GEOMETRIC SEQUENCES AND SERIES

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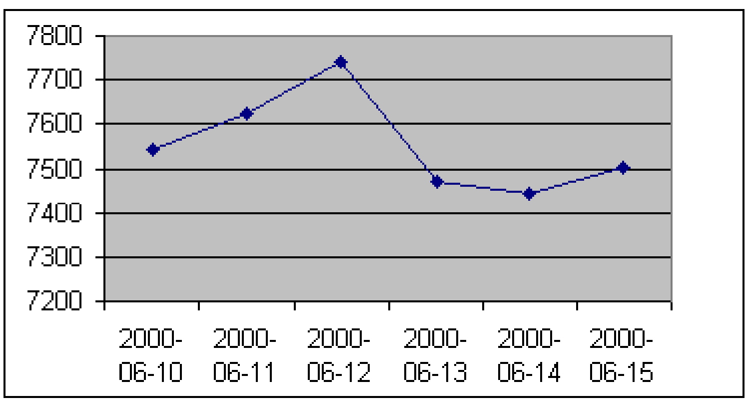
During the duration of an investment, the value of an investment can vary in function of time. The study of an investment at different dates produces a sequence of values. The market index, for example, represents a random sequence in itself. At some point, you surely must have observed a curve of market tendencies like this one :



This curve is merely a visualization of the chronological sequence of values:

10-juin	7542
11-juin	7623
12-juin	7743
13-juin	7471
14-juin	7443
15-juin	7501

This section will cover the study of sequences and series. We will particularly study geometric sequences and series since these are the subject of most bank contracts (investments, loans, mortgages).

1. Geometric sequences

Definition: A sequence $\{\mathrm{a_n}\}_{\mathrm{n=0}}^{\infty} = \{\mathrm{a_0, a_1, a_2, a_3}, \dots\}$ is an ordered set of numbers. The index of each term of the sequence indicates the position or order in which specific data is found. This order is very important. For example, the sequence $\{1,3,5,7,9, \dots\}$ differs from the sequence $\{9,7,5,3,1, \dots .\}$, even if the terms are the same.

Definition: A sequence $\{\mathrm{a_n}\}_{\mathrm{n=0}}^{\infty} = \{\mathrm{a_0, a_1, a_2, a_3}, \dots\}$ is said to be geometric with common ratio r if the terms satisfy the recurrent formula :

$$a_n = r\, a_{n-1}$$

Example 1

The sequence $\{1,2,4,8,16, \dots\}$ is a geometric sequence with common ratio 2, since each term is obtained from the preceding one by doubling.
The sequence $\{9,3,1,1/3, \dots\}$ is a geometric sequence with common ratio $1/3$.

Standard form

Generally, we prefer to express the term a_n of a geometric sequence in function of r and the initial term a_0, as in the formula:

$$a_n = a_0 r^n$$

Example 2

Stocks of a company are initially issued at the price of 10 $. The value of the stock grows by 25 % every year.
Show that the value of a stock follows a geometric sequence.
Calculate the value of the stock ten years after the initial public offering.
Plot a graph of the sequence over a period of 10 years after it was issued.

Solution

Each year, the value of the stock increases by 25 %, thus

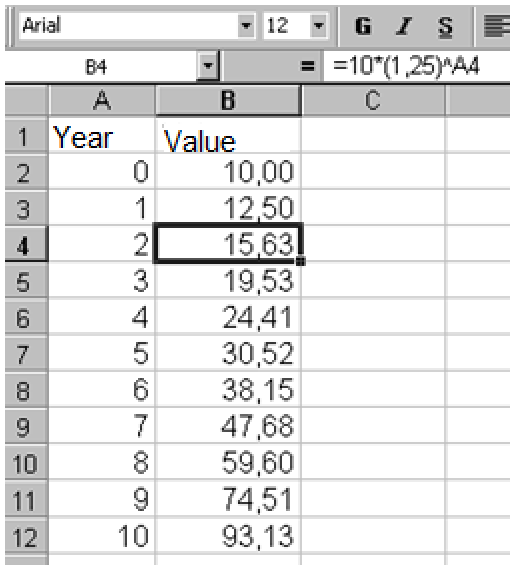
$$a_n = a_{n-1} + 0{,}25a_{n-1} = 1{,}25a_{n-1}$$

This expression satisfies the recurrent form of a geometric sequence of common ratio 1,25.

