

# **The Limitations of Diversification Return**

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## **Abstract**

Diversification return is the amount by which the geometric mean return (i.e., average compounded return) of a portfolio exceeds the weighted average of the geometric means of the portfolio's constituent assets. Diversification return has been touted as a source of added return even if markets are informationally efficient. Portfolio rebalancing has been advocated as a valuable source of diversification return. We demonstrate that diversification return is not a source of increased expected value. However, portfolio rebalancing can be an effective mean-reverting strategy. Any enhanced expected value from rebalancing emanates from mean-reversion rather than from diversification or variance reduction.

## **The Limitations of Diversification Return**

Booth and Fama (1992) originated the popular term for the concept known as “diversification return”. However, Fernholz and Shay (1982) previously developed the same general concept using continuous time mathematics and termed the concept “excess return”. Diversification return is a portfolio’s average compound return minus the weighted average of the compound returns on the assets in the portfolio. Booth and Fama claim that this “incremental return is due to diversification”. Numerous others have studied diversification return and the role of portfolio rebalancing in creating diversification return and have generally concluded that diversification return is a valuable source of added return. Absent mean-reversion in the underlying assets, we find no justification for believing that diversification return provides increased expected value.

### **Review of Diversification Return**

We begin with a review of the concept of diversification return as presented in previous studies such as Willenbock (2011) and Qian (2012). Our purpose in this section is not to analyze the strengths or weaknesses of the concept of diversification return, but rather to provide a foundation for analysis and discussion.

For simplicity, consider an asset with returns that are uncorrelated through time and that have two equally likely annual outcomes: +25% and -20%. The arithmetic mean return of this asset is 2.5% and its annual volatility is 22.5% both of which may be found using the two outcomes, the probabilities and the common definitions of expected return and standard deviation. The arithmetic mean of 2.5% indicates that the expected value of the asset at every point in time is expected to be 2.5% higher than its value one year earlier. Despite this positive annual expected growth, the high volatility of the asset causes the geometric mean of the asset,  $g$ , to equal only 0% - which is why this particular return combination, also used by Willenbrock, was selected. The exact geometric mean can be found directly from the annual returns as  $(1.25*0.80)^{0.5}-1$ , and can be approximated using the arithmetic mean return,  $\mu$ , and the population standard deviation of returns,  $\sigma$ , with the following approximation (which omits the higher order terms from a Taylor series approximation):

$$g \approx \mu - (\sigma^2/2)$$

Equation 1

Substituting the asset's arithmetic mean return and standard deviation into Equation 1 confirms that the asset's annual volatility of 22.5% lowers its arithmetic mean return from 2.5% to a geometric mean return of roughly 0%.

Now let's expand this single-asset example to a set of uncorrelated but otherwise identical assets in order to analyze equally weighted (and annually rebalanced) portfolios of assets each with the same parameters as the above asset. For simplicity we assume that all of the assets have returns that are uncorrelated through time and uncorrelated with each other. Exhibit 1 summarizes the returns of portfolios with various numbers of assets. Exhibit 1 depicts the decline in portfolio volatility and the increase in each portfolio's geometric mean obtained by diversifying into larger numbers of equally volatile but uncorrelated assets. In the limit, the portfolio becomes a riskless asset with a fixed return of 2.5%. Therefore, the infinitely diversified portfolio has no volatility and has both an arithmetic mean return and a geometric mean return equal to 2.5%.

Diversification return was defined by Booth and Fama as the difference between the geometric mean return of a portfolio,  $g_p$ , and the weighted average of the geometric returns of the portfolio's constituent assets. With the subscript  $i$  denoting assets within the portfolio and  $w_i$  as the weight of asset  $i$  (based on market values) diversification return may be expressed as:

$$\text{Diversification Return} = g_p - \sum w_i g_i$$

Equation 2

Willenbrock describes the summation on the right hand side of Equation 2 as the *strategic return*. Note that since each of the portfolio's assets in this example has a zero geometric mean return, the strategic return of every possible portfolio of these assets is zero and so in this particular case the diversification return of each portfolio is simply the geometric mean return of that portfolio. As depicted in Exhibit 1, the one-asset portfolio enjoys no diversification and therefore has no diversification return. The infinitely diversified portfolio has a diversification return equal to its arithmetic mean return (2.5%). Exhibit 1 confirms the relationship between diversification and diversification return since the portfolios with more assets have lower volatilities, greater diversification, higher geometric means and higher diversification returns.

Equation 3 depicts diversification return using Willenbrock's term "strategic return" to represent the weighted average of the geometric returns of the portfolio's constituent assets.

$$\text{Diversification Return} = \text{Geometric Mean Return} - \text{Strategic Return} \quad \text{Equation 3}$$

The strategic return is a key concept in diversification return but it has unclear economic meaning. It can be viewed as the geometric mean return of a rebalanced portfolio containing a set of hypothetical riskless (zero volatility) assets that have the same geometric returns as the actual assets in the portfolio. The diversification return then might be argued to serve as a measure of the added geometric return that diversification and/or rebalancing can generate through the reduction of risk caused by assembling imperfectly correlated risky assets into a portfolio.

It should be noted that the strategic return cannot be obtained through any portfolio rebalancing of the underlying assets unless both of the assets have zero return volatility. Brennan and Schwartz (1985) disproved the common misconception that a continuously balanced portfolio of risky assets earns a geometric return equal to the value-weighted average of the geometric mean returns of the portfolio's assets. In studies regarding diversification return the strategic return seems to be used as a hypothetical benchmark for evaluating the actual geometric mean returns of portfolios. Return above this "benchmark" is the diversification return and it is interpreted without formal proof as an enhanced return attributable to diversification and/or rebalancing.

