# THE SUMMATION SYMBOL

## Summary



## 1. Simple sum

The symbol $\Sigma$ (sigma) is generally used to denote a sum of multiple terms. This symbol is generally accompanied by an index that varies to encompass all terms that must be considered in the sum.

For example, the sum of $n$ first whole numbers can be represented in the following manner:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots . + n$$

More generally, the expression $\sum_{i=1}^{n} x_i$ represents the sum of n terms

$$x_1 + x_2 + x_3 + \cdots . + x_n.$$

***Example 1***

Given $x_1 = 3, x_2 = 5, x_3 = 6, x_4 = 2 \; and \; x_5 = 7.$

Evaluate $\sum_{i=1}^{5} x_i$ and $\sum_{i=2}^{4} x_i$.

**Solution**:

In the first sum, the index " $i$ " varies from 1 to 5. We must therefore include the 5 terms in the sum.

$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5 = 3 + 5 + 6 + 2 + 7 = 23$$

In the second case, the index " *i* "varies from 2 to 4. Only the terms $x_2, x_3$ and $x_4$ must therefore be considered.

$$\sum_{i=2}^{4} x_i = x_2 + x_3 + x_4 = 5 + 6 + 2 = 13$$

When we use the summation symbol, it is useful to remember the following rules:

$$\sum_{i=1}^{n} cx_i = c \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$$

***Example 2***

Given $x_1 = 3, x_2 = 5, x_3 = 6, x_4 = 2\ and\ x_5 = 7$ et $y_1 = 2, y_2 = 8, y_3 = 3,$

$y_4 = 1\ and\ y_5 = 6.$

Verify the three preceding rules with the following sums :

$a) \sum_{i=1}^{5} 4x_i$

$b) \sum_{i=1}^{5} 4$

$c) \sum_{i=1}^{5} (x_i + y_i)$

***Solution :***

$$a)\ \sum_{i=1}^{5} 4x_i = 4x_1 + 4x_2 + 4x_3 + 4x_4 + 4x_5 = 4\times 3 + 4\times 5 + 4\times 6 + 4\times 2 + 4\times 7$$
$$= 92$$

and

$$4\sum_{i=1}^{5} x_i = 4\times(3+5+6+2+7) = 4\times 23 = 92$$

b)

$$\sum_{i=1}^{5} 4 = 4+4+4+4+4 = 5\times 4 = 20$$

c)

$$\sum_{i=1}^{5}(x_i + y_i) = (3+2)+(5+8)+(6+3)+(2+1)+(7+6) = 43$$

And

$$\sum_{i=1}^{5} x_i + \sum_{i=1}^{5} y_i = (3+5+\cdots 7)+(2+8+\cdots+6) = 23+20 = 43$$

***Attention*** **:**

We must neither confound the expression

$$\sum_{i=1}^{n} x_i^2$$

with

$$\left(\sum_{i=1}^{n} x_i\right)^2$$

nor the expression

$$\sum_{i=1}^{n} x_i y_i$$

with

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

***Example 3***

Given $x_1 = 3, x_2 = 5, x_3 = 6, x_4 = 2 \; and \; x_5 = 7$ and $y_1 = 2, y_2 = 8, y_3 = 3,$

$y_4 = 1 \; and \; y_5 = 6.$

a)

$$\sum_{i=1}^{5} x_i^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 3^2 + 5^2 + 6^2 + 2^2 + 7^2 = 123$$

and

$$\left(\sum_{i=1}^{5} x_i\right)^2 = (3 + 5 + 6 + 2 + 7)^2 = 23^2 = 52 \neq 123$$

b)

$$\sum_{i=1}^{5} x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 + x_5 y_5 = (3 \times 2) + (5 \times 8) + \cdots + (7 \times 6)$$
$$= 6 + 40 + \cdots + 42 = 108$$

and

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) = (3 + 5 + \cdots + 7) \times (2 + 8 + \cdots + 6) = 23 \times 20 = 460 \neq 108$$

## 2. Double sum

In certain situations, using a double sum may be necessary. You must then apply the definition successively.

***Example***

Given $x_1 = 3, x_2 = 5, x_3 = 1 \; and \; y_1 = 2, y_2 = 4$

We will use the index *i* for the terms of $x$ and index *j* for the terms of $y$

$$\sum_{i=1}^{3}\sum_{j=1}^{2} x_i y_j = \sum_{=1}^{3}(x_i y_1 + x_i y_2) = x_1 y_1 + x_1 y_2 + x_2 y_1 + x_{2i} y_2 + x_3 y_1 + x_3 y_2$$

$$= (3 \times 2) + (3 \times 4) + (5 \times 2) + (5 \times 4) + (1 \times 2) + (1 \times 4) = 54$$

## 3. Double index

To represent the data of a table or a matrix, we often use a double index notation, like $x_{ij}$ where the first index (*i*) corresponds to the number of the row where the data is located and the second (*j*) to the column. For example, the term $x_{24}$ represents the data that is situated at the intersection of the 2$^{nd}$ row and the 4$^{th}$ column of the table or the matrix.

***Example***

Given

$$x_{11} = 4 \;\; x_{12} = 4 \;\; x_{13} = 1 \;\; x_{14} = 5$$

$$x_{21} = 0 \;\; x_{22} = 3 \;\; x_{23} = 1 \;\; x_{24} = 2$$

$$x_{31} = 1 \;\; x_{32} = 4 \;\; x_{33} = 2 \;\; x_{34} = 3$$

To carry out the sum of the terms of a row, we must fix the index of that row and vary, for all possible values, the index of the column. For example:

$\sum_{j=1}^{4} x_{1j} = x_{11} + x_{12} + x_{13} + x_{14} = 2 + 4 + 1 + 5 = 12$ (sum of the first row)

$\sum_{j=1}^{4} x_{2j} = x_{21} + x_{22} + x_{23} + x_{24} = 0 + 3 + 1 + 2 = 6$ (sum of the 2$^{nd}$ row)

To carry out the sum of the terms of a column, you must fix the index of this column and vary, for all possible values, the index of the row.

For example:

$\sum_{i=1}^{3} x_{i4} = x_{14} + x_{24} + x_{34} = 5 + 2 + 3 = 10$ (sum of the 4th column)

To carry out the sum of all terms of the table, you must vary both indices and use a double sum:

$$\sum_{i=1}^{3}\sum_{j=1}^{4} x_{ij} = \sum_{i=1}^{3}(x_{i1} + x_{i2} + x_{i3} + x_{i4})$$
$$= x_{11} + x_{12} + x_{13} + x_{14} + x_{21} + x_{22} + x_{23} + x_{24} + x_{31} + x_{32} + x_{33} + x_{34} = 2 + 4 + 1 + 6 + 0 + 3 + \cdots + 3 = 28$$