The initial stock value was $a_0 = \$ 10$. After 10 complete years, the stock is worth

$$a_{10} = a_0 r^{10} = 10(1{,}25)^{10} = 10 \times 9{,}313 = \$ 93{,}13$$

With the help of Excel, we can create the table of stock values at the end of each year.



	A	B	C
1	Year	Value	
2	0	10,00	
3	1	12,50	
4	2	15,63	
5	3	19,53	
6	4	24,41	
7	5	30,52	
8	6	38,15	
9	7	47,68	
10	8	59,60	
11	9	74,51	
12	10	93,13	

The value of the stock at the end of each year is therefore described by the geometric sequence $\{10 , 10.33 , 15.63 , \dots\}$.

The example we just presented describes an increasing geometric sequence. The sequence $\{16 , 8 , 4 , 2 , 1 , 1/2 , \dots\}$ is a decreasing geometric sequence of common ratio ½.

A geometric sequence is :

- **increasing** if and only if $r > 1$
- **decreasing** if and only if $0 < r < 1$

Example 3

Alberta's crude oil reserves are diminishing by 10 % each year. Knowing that 100 000 Ml were the initial reserves, show that the crude oil reserves describe a decreasing geometric sequence and find the common ratio for it.
Which volume will remain four years later?
Plot a graph of the sequence for a period of 20 years.

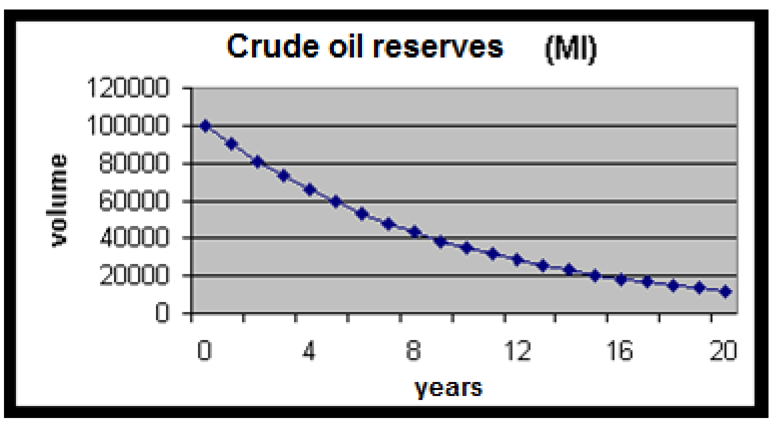
Solution
Each year, the volume decreases by 10 % compared to the previous year:

$$a_n = a_{n-1} - 0{,}10a_{n-1} = 0{,}90a_{n-1}$$

This relation satisfies the recurrent form of a geometric sequence of common ratio 0,90. Moreover, the sequence is decreasing since $0 < r < 1$. The initial volume of crude oil is $a_0 = 100\ 000$. After 4 complete years, the crude oil reserves are

$$a_4 = a_0 r^4 = 100\ 000(0{,}90)^4 = 100\ 000 \times 0{,}6561 = 65610$$

There are therefore 65 610 Ml of crude oil in the reserves after four years.



The recurrence formula also allows us to obtain the value of each element of a sequence without knowing a_0 but rather some element a_k. Any term a_n of a geometric sequence of common ratio r is obtained from the term a_k by the relation $a_n = r^{n-k}a_k$.

Example 4

Gill Bate's personal fortune doubles every year. If the value of his fortune was estimated at $ 32 000 000 in 2000, how much was it in 1995? At the end of which year will his fortune surpass one billion? ($ 1 000 000 000)?