### **Claims Regarding the Efficacy of Diversification Return**

This section summarizes the rationale and purported benefits of diversification return. Bouchy et al. (2012) refer to the gains from diversification return as "volatility harvesting" and claim that it is "the extra growth generated from systematically diversifying and rebalancing a portfolio". This extra growth is said to be available whenever assets have "volatilities greater than zero and correlations less than one". Diversifying and rebalancing are said to "enhance returns in the long run". Bouchy et al. demonstrate diversification return using serially correlated data and they note the ability of negative autocorrelations (i.e., mean-reverting returns) to enhance the gains. However, they do not limit the benefits of diversification return to mean-reverting assets and conclude "The principles presented here are mathematical in nature and apply to any set of sufficiently liquid investments that are volatile and uncorrelated". They add that volatility is "an opportunity that can be exploited through rebalancing. Just as it is possible to harness energy

from waves in the ocean, it is possible to harvest return from volatility in the market.” Erb and Harvey (2006) refer to diversification as “Turning Water into Wine” and add “Where does this incremental return come from? From variance reduction.” They claim that diversification return may be enhanced by mean-reversion but does not depend on mean-reversion.

Willenbrock (2011) provides clear definitions and demonstrations of diversification return. He notes that while diversification is often described as a free lunch, he concludes that “diversification return might be described as the only free dessert in finance because it is the incremental return earned while maintaining a constant risk profile.” Willenbrock provides an analysis of the role of rebalancing in causing commodity price indices to generate different returns and relates the analysis to diversification return.

Qian’s (2012) primary contribution is to demonstrate the effects of leverage and lending on diversification return. Qian describes portfolio balancing as “the simplest and clearest technique that with few exceptions adds incremental value to fixed-weight multi-asset portfolios”. Qian notes that “Portfolio rebalancing is essential for harvesting diversification return” and that the two (rebalancing and diversification) are “inseparable”.

The remaining sections of this paper analyze the purported benefits of diversification return. That analysis begins with subsections that lay important foundational material with regard to the underlying issues of diversification return including geometric mean returns, volatility, diversification, and determinants of expected long term growth.

### **The Geometric Mean Return**

Since the diversification return of a portfolio is formed by subtracting the weighted average of the geometric mean returns of a portfolio’s assets from the geometric mean return of the portfolio, a clear understanding of diversification return requires a clear understanding of the strengths and weaknesses of the geometric mean return as a performance metric.

A correct interpretation of the geometric mean return is that it is an average of multi-period compounded *rates*. A key misconception with regard to the expected geometric mean return is that it provides an accurate indication of long term expected future *wealth*.

The *realized* geometric mean return of an asset from period 0 to period T is the average compounded return that discounts the value of the asset at time T to its value at time 0 (assuming there are no intervening cash flows such as dividends). The realized geometric mean return correctly *ranks* assets relative to their total percentage growth. Thus if asset A experiences a higher realized geometric mean return than asset B, the total (i.e., non-annualized) percentage growth in asset A's value will exceed that of asset B over the same time period.

The geometric mean return is a concave transformation of an asset's total percentage growth (i.e., its non-annualized ratio of the change in its value to its initial value). Differences in realized geometric mean returns can have profoundly different effects on total growth due to this concavity. If asset A has a realized geometric mean return that is positive and is two times the geometric mean of asset B, then the total percentage growth in the value of asset A will exceed the total percentage growth in asset B by more than two times. The exact degree by which asset A's total percentage growth will exceed asset B's growth depends on the time interval involved. This concavity raises problems when the concept of *expected* geometric mean returns is used.

The distinction between the use of a *realized* geometric mean return to measure realized growth and the use of the *expected* geometric mean return to measure expected growth is crucial. The expected geometric mean return (i.e., the expected compounded rate of return) of an asset is the probability weighted average of all of the potential realized geometric mean returns. While realized geometric mean returns correctly rank realized total returns, the expected geometric mean return does not correctly rank the expected total returns.<sup>1</sup> Another potential misconception regarding geometric mean returns is that maximization of the expected geometric mean return of a portfolio is an optimal portfolio strategy.<sup>2</sup>

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<sup>1</sup> To prove this point consider a risky asset with expected return  $E(r_m)$ . Assuming serially uncorrelated returns, the asset's expected total (non-annualized) return over T periods is  $[1 + E(r_m)]^T - 1$ . Specifically,  $E[\prod(1 + r_{m,t})] = [1 + E(r_m)]^T$  since the expected values of each cross-product,  $E(r_{m,t} \cdot r_{m,t-k})$ , is zero. Thus all serially uncorrelated risky assets have multi-period expected total non-annualized returns directly related to their single period expected returns and unrelated to their volatility. In other words, the single period arithmetic mean return of serially uncorrelated assets correctly ranks the expected non-annualized total return of assets but the geometric mean return does not because geometric means depend on volatility.

