2}\mathbb{E}_t^{\mathbb{P}^A}\left(\int_u^\infty \frac{\xi_s}{\xi_u}\delta_s\left((\omega(\eta_s)-\omega(\eta_u))\int_t^u g_v\mathcal{D}_t g_v\mathrm{d}v - (1-\omega(\eta_s))\int_u^s g_v\mathcal{D}_t g_v\mathrm{d}v\right)\mathrm{d}s\right).
\end{aligned}$$

□

A.7 Proof of Proposition 8

The dynamics of Agent A's wealth satisfy

$$dV_t = r_t^f V_t \mathrm{d}t + \left(\mu_t - r_t^f\right) Q_t S_t \mathrm{d}t - c_{At}\mathrm{d}t + \sigma_t Q_t S_t \mathrm{d}\widehat{W}_t^A, \tag{45}$$

where r^f is the risk-free rate defined in Eq. (10), $\mu - r^f$ is the risk premium on the stock, Q is the number of shares held by Agent A, and σ is the diffusion of stock returns.

Applying Itô's lemma to the stock price yields the diffusion of stock returns, σ, as follows:

$$\sigma = \sigma_\delta + D_{\widehat{f}^A}\frac{\gamma}{\sigma_\delta} + D_g \frac{\gamma - \left(\widehat{f}^A - g - f^l\right)\left(f^h + g - \widehat{f}^A\right)}{\sigma_\delta} - D_\eta \frac{g\eta}{\sigma_\delta},$$

where $D_x \equiv \frac{1}{S}\frac{\partial S}{\partial x}$.

The risk premium, $\mu - r^f$, satisfies

$$\mu - r^f = \sigma\theta,$$

where σ and θ are defined in Eqs. (24) and (11), respectively.

Applying Ito's lemma to Agent A's wealth and matching resulting diffusion term with that in Eq. (45) gives the number of shares held by Agent A, Q, which satisfies

$$Q = \frac{V_\delta \sigma_\delta \delta + V_{\widehat{f}^A}\frac{\gamma}{\sigma_\delta} + V_g \frac{\gamma - \left(\widehat{f}^A - g - f^l\right)\left(f^h + g - \widehat{f}^A\right)}{\sigma_\delta} - V_\eta \frac{g\eta}{\sigma_\delta}}{\sigma S},$$

where $V_x \equiv \frac{\partial V}{\partial x}$. To quantify the profit generated by exploiting exclusively predictability, we break the number of shares into two components, as follows:

$$Q \equiv M + H,$$

where M is the myopic component and H the hedging component. We characterize the myopic and the hedging components using the Martingale Representation Theorem along with the the Clark-Ocone Theorem. Specifically, we define the following martingale

$$\begin{aligned}
\mathrm{d}\left(\xi_t V_t + \int_0^t \xi_s c_{As} ds\right) &= \phi_t \mathrm{d}\widehat{W}_t^A \\
&= \mathbb{E}_t^{\mathbb{P}^A}\left(\int_t^\infty \mathcal{D}_t\left(\xi_s c_{As}\right) \mathrm{d}s\right) \mathrm{d}\widehat{W}_t^A \\
&= \left(\xi_t \sigma_t Q_t S_t - V_t \theta_t \xi_t\right) \mathrm{d}\widehat{W}_t^A,
\end{aligned}$$

where the first and second equalities follow from the Martingale Representation Theorem and the Clark-Ocone Theorem, respectively. Therefore, the number of shares Q satisfies

$$Q_t = \frac{\mu_t - r_t^f}{\sigma_t^2}\frac{V_t}{S_t} + \frac{1}{\xi_t \sigma_t S_t}\mathbb{E}_t^{\mathbb{P}^A}\left(\int_t^\infty \mathcal{D}_t\left(\xi_s c_{As}\right) \mathrm{d}s\right). \tag{46}$$

Using the fact that

$$\begin{aligned}\xi_s c_{As} &= (\phi_A e^{\rho s})^{-1/\alpha}\, \xi_s^{\frac{\alpha-1}{\alpha}} \text{ and thus}\\ \mathcal{D}_t\left(\xi_s c_{As}\right) &= \frac{\alpha-1}{\alpha}\mathcal{D}_t(\xi_s) c_{As},\end{aligned}$$

we can rewrite Eq. (46) as follows

$$\begin{aligned}Q_t &= \frac{\mu_t - r_t^f}{\sigma_t^2}\frac{V_t}{S_t} + \frac{\alpha-1}{\alpha\sigma_t S_t}\mathbb{E}_t^{\mathbb{P}^A}\left(\int_t^\infty \frac{\xi_s c_{As}}{\xi_t}\frac{\mathcal{D}_t\xi_s}{\xi_s}\mathrm{d}s\right)\\ &= \frac{\mu_t - r_t^f}{\sigma_t^2}\frac{V_t}{S_t} + \frac{\alpha-1}{\alpha\sigma_t S_t}\mathbb{E}_t^{\mathbb{P}^A}\left(\int_t^\infty \frac{\xi_s c_{As}}{\xi_t}\left(\frac{\mathcal{D}_t\xi_s}{\xi_s} - \frac{\mathcal{D}_t\xi_t}{\xi_t} + \frac{\mathcal{D}_t\xi_t}{\xi_t}\right)\mathrm{d}s\right)\\ &= \frac{\mu_t - r_t^f}{\sigma_t^2}\frac{V_t}{S_t} + \frac{\alpha-1}{\alpha\sigma_t S_t}\mathbb{E}_t^{\mathbb{P}^A}\left(\int_t^\infty \frac{\xi_s c_{As}}{\xi_t}\left(\frac{\mathcal{D}_t\xi_s}{\xi_s} - \frac{\mathcal{D}_t\xi_t}{\xi_t}\right)\mathrm{d}s\right) - \frac{(\alpha-1)\theta_t V_t}{\alpha\sigma_t S_t}\\ &= \frac{\mu_t - r_t^f}{\alpha\sigma_t^2}\frac{V_t}{S_t} + \frac{\alpha-1}{\alpha\sigma_t S_t}\mathbb{E}_t^{\mathbb{P}^A}\left(\int_t^\infty \frac{\xi_s c_{As}}{\xi_t}\left(\frac{\mathcal{D}_t\xi_s}{\xi_s} - \frac{\mathcal{D}_t\xi_t}{\xi_t}\right)\mathrm{d}s\right)\\ &\equiv M_t + H_t.\end{aligned}$$

□