Solution

Each year, the amount of his fortune doubles with regards to the previous year $a_n = 2a_{n-1}$. This is a geometric sequence of common ratio $r = 2$. The initial value of the fortune is unknown, but this information is of no importance thanks to the relation $a_n = r^{n-k}a_k$:

$$a_{1995} = 2^{1995-2000}a_{2000} = 2^{-5}.32\,000\,000$$
$$a_{1995} = \frac{1}{2^5}.32\,000\,000 = \frac{1}{32}.32\,000\,000$$
$$a_{1995} = 1\,000\,000$$

To obtain the date when one billion will be surpassed, we need to find *n* such that $a_n = 1\,000\,000\,000$.

$$a_n = 2^{n-2000}a_{2000}$$
$$1\,000\,000\,000 = 2^{n-2000}a_{2000}$$
$$2^{n-2000} = \frac{1000}{32}$$

We have an equation in which the variable we want to solve for is in the exponent (see exponential equations). We must use a logarithmic transformation to solve this equation.

$$2^{n-2000} = \frac{1000}{32} \to \ln(2^{n-2000}) = \ln\left(\frac{1000}{32}\right)$$
$$\to (n-2000)\ln 2 = \ln\left(\frac{1000}{32}\right)$$
$$\to (n-2000) = \frac{\ln\left(\frac{1000}{32}\right)}{\ln 2}$$
$$\to n = 2000 + \frac{\ln\left(\frac{1000}{32}\right)}{\ln 2}$$
$$\to n = 2000 + 4{,}966 = 2004{,}966$$

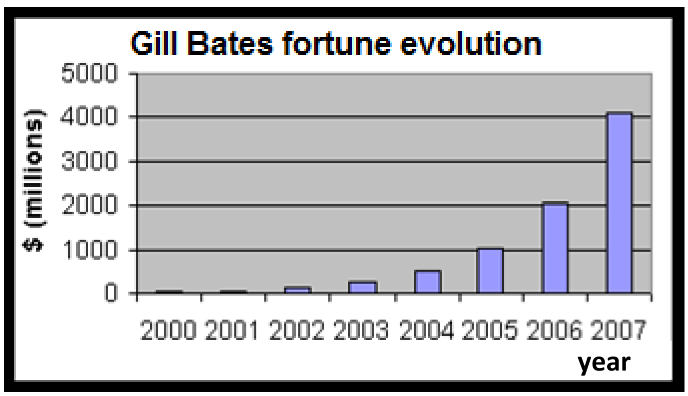
The billion will be reached at the end of 2005.

In this case, it would be faster to create an iterative table in Excel allowing us to observe the temporal evolution of Gill Bate's fortune. The recurrence formula $a_n = 2a_{n-1}$ is easily programmed:

Arial 12 G I S

B7 = =B6*2

	A	B	C
1	Year	Fortune	
2	2000	32000000	
3	2001	64000000	
4	2002	128000000	
5	2003	256000000	
6	2004	512000000	
7	2005	1024000000	
8	2006	2048000000	
9	2007	4096000000	



2. Exercise

Each year the annual global demand for figs increases by 5 %.

a) Show that the demand for figs can be represented by a geometric sequence.
b) If the demand for figs was 2,3 tonnes in 1997, what will the demand be in 2003? (answer: 3,08 tonnes)
c) In which year did the demand pas 1 tonne for the first time? (ans : 1980)
d) With Excel, create a table describing the temporal evolution of the demand in figs and plot the graph as a histogram.

3. Geometric sequence applications to financial mathematics

A widespread application of geometric sequences is found in bank transactions (loans, investments). For example, a person deposits an amount of 1 000 $ at the bank. The bank offers this person an annual return of 6 % of his investment, i.e. the deposited sum will increase by an interest of 6 % at the end of each year. If the person leaves the interest in the account, the annual evolution of the investment is given in the following table:

Time passed	deposit	interest	Balance
-	1000$	0	$a_0 = 1000\$$
1 year	0	$0{,}06(1000) = 60\$$	$a_1 = 1060\$$
2 years	0	$0{,}06(1060) = 63{,}60\$$	$a_2 = 1123{,}60\$$
3 years	0	$0{,}06(1123{,}60) = 67{,}42\$$	$a_3 = 1191{,}02\$$
4 years	0	$0{,}06(1191{,}02) = 71{,}46\$$	$a_4 = 1262{,}48\$$

The temporal evolution of the investment is a geometric sequence. Since $a_n = a_{n-1} + 0{,}06a_{n-1} = 1{,}06a_{n-1}$, the sequence of accumulated values of the investment is geometric of common ratio $1{,}06$.