<sup>2</sup> Latane's (1959) pioneering work on using geometric mean return maximization as a portfolio optimization criterion merely claims that the strategy "falls within the generally accepted range of rational behavior" and that it is "a useful criterion". Samuelson (1971) used a gambling analogy to criticize the optimization of the geometric mean return as a criterion for choice amongst risky ventures in his paper entitled "The 'Fallacy' of Maximizing the Geometric Mean in Long Sequences of Investing or Gambling". As Samuelson (1971) notes: "The novel criterion of maximizing the expected average compound return, which asymptotically leads to maximizing the geometric mean, is shown to be arbitrary." A focus on

In summary, the geometric mean return is a concave transformation of wealth ratios. The expected geometric mean return is a poor indicator of expected long term growth when growth is viewed in terms of total non-annualized returns rather than annualized rates of return. Differences in volatility cause differences in geometric mean returns but differences in volatility do not cause differences in expected long term growth. This point is essential to correcting misconceptions that arise with the analysis of diversification return and is the subject of the next section.

### **Volatility Does Not Diminish Expected Wealth**

One of the most widespread misconceptions related to the analysis of diversification returns and geometric mean returns is that return volatility diminishes expected long term growth in wealth. For example, it is often incorrectly believed that an asset that has equally likely +10% returns and -10% returns will have diminished long term growth compared to an asset with the same arithmetic mean return but with less volatility. The justification focuses on the idea that making 10% one year and losing 10% the next year (or vice versa) results in a 1% loss. The asset's profits when it earns 10% *per year* during consecutive years of gains appears to be offset by the equally likely losses suffered when the asset loses 10% *per year* during consecutive losses. Thus the asset appears to lose value in the long run due to the periods of alternating gains and losses, and it appears that higher volatility hastens the decline in expected value.

However, assuming that the asset's returns are not serially correlated, the asset's expected value does not decline through time. Each time that the asset experiences two periods of 10% growth there is a profit of 21%. Each time the asset experiences two periods of 10% decline there is a loss of only 19%. It is often overlooked that the periods of consecutive 10% growth generate compounded gains of 21% that fully offset the expected losses from the other three equally likely paths (19%+1%+1%). When viewed in terms of wealth rather than compounded rates, it is clear that the asset's expected value does not diminish. The return dispersion does not create or destroy expected wealth or expected growth. The arithmetic mean correctly predicts zero expected growth. While the expected geometric mean return is a valid indicator of expected compounded *rates*, it is not an accurate indicator of non-annualized measures of the changes in expected *wealth*. And it is wealth that people use to purchase goods, not compound rates of return.

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geometric mean return may lead to a potentially useful tradeoff between risk and return, but it is an arbitrary tradeoff.

As an expected compounded rate, an expected geometric mean creates an illusion that volatility diminishes expected growth (see Equation 1) because volatility is subtracted from the arithmetic mean to form the geometric mean return and the realized geometric mean return is associated with long term growth. But return volatility does not diminish (or increase) short term or long term expected growth. The corrected statement is that return volatility diminishes expected *rates* of growth when the rate of growth is measured as a *compounded rate*. The difference is important because average compound annual rates of growth can be deceptive and it is that deception that has caused the misinterpretation of diversification return as being a form of added value to a portfolio. When expected long term growth in a portfolio's value is analyzed properly with total growth percentages or arithmetic mean returns, the illusion of diversification return vanishes.

Willenbrock (2011) asserts the opposite by criticizing average arithmetic returns: "Given the misleading nature of the arithmetic average return, it generally cannot be used by itself to judge the performance of an asset or portfolio." The issue is complex and can be confusing especially when the analyses alternate between a focus on realized mean returns and expected returns and when the term "expected growth" is used interchangeably with "expected annualized growth rate". An asset that offers the highest expected arithmetic return offers the highest expected long term non-annualized growth.

To illustrate the deception of expected annualized rates in making decisions regarding expected long term growth, consider the following example:

*Suppose that a bank offers zero coupon insured certificates of deposit (CDs) that accrue interest at a competitive rate that is payable at maturity in 18 years. Thus a \$10,000 CD offering a yield of 4% would mature in eighteen years with a payoff of approximately \$20,000. Now let's suppose that the bank decides to offer its CD owners a gamble. Based on the flip of a fair coin the bank will go "double or nothing" on the CD's yield. Thus the bank will either add 4% to the original 4% yield or take away 4% depending on the outcome of the flip. Although from an annualized rate perspective the gamble seems even (4% equals the average of 8% and 0%) from a dollar and cents perspective the gamble very much favors the investor. If the interest rate is doubled to 8% the final payoff is roughly \$40,000. If the interest rate is cut to zero the payoff is \$10,000. The expected payoff of the CD rises from \$20,000 to roughly \$25,000 if the gamble is accepted even though the expected yield is unchanged (i.e., the expected yield is equal to the original*

*yield). The point is that expected annualized or compounded rates of growth can be misleading indicators of non-annualized expected growth amounts.*<sup>3</sup>

The distortion caused by expected compounded growth rates can be further illustrated by returning to the previously described asset that offers 50%/50% chances each period of either rising by 25% or falling by 20%. As previously discussed, this asset has a one period arithmetic mean return of +2.5% and a geometric mean return of 0%. Exhibit 2 contains a four period tree of prices and an analogous four period tree of annualized compounded rates of returns for this asset assuming a \$1 starting value and zero serial return correlation. Underneath the top tree are the probability weighted averages of each period's *price* along with the growth rate corresponding to that average price through time. Underneath the bottom tree are the probability weighted averages of each period's compound growth *rates*.

