RESOLUTION OF SYSTEMS WITH TWO UNKNOWNS

Summary

This help section will deal with solving problems with systems of two linear equations and two variables. The methods presented are essential for the Microeconomic analysis course, the Managerial economy course, as well as for all linear programming and operational research courses.

Solution of a system of equations

Given the linear equation system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

The solution of a system is the set of values for x and y that satisfies both equations simultaneously.

Example

$x = 1, y = 2$ is a solution of the linear system of equations

$$\begin{cases} 2\boldsymbol{x} + 3\boldsymbol{y} = 8 \\ \quad 3y - 1 = 1 \end{cases}$$

When the variables x and y are substituted by 1 and 3 respectively, both equations are satisfied, i.e.

$2(1) + 3(2) = 2 + 6 = 8$;

$3(1) - (2) = 3 - 2 = 1.$

It would be false to say that $x = 4$, $y = 0$ is also a solution for this system.

$2(4) + 3(0) = 8 + 0 = 8$;

$3(4) - (0) = 12 \neq 1$.

Even if the first equation in the system is satisfied, the second is not. Be reminded that by definition, the solution of a system must simultaneously satisfy both equations. For this reason, the values $x = 3, y = 8$, that make the second equation true but not the first, would not be a solution either.

Note

It is important to mention that a solution is made up of two values, one being the value of variable x and the other, of y. We will avoid saying that $x = 1$ is a solution and that $y = 2$ is another solution. A solution is made up of the set of values jointly taken by the variables to satisfy the system's equations.

1- Methods to resolve

Essentially, there are two distinct methods to resolve systems of two equations with two unknowns. Whichever method you choose to use, know that the solution found will be the same. You need to determine which method you prefer and use it religiously!

1.1. Substitution method

The problem of solving a system like

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

comes from the fact that two variables are present in each equation. The method of substitution will allow you to use the information in one of the equations to reduce the second equation to one variable. The following steps must be followed:

1. In the first equation, isolate x. It is normal that you don't obtain a precise value just yet. You should have an expression in which x depends on y ;
2. Substitute x in the second equation by the expression found in the previous step. Normally, you should obtain an expression containing only the variable y;
3. Solve for y ;
4. Find x by using the expression found in 1) and the value of y now found.

Know that these steps are not absolute. It will sometimes be simpler to isolate y in the second equation and to replace the expression obtained in the first...

Example

Solve the following system with two unknowns

$$\begin{cases} x + 3y = 5 \\ 2x + 5y = 9 \end{cases}$$

Solution

1. Isolate x in the first equation…

$x = 5 - 3y$

2. Substitute x in the second equation by $5 - 3y$ …

$2(5 - 3y) + 5y = 9$

3. Solve for y…

$10 - 6y + 5y = 9$

$-y = 9 - 10$

$y = 1$

we discovered that $x = 5 - 3y$ and we now know that $y = 1$.

$x = 5 - 3(1)$

$x = 5 - 3$

$x = 2$

The solution of the system

$$\begin{cases} x + 3y = 5 \\ 2x + 5y = 9 \end{cases}$$

is therefore $x = 2,\ y = 1$.

Example

Solve the system with two variables

$$\begin{cases} 4\boldsymbol{x} + \boldsymbol{y} = 7 \\ 2x + 3y = 11 \end{cases}$$

Solution

As we mentioned earlier, the solution steps are not absolute. For example, in this case, to isolate x from the first equation will produce fractions (eventually, you will need to divide by 4 to remove the coefficient of x). It would be easier to isolate y from the first equation since its coefficient (hidden) is 1.

1. Isolate y in the first equation…

$y = 7 - 4x$

2. Substitute y in the second equation by $7 - 4x$ …

$2x + 3\,(7 - 4x) = 11$

3. Solve for y…

$2x + 21 - 12x = 11$

$-10x + 21 = 11$

$-10x = -10$

$x = 1$

4. Find y…

In 1) we discovered that $y = 7 - 4x$ and we now know that $x = 1$.

$y = 7 - 4(1)$

$y = 5 - 4$

$y = 3$

The solution of the system

$$\begin{cases} 4\boldsymbol{x} + \boldsymbol{y} = 7 \\ 2x + 3y = 11 \end{cases}$$

is therefore $x = 1, y = 3$.

Note that the comparison method could prove to be difficult to use if none of the variables are easy to isolate because of their coefficients. For example, the system

$$\begin{cases} 7x - 4y = 11 \\ 5x + 3y = 15 \end{cases}$$

will create fractions no matter what the variable we isolate first. For these cases, we suggest the following method.

1.2. Method of linear combinations

Let us consider the following system with two equations and two unknowns:

$$\begin{cases} 6\boldsymbol{x} + 2\boldsymbol{y} = 12 \\ -6x + 3y = 8 \end{cases}$$

Here, the substitution method would produce fractions that are both superfluous and difficult to manipulate. We can see that the coefficient of x is 6 in both equations. Wouldn't it be great to add the $6x$ from the first equation to the $-6x$ of the second so that they cancel each other? In fact, we can do it... by following certain rules. To carry out linear combinations of two equations consists of adding (or subtracting) ALL of an equation to another, not just specific terms. For example, in the case presented below, the addition of the two equations would produce the following

$$6x + 2y = 12$$

$$-6x + 3y = 8$$

$$5y = 20$$

Where we find $y = 4$. Following this, substituting y by 4 in one of the equations will allow us to find x.