4. Vocabulary

- Interest dates : dates when the interests are deposited;
- Interest period : time interval between two interest dates;
- Capitalization : adding interests to the capital;
- Periodic interest rate(i) : real interest rate per interest period;
- Nominal interest rate (j) : This rate, calculated on an annual basis, is used to determine the periodic rate. IT is generally this rate that is posted. It should always be accompanied by a precision on the type of capitalization. Given

  $m =$ number of interest periods in the year
  $d =$ duration of the period in the fraction of a year
  $j =$ nominal rate

  Then the periodic rate is given by $i = j/m = d \times j$. For example, a rate of "8 % biannually capitalized" signifies that the interest period is the half-year ($m = 2$ or $d = 1/2$) and that the periodic rate (biannually) is $i = 8\%/2 = 4\%$. The nominal rate does not correspond to the real annual rate, unless the capitalization is annual;
- Effective rate : real annual interest rate;

In general, if V_0 is the initial amount invested at the periodic interest rate "i", then the value of the investment after n interest periods V_n, is described by the relation

$$V_n = V_0(1+i)^n$$

(if we let the interests capitalize). The sequence of the value of the investment $\{V_0, V_1, V_2, \ldots\}$ is geometric of common ratio $1 + i$.

Example

A student borrows 2 500 $. The bank loans this money at a rate of 9 %, capitalized monthly. What amount will the student have to reimburse two years later?

Solution
When the interest rate is stated this way, it is the nominal rate. Since the capitalization is monthly, the interest period is one month and the number of periods in the year is $\mathrm{m} = 12$. The periodic rate is then $\mathrm{i} = 0{,}09 / 12 = 0{,}0075$ per month. The student must reimburse the loan in two years, $n = 24$ interest periods later. He needs to reimburse

$$\mathrm{V}_{24} = \mathrm{V}_0(1+\mathrm{i})^{24}$$
$$\mathrm{V}_{24} = 2500(1+0{,}0075)^{24}$$
$$\mathrm{V}_{24} = 2500 * 1{,}1964 = 2991{,}03$$

5. Exercises

Problem 1
An investor deposits $ 15 000 in a bank account. The bank offers an interest rate of $i = 4{,}1\ \%$ per year.

a) What is the value of the investment 4 years later? (answer : $ 17615,47)
b) How much time is needed for the amount to double? (answer : 18 years)

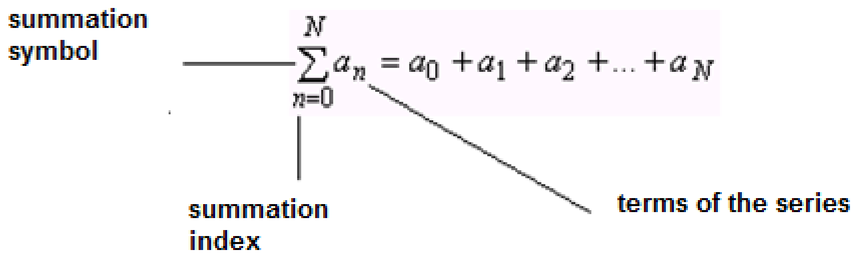
Problem 2
A person wishes to buy a motorcycle worth $ 12 000. In order to collect this amount, he deposits an amount V_0 at the bank, and lets it flourish for 5 years at an interest rate of 5 %, capitalized biannually. Find V_0. (answer : 9 374,38 $)

Problem 3
The price of a liter of milk in 1990 was $ 0,95. In 2000, the price of milk was fixed at $ 1,42. What was the annual inflation rate for this period? (answer : 4,1 %)

6. Geometric series

Given $\{a_n,\} = \{a_0, a_1 r, a_2 r^2, \ldots\}$, a geometric sequence of common ratio r. A geometric series is the sum of the elements of a geometric sequence $a_0 + a_1 r + a_2 r^2 +$..