Exhibit 2 demonstrates that volatility does not diminish expected value. The top tree in Exhibit 2 uses dollar values and shows steady annual growth in the asset's expected value equal to 2.5% compounded each year. The bottom tree uses compounded rates of return and shows slowing growth in averaged annualized rates over the four periods. It is a mistake to infer that volatility is diminishing the asset's expected growth – it is only diminishing the mean annualized rates. Simply put, the growth in dollar value reflects the true and steady growth in the expected value of the asset through time at its arithmetic mean growth rate of 2.5%. However, the average compounded rates of return create an illusion of slower growth due to volatility. The expected compounded rates of return start in the first period at 2.5% but decline after the first year to 0.6% by the end of the fourth year. In the long run the expected compounded growth rate would approach the asset's geometric mean return of 0%. A focus on average compounded rates leads to the misconception that the asset does not offer any expected long term growth. But the top tree containing the possible prices shows the expected growth. The illusion of reduced growth due to volatility results from the concavity of expected annualized rates of return as a function of expected wealth changes. Just as in the previous CD example, average annualized rates are a poor indicator of expected wealth.

The false interpretation of expected geometric mean returns as indicators of expected long term total growth is not limited to analyses of diversification return. The deception caused by a focus

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<sup>3</sup> Note further that any continuously rebalanced portfolio has an infinitely negative continuously compounded geometric mean return if any of its assets have a probability of becoming worthless.

on expected compound rates is well established as the fallacy of time diversification by Bodie(1995). Bodie builds on work by Samuelson (1971) in showing that time diversification in a serially uncorrelated market is a fallacy that is caused by a focus on dispersion of compounded rates rather than on the dispersion of wealth. Note that the dispersion of the annual growth *rates* in Exhibit 2 diminishes through time. After one period the asset's growth rate (25% or -20%) differs from its expected value by 22.5% with 100% probability. But after ten periods there would only be a 1 in 1,024 chance that the asset's annualized growth rate would exceed the expected growth rate by 22.5% and the same probability that it would be 22.5% under its expected growth rate. But remaining in a risky asset for longer periods of time does not diminish risk; it only diminishes the volatility of the annualized rates of return.

The average compounded growth rates in Exhibit 2 decline through time due to the muting of the magnitude of very large wealth increases through their inclusion as relatively moderate compounded rates of return. The potential dollar values in Exhibit 2 clearly indicate that the asset's value is becoming *more* dispersed through time. Note that the maximum spread between values is \$0.45 after one year and over \$2 by the end of the fourth year. But the dispersion in the potential realized rates of return is diminishing. The concave transformation of large wealth increases into relatively modest annualized rates of return diminishes the average of the annualized rates and gives the false impression that volatility reduces expected future wealth.

### **Diversification Return and Asset Volatility**

Advocates of diversification return tend to claim that diversification return emanates from reduced volatility and that diversification is the source of the reduced volatility that generates diversification return. This section demonstrates that diversification is not necessary to generate diversification return by investigating the two-asset case of combining a riskless asset and a risky asset. For simplicity the riskless asset is assumed to have a return of 0% and the risky asset continues to be the previous case of an asset that has a 50%/50% chance of rising by 25% or falling by 20% each period. Both the riskless and risky assets have geometric means of 0% and so the strategic returns of all combinations of the two assets are also equal to 0% and the diversification return equals the geometric mean. Exhibit 3 summarizes results for five asset allocations ranging from 0% in the risky asset to 100% in the risky asset. Given that the riskless asset has an arithmetic mean return of zero, the arithmetic mean return of each portfolio is simply  $w$  times 2.5% where  $w$  is the proportion of the portfolio invested in the risky asset. Similarly, the

standard deviation of the rebalanced portfolio is  $w$  times the standard deviation of the risky asset (22.5%).

As would be expected, there is no diversification and therefore no diversification return when  $w$  is equal to either zero or one since in both cases the portfolio contains only one asset. But the most interesting result of Exhibit 3 is that diversification return emerges in the other three cases even though there is no diversification taking place since it is a riskless asset that is being combined with a risky asset. This result is shown by Qian (2012) who also notes that the diversification return becomes negative when  $w$  is less than zero (shorting the risky asset) or greater than one (leveraging the risky asset). The analysis in Exhibit 3 demonstrates that diversification return does not require that a portfolio offset idiosyncratic risks by combining imperfectly correlated assets. Rather, diversification return results simply from dampening the dispersion in a particular measure of performance: multi-period compounded rates of return. The diversification return available in Exhibit 3 raises serious questions as to whether it should be called diversification return since it does not emanate from diversification as defined in the traditional sense.

The diversification return in this case of a two-asset portfolio of one riskless and one risky asset is due to risk-dampening from rebalancing. As noted by Fernholz and Shay (1982), when the risky asset experiences a positive return (an uptick) a portion of the growth is rebalanced from the risky asset to the riskless asset. When the risky asset experiences a negative return (a downtick) a portion of the balance in the riskless asset is transferred to the risky asset. This rebalancing lowers the most extreme outcomes and lowers dispersion. It is the reduced volatility of rates, not diversification as traditionally defined, that generates diversification return.

But a closer look indicates that the phenomenon is arbitrary. Because this case involves combining a riskless asset with a risky asset and because both assets have a geometric mean return equal to zero, the formula for the diversification return of the portfolio reduces to a simple formula for the geometric mean return of the portfolio.<sup>4</sup> Substituting and solving with Equations 1, 2, and 3 with  $w$  as the weight of the risky asset and using the arithmetic mean (.025) and variance (.225<sup>2</sup>) from the example generates Equation 4 for the diversification return:

$$\text{Diversification return} = \mu - (\sigma^2/2) = .025 w - .225^2 w^2/2 \quad \text{Equation 4}$$

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<sup>4</sup> Fernholz and Shay (1982) derive the same relationship (their equation number 20) in concluding that rebalancing “produces a constant accrual of revenues” that would “be absent in a passive portfolio.

