THE SUMMATION SYMBOL

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1. Simple sum

The symbol Σ (sigma) is generally used to denote a sum of multiple terms. This symbol is generally accompanied by an index that varies to encompass all terms that must be considered in the sum.

For example, the sum of \( n \) first whole numbers can be represented in the following manner:

\[
\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n
\]

More generally, the expression \( \sum_{i=1}^{n} x_i \) represents the sum of \( n \) terms

\[x_1 + x_2 + x_3 + \cdots + x_n.\]

Example 1

Given \( x_1 = 3, x_2 = 5, x_3 = 6, x_4 = 2 \) and \( x_5 = 7 \).

Evaluate \( \sum_{i=1}^{5} x_i \) and \( \sum_{i=2}^{4} x_i \).

Solution:

In the first sum, the index " \( i \) " varies from 1 to 5. We must therefore include the 5 terms in the sum.

\[
\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5 = 3 + 5 + 6 + 2 + 7 = 23
\]
In the second case, the index "i" varies from 2 to 4. Only the terms $x_2, x_3$ and $x_4$ must therefore be considered.

$$\sum_{i=2}^{4} x_i = x_2 + x_3 + x_4 = 5 + 6 + 2 = 13$$

When we use the summation symbol, it is useful to remember the following rules:

$$\sum_{i=1}^{n} c x_i = c \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$$

**Example 2**

Given $x_1 = 3, x_2 = 5, x_3 = 6, x_4 = 2$ and $x_5 = 7$, and $y_1 = 2, y_2 = 8, y_3 = 3, y_4 = 1$ and $y_5 = 6$.

Verify the three preceding rules with the following sums:

a) $\sum_{i=1}^{5} 4x_i$

b) $\sum_{i=1}^{5} 4$

c) $\sum_{i=1}^{5} (x_i + y_i)$
Solution:

\[ a) \sum_{i=1}^{5} 4x_i = 4x_1 + 4x_2 + 4x_3 + 4x_4 + 4x_5 = 4 \times 3 + 4 \times 5 + 4 \times 6 + 4 \times 2 + 4 \times 7 = 92 \]

and

\[ 4 \sum_{i=1}^{5} x_i = 4 \times (3 + 5 + 6 + 2 + 7) = 4 \times 23 = 92 \]

b)

\[ \sum_{i=1}^{5} 4 = 4 + 4 + 4 + 4 + 4 = 5 \times 4 = 20 \]

c)

\[ \sum_{i=1}^{5} (x_i + y_i) = (3 + 2) + (5 + 8) + (6 + 3) + (2 + 1) + (7 + 6) = 43 \]

And

\[ \sum_{i=1}^{5} x_i + \sum_{i=1}^{5} y_i = (3 + 5 + \cdots + 7) + (2 + 8 + \cdots + 6) = 23 + 20 = 43 \]

Attention:

We must neither confound the expression

\[ \sum_{i=1}^{n} x_i^2 \]

with

\[ \left( \sum_{i=1}^{n} x_i \right)^2 \]
nor the expression

$\sum_{i=1}^{n} x_i y_i$

with

$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$

**Example 3**

Given $x_1 = 3, x_2 = 5, x_3 = 6, x_4 = 2$ and $x_5 = 7$ and $y_1 = 2, y_2 = 8, y_3 = 3, y_4 = 1$ and $y_5 = 6$.

a)

$$\sum_{i=1}^{5} x_i^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 3^2 + 5^2 + 6^2 + 2^2 + 7^2 = 123$$

and

$$\left(\sum_{i=1}^{5} x_i\right)^2 = (3 + 5 + 6 + 2 + 7)^2 = 23^2 = 52 \neq 123$$

b)

$$\sum_{i=1}^{5} x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 + x_5 y_5 = (3 \times 2) + (5 \times 8) + \cdots + (7 \times 6)$$

\[= 6 + 40 + \cdots + 42 = 108\]

and

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) = (3 + 5 + \cdots + 7) \times (2 + 8 + \cdots + 6) = 23 \times 20 = 460 \neq 108$$
2. **Double sum**

In certain situations, using a double sum may be necessary. You must then apply the definition successively.

**Example**

Given \(x_1 = 3, x_2 = 5, x_3 = 1\) and \(y_1 = 2, y_2 = 4\)

We will use the index \(i\) for the terms of \(x\) and index \(j\) for the terms of \(y\)

\[
\sum_{i=1}^{3} \sum_{j=1}^{2} x_i y_j = \sum_{i=1}^{3} (x_i y_1 + x_i y_2) = x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2 + x_3 y_1 + x_3 y_2
\]

\[
= (3 \times 2) + (3 \times 4) + (5 \times 2) + (5 \times 4) + (1 \times 2) + (1 \times 4) = 54
\]

3. **Double index**

To represent the data of a table or a matrix, we often use a double index notation, like \(x_{ij}\) where the first index \((i)\) corresponds to the number of the row where the data is located and the second \((j)\) to the column. For example, the term \(x_{24}\) represents the data that is situated at the intersection of the 2\(^{nd}\) row and the 4\(^{th}\) column of the table or the matrix.

**Example**

Given

\[
x_{11} = 4 \quad x_{12} = 4 \quad x_{13} = 1 \quad x_{14} = 5
\]

\[
x_{21} = 0 \quad x_{22} = 3 \quad x_{23} = 1 \quad x_{24} = 2
\]

\[
x_{31} = 1 \quad x_{32} = 4 \quad x_{33} = 2 \quad x_{34} = 3
\]

To carry out the sum of the terms of a row, we must fix the index of that row and vary, for all possible values, the index of the column. For example:

\[
\sum_{j=1}^{4} x_{1j} = x_{11} + x_{12} + x_{13} + x_{14} = 2 + 4 + 1 + 5 = 12 \quad \text{(sum of the first row)}
\]

\[
\sum_{j=1}^{4} x_{2j} = x_{21} + x_{22} + x_{23} + x_{24} = 0 + 3 + 1 + 2 = 6 \quad \text{(sum of the 2\(^{nd}\) row)}
\]

To carry out the sum of the terms of a column, you must fix the index of this column and vary, for all possible values, the index of the row.
For example:

\[ \sum_{i=1}^{3} x_{i4} = x_{14} + x_{24} + x_{34} = 5 + 2 + 3 = 10 \text{ (sum of the 4th column)} \]

To carry out the sum of all terms of the table, you must vary both indices and use a double sum:

\[
\sum_{i=1}^{3} \sum_{j=1}^{4} x_{ij} = \sum_{i=1}^{3} (x_{i1} + x_{i2} + x_{i3} + x_{i4}) \\
= x_{11} + x_{12} + x_{13} + x_{14} + x_{21} + x_{22} + x_{23} + x_{24} + x_{31} + x_{32} + x_{33} \\
+ x_{34} = 2 + 4 + 1 + 6 + 0 + 3 + \cdots + 3 = 28
\]