A series can be finite (with a finite number of terms) or infinite. In order to reduce the writing of a series, we use the summation symbol (see section : The summation symbol) :



In the case of a geometric series, the terms of the sum are those of a geometric sequence, i.e. $a_n = \ a_0 r^n$. Thus, a geometric series of common ratio r has the following form

$$\sum_{n=0}^{N} a_n = \sum_{n=0}^{N} a_0 r^n = \ a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^N$$

Example

Write the following using the summation symbol: $1 \ + \ 2 \ + \ 4 \ + \ 8 \ + \ \ldots \ + \ 128$

Solution

The terms of the sum are those of the geometric sequence $\{1\,, 2\,, 4\,, 8\,, \ldots, 128\,\}$
The initial term of the suite is $a_0 = 1$ and its common ratio is $r = 2$ (since each term is double the preceding term). All terms can be represented by the relation

$$\mathrm{a_n} = \ \mathrm{a_0 r^n} \rightarrow \mathrm{a_n} = \ 1 * 2^{\mathrm{n}} = 2^{\mathrm{n}}$$

The first term (1) of the series is obtained when the exponent of 2 is $n = 0$ and the last term (128), when the exponent of 2 is $n = 7$. The limits of summation are therefore 0 and 7.

Thus, the geometric series can be written as follows:

$$\sum_{\mathrm{n}=0}^{7} 2^{\mathrm{n}} = 1 + 2 + 4 + 8 + \cdots + 128$$

Formula to evaluate a finite geometric series :

Given $\sum_{n=0}^{N} a_0 r^n = a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^N$, a finite geometric series of common ratio r and of initial a_0. Then

$$\sum_{n=0}^{N} a_0 r^n = a_0 \frac{1 - r^{N+1}}{1 - r}$$

Example 1

Evaluate the geometric series $\sum_{n=0}^{4} 4(\frac{1}{2})^n$

Solution

You need to use the formula for the sum of a geometric series:

$$\sum_{n=0}^{4} 4(\frac{1}{2})^n = 4\frac{1-(\frac{1}{2})^{4+1}}{1-(\frac{1}{2})} = 4\frac{1-\frac{1}{32}}{1-(\frac{1}{2})} = 4\frac{\frac{31}{32}}{\frac{1}{2}} = 4 * \frac{31}{32} * \frac{2}{1} = 7{,}75$$

You can also verify that this answer is correct by adding the terms of the geometric series $\sum_{n=0}^{4} 4(\frac{1}{2})^n = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} = 7{,}75$

Example 2

Calculate the sum of the series $9 + (-3) + 1 + \left(-\frac{1}{3}\right) + \ldots + \left(-\frac{1}{243}\right)$

Solution

This is a geometric series of common ratio $r = 1/3$ with initial term $a_0 = 9$. We must also identify the upper limit of summation (the exponent of r of the last term. The last term, identified by a_N is $-1/243$:

$$a_N = a_0 r^N \Rightarrow -\frac{1}{243} = 9\left(-\frac{1}{3}\right)^N$$
$$\Rightarrow -\frac{1}{9.243} = \left(-\frac{1}{3}\right)^N$$

$$\Rightarrow -\frac{1}{3^2.3^5} = \left(-\frac{1}{3}\right)^N$$
$$\Rightarrow \left(-\frac{1}{3}\right)^7 = \left(-\frac{1}{3}\right)^N$$
$$\Rightarrow N = 7$$

We can therefore represent the series in "sigma" notation and calculate the sum using the formula seen previously:

$$\sum_{n=0}^{7} 9(\frac{-1}{3})^n = 9\frac{1-(\frac{-1}{3})^{7+1}}{1-(\frac{-1}{3})} = 9\frac{1-\frac{1}{6561}}{\frac{4}{3}} = 9 * \frac{6560}{6561} * \frac{3}{4} = 6{,}74897119$$

7. Exercises

Problem1 :

Evaluate the following geometric series :

$$\sum_{i=0}^{6} 4(\frac{2}{3})^i$$
$$\sum_{k=0}^{10} 2{,}5(\frac{1}{\sqrt{10}})^k$$
$$\sum_{n=0}^{5} \frac{1}{2}(\frac{-3}{2})^n$$