Equation 4 shows that the arithmetic return of the risky asset adds to the diversification return and the variance of the risky asset lowers the diversification return. As the portfolio weight of the risky asset,  $w$ , increases, the diversification return of the portfolio increases linearly due to the 2.5% arithmetic mean of the risky asset. But increases in  $w$  have a quadratic relationship in penalizing the diversification return for the variance of the risky asset. As indicated in Exhibit 3 the diversification return (i.e., the geometric mean return) rises, reaches a maximum and then declines as the weight of the risky asset moves from zero to one.

Note that higher values of  $w$  always cause higher expected wealth but that diversification return reaches a maximum at  $w = \mu / \sigma^2$  and becomes negative beyond  $w = 2\mu / \sigma^2$ . The diversification return in Equation 4 is nothing more than a manifestation of the difference between the arithmetic mean returns of the two assets and the reduction in the geometric mean caused by the higher volatility of assigning a higher portfolio weight to the risky asset. There is no reason to believe that the magnitude of the diversification return has any relevant economic meaning.

Diversification return simply reflects the downward bias in a geometric mean return attributable to the concave relationship between prices and compounded yields. Diversification return does not generate higher expected wealth; it only reflects the concavity of the wealth transformation.

### **Diversification Return, Portfolio Rebalancing and Serial Correlation**

The effect of portfolio rebalancing on diversification return is an issue of importance. The effect of rebalancing on expected portfolio growth depends on the autocorrelation or serial correlation of the underlying assets. One substantial point of agreement amongst commentators on diversification return is that the active management process of rebalancing portfolios back towards some fixed weights is a mean-reverting strategy. Rebalancing generates higher expected portfolio value when asset prices mean-revert and lower expected portfolio value when asset prices trend. The explanation is simple: rebalancing a portfolio to fixed weights involves selling assets that have experienced superior returns and buying assets that have experienced inferior returns. If the returns are mean-reverting (i.e., exhibit negative serial correlation) then rebalancing sells prior to relatively poor returns and buys prior to relatively high returns.

Advocates of the benefits of portfolio rebalancing in the context of diversification return do not require that asset returns be mean-reverting – they simply note that mean-reversion enhances diversification return. Willenbrock states “...the underlying source of diversification return is contained in the rebalancing. Rebalancing a portfolio involves selling assets that have appreciated in relative value and buying assets that have declined in relative value, as measured by their weights in the portfolio. This contrarian activity generates incremental returns as the assets fluctuate in value.”

Exhibit 4 refutes the claim that portfolio rebalancing adds expected growth in wealth when the underlying assets are serially uncorrelated. We return to the case of assets experiencing two equally likely returns each year: +25% or -20%. Consider an equally-weighted portfolio of two such assets with zero correlation that begins with \$1 and annually rebalances. Exhibit 4 depicts the 16 possible states over two periods for both a buy-and-hold strategy and an annually rebalanced strategy. The results are shown both in dollar values and compounded rates of return. Note that the average terminal value of both strategies is the same: \$1.0506. This expected value reflects the arithmetic mean growth rate of 2.5% for each asset compounded for two periods.

The average annualized realized rate of return is 1.874% for the rebalancing strategy and 1.867% for the buy-and-hold strategy. In the parlance of diversification return the “added return” (which is really just an illusion from averaging rates) is from rebalancing. Exhibit 4 marks with double and triple asterisks the four of the sixteen states that have different returns for the two strategies. The four states with different returns occur whenever the assets have different returns in both time periods. The difference in returns during the first time period causes the weights to differ going into the second period. The difference in the returns during the second time period causes the performance of the portfolios to differ. Note that the dollar differences between the strategies in each of the four states are equal to \$0.0506. However, the \$0.0506 differences form different percentages. The rebalanced strategy earns the incremental \$0.0506 when its dollar base is smaller (\$1.00) which causes the profit to be +2.50%. The buy-and-hold strategy earns the incremental \$0.0506 when its dollar base is higher (\$1.1013) which causes the profit to be only +2.44%. Exhibit 4 demonstrates the equality of the expected values of rebalancing and buy-and-hold strategies with serially uncorrelated returns. Exhibit 4 also demonstrates that the expected geometric mean returns of the strategies differ not because of higher expected values but rather from the averaging of rates. Diversification return is nothing more than a mirage in which the

same dollar benefits appear larger because they are calculated over different bases and then averaged.

Exhibit 4 illustrates an important point: an asset with a relatively high geometric mean return cannot be successfully arbitrated against an asset with a relatively low geometric mean returns if both assets have the same arithmetic mean returns. An arbitrageur that is long the rebalanced portfolio in Exhibit 4 and is short the buy-and-hold portfolio experiences two paths (marked \*\*\*) of \$0.0506 profit and two paths (marked \*\*) of \$0.0506 losses. There is zero expected profit to an arbitrageur from being long the higher average geometric mean strategy (rebalancing) and being short the lower average geometric mean strategy (buy and hold).

Exhibit 5 expands the same model used in Exhibit 4 by providing time horizons that vary from 1 period to 12 periods<sup>5</sup> (and omitting the listing of the detailed outcomes). The expected value of the portfolio for each strategy grows at the same fixed compound growth rate of 2.5% per year. However, the geometric mean returns decline from the first period value of 2.50% when the geometric mean return equals the arithmetic mean return towards 0.0% at the infinite time horizon (recall that the risky assets each have long term geometric mean returns equal to 0.0%).