As in substitution, you will see that the method of linear combinations transforms a system of two variables into one equation with one unknown. You will need to follow the following instructions :

1. Multiply one of the equations (or both, if necessary) so that the x variable has opposite coefficients ;
2. Add the new equations together. The x variable should cancel itself ;
3. Solve for y with the help of the expression obtained in 2) ;
4. Substitute y by the value obtained in 3) in one of the equations you began with.

Example

Solve the following system with two variables

$$\begin{cases} 2x - 3y = 8 \\ 3x = 4y = -5 \end{cases}$$

Solution

1. By multiplying the first equation by 3 and the second by -2, the coefficients of x will be opposites (6 and -6, respectively) ;

$$3\,(2x - 3y = 8) \rightarrow 6x - 9y = 24$$

$$-2\,(3x + 4y = -5) \rightarrow -6x - 8y = 10$$

2. Add the new equations obtained in the previous step …

$$6x - 9y = 24$$

$$-6x - 8y = 10$$

$$-17y = 34$$

3. Solve for y…

$$-17y = 34$$

$$y = -2$$

4. Substitute the value of y in one of the equations you began with …

From the first equation, we know that

$$2x - 3y = 8$$

$$2x - 3(-2) = 8$$

$$2x + 6 = 8$$

$2x = 2$

$x = 1$

The solution of this system is therefore $x = 1, y = -2$.

It may be simpler to multiply the equations so that the coefficients cancel each other out. In such a case, the steps to follow would be equivalent to those we just demonstrated.

Example

Solve the system with two variables

$$\begin{cases} 7x - 2y = 8 \\ 3x + 4y = 18 \end{cases}$$

Solution

Note that the coefficients for y already have opposite signs. You would only need to multiply the first equation by 2 in order for the y to cancel out during the addition of the equations. In this case, it wouldn't even be necessary to multiply the second equation. By multiplying the first equation by 2, the coefficients of y will be opposites (4 and -4, respectively) ;

$$2\,(\,7x - 2y = 8\,) \rightarrow 4x - 4y = 16$$

$$3x + 4y = 18 \rightarrow 3x + 4y = 18$$

1. Add the new equations obtained in the previous step…

   $14x - 4y = 16$

   $3x + 4y = 18$

   $17x = 34$

2. Solve for x…

   $17x = 34$

   $x = 2$

3. Substitute the value of x in one of the equations you began with …

   From the first equation we find that

   $7x - 2y = 8$

   $7(2) - 2y = 8$

   $14 - 2y = 8$

   $-2y = 8 - 14$

   $-2y = -6$

   $y = 3$

The system solution is therefore $x = 2, y = 3$.

The advantage of the linear combination method is that it adapts easily to more complex cases...

Example

Solve the following system

$$\begin{cases} 1{,}25\,\boldsymbol{x} + 3{,}25\,\boldsymbol{y} = 15{,}50 \\ 1{,}5\,x + 2{,}75y \qquad = 14 \end{cases}$$

Solution

In order to obtain opposite coefficients for x, you need to multiply the first equation by 1,5 and the second by -1,25. The x will cancel out when the equations are added together.

1. $1{,}5\,(\,1{,}25\,x + 3{,}25\,y = 15{,}50\,) \rightarrow 1{,}875\,x + 4{,}875\,y = 23{,}25$

   $-1{,}25\,(\,1{,}5\,x + 2{,}75\,y = 14\,) \rightarrow -1{,}875\,x - 3{,}4375\,y = -17{,}50$

2. Add the new equations obtained from the previous step …

$1{,}875\,x\ +\ 4{,}875\,y\ =\ 23{,}25$

$-1{,}875\,x\ -\ 3{,}4375\,y\ =\ -17{,}50$

$1{,}4375y = 5{,}75$

3. Solve for y…

   $1{,}4375\,y =\ 5{,}75$

   $$y\ = \frac{5{,}75}{1{,}4375}$$

   $y = 4$

4. Substitute the value of y in one of the equations you began with…

   From the first equation, we know that

   $1{,}25\,x\ +\ 3{,}25\,y\ =\ 15{,}50$

   $1{,}25\,x\ +\ 3{,}25\,(4)\ =\ 15{,}50$

   $1{,}25\,x\ +\ 13\ =\ 15{,}50$

   $1{,}25\,x\ =\ 2{,}50$

   $$x\ = \frac{2{,}50}{1{,}25}$$

   $x\ =\ 2$

The system solution is therefore $x\ =\ 2, y\ =\ 4$.

It is necessary, in cases like this one, to be extremely prudent when calculating ; a calculator can be very useful. However, the method itself for resolving the system is not modified in any way.

Exercises

Solve the following systems :

1. $\begin{cases} 2x + 3y = 1 \\ 3x - 2y = 8 \end{cases}$
2. $\begin{cases} x - 7y = 15 \\ 3x = 5y = -7 \end{cases}$
3. $\begin{cases} 4x + 3y = 29 \\ 2x - y = 7 \end{cases}$

4. $\begin{cases} 3x + 5y = 1 \\ 5x - 3y = -21 \end{cases}$

Solutions

1. $x = 2, y = -1$
2. $x = 1, y = -2$
3. $x = 5, y = 3$
4. $x = -3, y = 2$