Problem 2 :

Evaluate the following geometric series :

a) $4 + 1 + 1/4 + 1/16 + \ldots + 1/4096$

b) $(\ \ 1/3) + 1 + (\ \ 3) + 9 + \ldots + 729$

c) $6 + 3/2 + 3/8 + 3/32 + \ldots + 3/512$

8. Geometric series applications in financial mathematics

A person decides to deposit an amount V_0 in a savings account, the last day of every month for a full year. The first payment is done January 31 and the last, December 31. The bank where the deposits are made offers a monthly interest rate i for this type of account. How much will the person have accumulated at the end of the year, immediately after having deposited the last payment the 31 of December?

This type of problem, where we need to consider a certain number of equal deposits at regular intervals, can be resolved with geometric series.

Instead of making 12 monthly deposits in one account, we could open 12 different accounts, as long as the interest rate is the same in each. You would need to deposit an amount V_0 in the first account January 31, an amount in the second account February 30, and so forth until the 12th deposit. We would accumulate the same amount of money as if we had deposited the 12 payments in the same account.

The calculation of the accumulated amount on the 31 of December is done by the sum of the individual acquired values (or cumulated) per deposit. Note that even if the deposits are equal, they do not all capitalize for the same period. For example, the deposit of January 31 capitalizes on 11 full months as opposed to the deposit of December 31, which does not result in any interest.

As we demonstrated in the previous section, the value of a deposit after *n* capitalization periods at rate i is obtained from the relation $V_n = V_0(1+i)^n$.

Deposit date	**Initial value of deposit**	**Number of months of capitalization until December 31**	**Acquired value of deposit December 31**
31 January	$\boldsymbol{V_0}$	11	$V_0(1+i)^{11}$
28 February	$\boldsymbol{V_0}$	10	$V_0(1+i)^{10}$
31 March	$\boldsymbol{V_0}$	9	$V_0(1+i)^9$
30 april	$\boldsymbol{V_0}$	8	$V_0(1+i)^8$
31 May	$\boldsymbol{V_0}$	7	$V_0(1+i)^7$
30 Juin	$\boldsymbol{V_0}$	6	$V_0(1+i)^6$
31 July	$\boldsymbol{V_0}$	5	$V_0(1+i)^5$
31 August	$\boldsymbol{V_0}$	4	$V_0(1+i)^4$
30 September	$\boldsymbol{V_0}$	3	$V_0(1+i)^3$
31 October	$\boldsymbol{V_0}$	2	$V_0(1+i)^2$
30 November	$\boldsymbol{V_0}$	1	$V_0(1+i)^1$
31 December	$\boldsymbol{V_0}$	0	$V_0(1+i)^0$

The total amount available in the savings account of the person is the sum of the acquired (cumulated) values of each deposit:

$$M = V_0(1+i)^0 + V_0(1+i)^1 + V_0(1+i)^2 + \cdots + V_0(1+i)^{11} = \sum_{n=0}^{11} V_0(1+i)^n$$

You will have noticed the form of a geometric series of common ratio $1+i$. By applying the general formula

$$\sum_{n=0}^{N} a_0 r^n = a_0 \frac{1-r^{N+1}}{1-r}$$

We obtain :

$$M = \sum_{n=0}^{11} V_0(1+i)^n = V_0\left[\frac{1-(1+i)^{11+1}}{1-(1+i)}\right] = V_0\left[\frac{1-(1+\ \ )^{12}}{-i}\right] = V_0\left[\frac{(1+i)^{12}-1}{i}\right]$$

Note :
In financial mathematics, we generally denote the value of each deposit by "PMT = principal payment" and the number of deposits by "n". Immediately after the last deposit, the acquired value ($FV = future\ value$) after a sequence of n equal deposits done at regular intervals at a periodic interest rate "i", is given by the formula :

$\mathrm{FV} = \mathrm{PMT}s_{n\rceil i}$ with $s_{n\rceil i} = \frac{(1+i)^n - 1}{i}$

The interest period considered for the rate "i" must correspond to the period between two consecutive deposits.