Exhibit 5 displays means and volatilities of the annualized rates of return for both rebalancing and buy-and-hold strategies. First, note that the rebalanced strategy has higher expected geometric means for all time horizons beyond one year. Second, note that the rebalanced strategy has smaller dispersion in the realized growth rates. Rebalancing reduces wealth dispersion. Reduced wealth dispersion increases geometric mean returns. However, the expected value of each portfolio grows at the same compound rate of 2.5%.

Exhibits 4 and 5 explore the roll of rebalancing using hypothetical data in which serial correlation is set to zero. Booth and Fama, Willenbrock, Bouchy et al. and Qian demonstrate that portfolio rebalancing generally resulted in improved geometric mean returns using actual market returns from various time periods, markets and asset allocation levels. Part of these results can be explained by the illusion of geometric returns, but most is attributable to mean-reversion in the underlying market data and even in the hypothetical examples that are used. Consistently higher risk-adjusted growth in value from portfolio rebalancing in practice requires mean-reversion.

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<sup>5</sup> The time horizon was not extended beyond 12 years because a four path tree with non-recombining nodes reaches 16,777,216 nodes after only 12 periods.

Bouchy et al. reported that rebalancing worked for a two stock portfolio (Apple and Starbucks) over the interval 1994-2011, but they noted that the strategy underperformed over the first four years when apparently the returns were trending. Nevertheless, the reported robustness of the rebalancing strategy over various studies using a variety of assets and time intervals based on actual market data is impressive. These empirical studies attest to the effectiveness of rebalancing in markets such as equities when mean-reversion is prevailing, but err in attributing the source of enhanced value to diversification or diversification return.

Diversification return advocates argue that rebalancing creates diversification return through maintaining better diversification than is obtained using a buy-and-hold strategy. But rebalancing does not inherently keep a portfolio better diversified. Whether rebalancing a portfolio towards original weights increases or decreases diversification depends on the definition of diversification and on the original weights. In most equilibrium capital market theories the most diversified portfolio is the market portfolio since it contains no idiosyncratic risk. Generally, a market portfolio needs little or no rebalancing through time because the weights naturally remain market weights (in the absence of differential dividend yields, share repurchases or new offerings). Rebalancing a portfolio that began with market-weights to maintain fixed weights (fixed at their original values) would tend to move the portfolio away from the market weights and therefore cause the portfolio to be less well diversified according to most theories of diversification.

It is not clear that a portfolio is made more or less diversified by rebalancing. Rebalancing restores the original weights after market forces drive some weights higher and some lower but who is to say that the original weights provide better or worse diversification than the new weights? If a small company with a very small original weight soars in value, it is reasonable to believe that the portfolio can be better diversified by allowing its weight to increase. Similarly, allowing the very large weight of a large firm to fall when its value has fallen would likely improve diversification relative to rebalancing. Thus, rebalancing can improve diversification in some cases and can increase idiosyncratic risk in other cases.

A more clear description of the effect of rebalancing is to describe the effect on return as “rebalancing return” and to note that rebalancing return should generally be positive when asset prices are mean-reverting and negative when asset prices are trending.

## Summary and Conclusions

The expected compound rate of return is a misunderstood measure of performance because it focuses on rates and creates an illusion that volatility “punishes” expected growth (not just growth rates). Bodie debunked the illusion of time diversification created by the use of averaged annual rates. Without serial correlation, it is clear that holding a risky asset for a longer period of time increases the dollar risk even though it provides an illusion of reduced risk when the risk is measured by geometric mean returns. Our criticism of “diversification return” follows the same logic and criticizes the same type of illusory analysis. It is through the distorted lens of geometric mean analysis that reduced volatility of and by itself can be interpreted as generating higher expected portfolio value.

The illusion that diversification return generates incremental expected wealth gains comes from comparing a portfolio’s realized or expected geometric mean return to a flawed and meaningless benchmark – the weighted average of the geometric means of the portfolio’s assets.

The primary conclusions of our analysis are threefold:

1. It is true that portfolio rebalancing tends to generate higher geometric mean returns (i.e., diversification return) even when returns are serially uncorrelated. But the higher geometric mean returns do not cause higher expected portfolio values. Expected portfolio values are governed by arithmetic means, not geometric means or volatility.
2. Portfolio rebalancing tends to increase the expected value of a portfolio when asset prices are mean-reverting. This enhanced growth emanates from applying a mean-reverting strategy (i.e., rebalancing) to prices that are mean-reverting. The added expected portfolio value is not attributable to either reduced volatility or increased diversification.
3. The higher expected geometric mean of a low volatility portfolio cannot be arbitrated against a high volatility portfolio when both portfolios have the same arithmetic mean returns and when prices are Markov. Rebalancing generates arbitrage opportunities only when prices are mean-reverting.

The divergence of opinion with regard to the efficacy of diversification return originates from the difference between applying a rate-focused view and a value-focused view of expected growth. In other words, the controversy relates to whether an investor should be more concerned about expected long term growth expressed as an expected annualized rate or expressed as an expected total percentage change in value. The former is an arbitrary<sup>6</sup> nonlinear transformation of wealth, the latter is not.

Consider the following approximation previously detailed:  $g \approx \mu - (\sigma^2/2)$ .

A rate-based approach views the geometric mean return,  $g$ , as the best measure of expected long term growth and views the arithmetic mean,  $\mu$ , as ignoring volatility. A value-based approach views the arithmetic mean return,  $\mu$ , as the best measure of expected long term growth and views the geometric mean,  $g$ , as being distorted downward by volatility.

Elton and Gruber (1974) carefully refute the optimality of a rate-based approach to selecting portfolios and conclude: “portfolio decisions based on...the geometric mean of multiperiod returns are often...inferior to decisions based on consideration of returns sequentially over time...even when the distribution of returns is expected to be identical in each future period”. Simply put, wealth or some *linear* transformation of wealth serves as a better argument for a utility function than a geometric mean.

In the rate-based view of diversification return, by holding the arithmetic mean constant and reducing the volatility of realized rates through portfolio rebalancing, an investor can increase the geometric mean return of a portfolio and therefore can create added averaged return through rebalancing. In the value-based view of diversification return the arithmetic mean governs the expected value and volatility only plays a meaningful role in the context of risk-aversion.

To ascertain whether a rate-based or value based approach is better merely requires returning to the example of the bank offering eighteen-year CDs that offer either a guaranteed yield of 4% or a 50%-50% chance of 0% and 8% yields. The rate-based view sees the equal chances of a 0% and 8% yield as having the same expected realized rates of return (i.e., geometric mean return). The value-based view sees the equal chances of a 0% and 8% yield as having a much higher expected value (roughly \$25,000) than the certain 4% yield (\$20,000). As advocates of the value-based

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<sup>6</sup> The magnitude of the effect is driven by the time it takes the planet to orbit the sun.

view we believe this can be arbitrated and will offer to borrow from anyone at a long-term fixed rate of return of 4% if they will allow us to lend to them at a 50%-50% chance of receiving long term compounded rates of either 0% or 8%.<sup>7</sup>

The CD example provides a clear analogy and a clear answer: expected annualized rates are a deceptive measure of expected long term growth. The rate-based approach is flawed and although diversification return indicates increased expected annualized rates of return it does not indicate increased expected value. However, portfolio rebalancing can serve as an effective mean-reverting strategy. When underlying returns are mean-reverting, rebalancing offers a “free dessert”. It does so through allocating away from previously-high performing assets towards previously-low performing assets – not through diversification or volatility reduction. The expected gains of rebalancing mean-reverting assets come from the expected losses of other traders who are implementing trending strategies, not from “turning water into wine”.

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<sup>7</sup> Based on a \$10,000 initial principal amount and a ten year horizon, this wager would commit us to paying off the debt with roughly \$20,000 in ten years, but it would enable us to receive a 50%-50% chance of receiving \$10,000 or roughly \$40,000 at the same 10 year horizon.

## REFERENCES

- Bodie, Zvi (1995) "On the Risks of Stocks in the Long Run", *Financial Analysts Journal*, Vol. 51, No. 3, pp. 18-22 (May/June).
- Booth, David G. and Eugene F. Fama (1992) "Diversification Returns and Asset Contributions", *Financial Analysts Journal*, Vol. 48, No. 3 (May - Jun.), pp. 26-32
- Bouchev, Paul, Vassilii Nemtchinov, Alex Paulsen, and David M. Stein (2012) "Volatility Harvesting: Why Does Diversifying and Rebalancing Create Portfolio Growth?" *The Journal of Wealth Management*, Fall, Vol. 15, No. 2: pp. 26-35.
- Brennan, Michael and Eduardo Schwartz (1985) "On the Geometric Mean Index: A Note", *Journal of Financial and Quantitative Analysis*, Vol. 20, No. 1pp. 119-122.
- Erb, Claude B., and Campbell R. Harvey. (2006) "The Tactical and Strategic Value of Commodity Futures." *Financial Analysts Journal*, vol. 62, no. 2 (March/April): 69-97
- Elton, Edwin T. and Martin J. Gruber (1974) "On the Optimality of Some Multiperiod Portfolio Selection Criteria", *The Journal of Business*, Vol. 47, No. 2 (April), pp 231-243.
- Fernholz, Robert and Brian Shay (1982) "Stochastic Portfolio Theory and Stock Market Equilibrium", *Journal of Finance*, Vol. 27, No. 2 pp 615-624.
- Latane, Henry (1959) "Criteria for Choice Among Risky Ventures." *Journal of Political Economy*, 67, pp. 144-155.
- Qian, Edward (2012) "Diversification Return and Leveraged Portfolios" *The Journal of Portfolio Management*, Summer, Vol. 38, No. 4: pp. 14-25
- Samuelson, Paul A. (1971) "The 'Fallacy' of Maximizing the Geometric Mean in Long Sequences of Investing or Gambling" *Proceedings of the National Academy of Science*, Vol. 68, No. 10, pp. 2493-2496 (1971).
- Willenbrock, Scott (2011) "Diversification Return, Portfolio Rebalancing, and the Commodity Return Puzzle", *Financial Analysts Journal*, Vol. 67, No. 4, pp. 42-49, July/August .

**Exhibit 1 Portfolios of uncorrelated assets with 50%/50% chances of +25% and -20% returns**

	Number of Securities				
	1	4	25	100	$\infty$
Arithmetic mean	2.5%	2.5%	2.5%	2.5%	2.5%
Standard deviation	22.50%	11.25%	4.50%	2.25%	0%
Geometric mean	0%	1.87%	2.40%	2.47%	2.50%
Strategic return	0%	0%	0%	0%	0%
Diversification return	0%	1.87%	2.40%	2.47%	2.50%

**Exhibit 2 Four Period Trees of Prices, Rates and Their Means**

				\$2.4414
			\$1.9531 <	\$1.5625
	\$1.25 <	\$1.5625 <	\$1.25 <	\$1.00
\$1.00 <	\$1.00 <	\$1.00 <	\$1.00 <	\$1.00
	\$0.80 <	\$0.80 <	\$0.80 <	\$0.64
		\$0.64 <	\$0.64 <	\$0.64
			\$0.5120 <	\$0.4096
				\$0.4096

Means:	\$1.025	\$1.05625	\$1.0769	\$1.1038
Growth:	2.50%	2.50%	2.50%	2.50%

				25.00%
			25.00% <	11.80%
	25% <	25% <	7.72% <	0.00%
0% <	0% <	0% <	0% <	0.00%
	-20% <	-20% <	-7.17% <	-10.56%
		-20% <	-10.56% <	-10.56%
			-20.00% <	-20.00%
				-20.00%

Mean				
Growth:	2.50%	1.25%	0.83%	0.62%

Note: The underlying asset has a 50%-50% chance of a +25% return and a -20% return. Means are probability weighted based on binomial probabilities (e.g., end of second period outcomes are weighted 25%, 50% and 25%).

**Exhibit 3 Portfolio of cash plus a security with a 50%/50% chances of +25% and -20% returns**

	Weight of Risky Asset				
	0%	25%	50%	75%	100%
Arithmetic mean	0.00%	0.63%	1.25%	1.88%	2.50%
Standard Deviation	0.00%	5.63%	11.25%	16.88%	22.5%
Geometric mean	0.00%	0.47%	0.63%	0.45%	0.00%
Strategic Return	0.00%	0.00%	0.00%	0.00%	0.00%
Diversification Return	0.00%	0.47%	0.63%	0.45%	0.00%

**Exhibit 4 Equally-weighted portfolios of two assets after two periods\***

Asset Num.		Rebalanced		Buy and Hold		Differences	
1	2	Value	Rate	Value	Rate	Value	Rate
+	+	\$1.5625	25.00%	\$1.5625	25.00%		
+	+	\$1.2813	13.19%	\$1.2813	13.19%		
+	+	\$1.2813	13.19%	\$1.2813	13.19%		
+	+	\$1.0506	2.50%	\$1.1013	4.94% **	\$0.0506	2.44%
+	-	\$1.2813	13.19%	\$1.2813	13.19%		
+	-	\$1.0000	0.00%	\$1.0000	0.00%		
+	-	\$1.0506	2.50%	\$1.0000	0.00% ***	\$0.0506	2.50%
+	-	\$0.8200	-9.45%	\$0.8200	-9.45%		
-	+	\$1.2813	13.19%	\$1.2813	13.19%		
-	+	\$1.0506	2.50%	\$1.0000	0.00% ***	\$0.0506	2.50%
-	+	\$1.0000	0.00%	\$1.0000	0.00%		
-	+	\$0.8200	-9.45%	\$0.8200	-9.45%		
-	-	\$1.0506	2.50%	\$1.1013	4.94% **	\$0.0506	2.44%
-	-	\$0.8200	-9.45%	\$0.8200	-9.45%		
-	-	\$0.8200	-9.45%	\$0.8200	-9.45%		
-	-	\$0.6400	-20.00%	\$0.6400	-20.00%		
Means:		\$1.0506	1.874%	\$1.0506	1.867%		

Note: The first four columns denote the return for the two periods with each pair in chronological order and with “+” denoting +25% and “-” denoting -20%.

\* Each asset with equal probabilities of +25% and -20% returns and with no cross-sectional or serial correlations

\*\* Buy and hold strategy performs better because assets “trended”

\*\*\* Rebalancing strategy performs better because assets “mean reverted”

**Exhibit 5 Equally-weighted portfolios of two assets for up to 12 periods\***

Number of Periods	<u>Rebalanced Portfolio</u>			<u>Buy and Hold Portfolio</u>		
	Mean Ret.	E(Value)	Volatility	Mean Ret.	E(Value)	Volatility
1	2.50%	\$1.0250	0.184	2.50%	\$1.0250	0.184
2	1.87%	\$1.0506	0.117	1.87%	\$1.0506	0.118
3	1.66%	\$1.0769	0.093	1.64%	\$1.0769	0.094
4	1.56%	\$1.1038	0.080	1.52%	\$1.1038	0.082
5	1.50%	\$1.1314	0.072	1.45%	\$1.1314	0.073
6	1.45%	\$1.1597	0.065	1.40%	\$1.1597	0.067
7	1.42%	\$1.1887	0.061	1.36%	\$1.1887	0.062
8	1.40%	\$1.2184	0.057	1.32%	\$1.2184	0.058
9	1.38%	\$1.2489	0.053	1.29%	\$1.2489	0.055
10	1.37%	\$1.2801	0.051	1.27%	\$1.2801	0.053
11	1.36%	\$1.3121	0.048	1.25%	\$1.3121	0.050
12	1.35%	\$1.3499	0.046	1.23%	\$1.3499	0.048

\*Each asset has equal probabilities of +25% and -20% returns with no cross-sectional or serial correlations. Mean return is the average geometric return, E(Value) is the expected value of the portfolio from a starting value of \$1 and volatility is the standard deviation of the realized rates of return.