PROBABILITY NOTIONS

Summary

1. Random experiment

A random experiment is a process in which chance intervenes and is susceptible of producing different results; it is characterized in four ways:

1. we **cannot** predict the result with **certainty**,
2. we can describe a priori the set of all possible results,
3. it can be repeated,
4. it has a precise goal.

Random experiment examples

1. Throw two die and observe the total
2. Have the winning lottery number
3. Flip a coin twice and write down the outcomes.

2. Sample space

The **sample space** of a random experiment is the set of all possible results of this experiment, denoted Ω

Examples

The sample spaces associated to the random experiments presented in the previous example are respectively:

1. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
2. $\Omega = \{1000, 1001, \dots, 9999\}$
3. $\Omega = \{\text{ff}, \text{fp}, \text{pf}, \text{pp}\}$

3. Event

An **event** linked to a random experiment is a subset of the sample space Ω. We usually denote events by $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$

Example

Consider the experiment that consists of flipping a coin twice and noting the outcome. The sample space is

$$\Omega = \{\text{hh}, \text{ht}, \text{th}, \text{tt}\}$$

Below are a few examples of events:

$\boldsymbol{A}$ = "tossing heads on the first flip " = $\{ht, hh\}$;

$\boldsymbol{B}$ = "tossing heads on the second flip " = $\{th, hh\}$;

$\boldsymbol{C}$ = "tossing the same face for both flips " = $\{tt, hh\}$;

$\boldsymbol{D}$ = "tossing different faces for both flips " = $\{th, ht\}$;

Using the operations on sets, we can form new events from one or more events.

Example

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two events, then:

1. $\overline{\boldsymbol{A}}$ is the event that comes true if the event $\boldsymbol{A}$ does not. We say that $\overline{\boldsymbol{A}}$ is the complementary event of event $\boldsymbol{A}$.
2. $\boldsymbol{A} \cap \boldsymbol{B}$ is the event for which both events will come true (If $\boldsymbol{A} \cap \boldsymbol{B} = \emptyset$, we say that $\boldsymbol{A}$ and $\boldsymbol{B}$ are mutually exclusive events)
3. $\boldsymbol{A} \cup \boldsymbol{B}$ is the event for which at least one of the events $\boldsymbol{A}$ or $\boldsymbol{B}$ comes true.
4. $\boldsymbol{A} \setminus \boldsymbol{B}$ is the event for which $\boldsymbol{A}$ comes true but not $\boldsymbol{B}$.

4. Probability calculations

4.1. The fundamental sample space

A sample space is said to be ***fundamental*** if each of its results has equal probability to come true.

Example

- If we throw a regular dice we have as much chance of observing a 6 as any other number.
- If we flip a coin, the *heads* result has as much chance of occurring as the *tails* result.

4.2. Calculation of probability

If a sample space is *fundamental*, then the **probability** that an event $A \subseteq \Omega$ will come true, denoted $\boldsymbol{P}(\boldsymbol{A})$, is obtained from the following formula :

$$P(A) = \frac{\#A}{\#\Omega} = \frac{Number\ of\ elements\ in\ A}{Number\ of\ elements\ in\ \Omega}$$

Example

We pick a card randomly from a conventional pack of 52 cards.

$\Omega = \{$ 2♠ to A♠ , 2♥ to A♥ , 2♦ to A♦ , 2♣ to A♣ $\}$

Given the event $\boldsymbol{A}$ = "obtain a ♠". Then,

$$P(A) = \frac{\#A}{\#\Omega} = \frac{Number\ of\ spade\ cards}{Total\ number\ of\ cards} = \frac{13}{52} = 0.25$$

4.3. Non fundamental sample space

A sample space for which at least one element has more or less chances of occurring than others is said to be ***non-fundamental***.

Example

Consider the event "throw two regular die and add the results". The sample space is described by the set

$$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

The elements of this set are not equiprobable. For example, there are three different combinations to obtain a sum of 4: throw 1-3, 3-1 or 2-2. However, only the throw 1-1 gives us the sum of 2. We will let you verify that the different ways of obtaining a sum of 7 are even more numerous (there are 6 in all).

When a sample space is *non-fundamental,* the probability that the event $A \subseteq \Omega$ occurs, denoted $\boldsymbol{P(A)}$, is obtained from the following formula:

$$\boldsymbol{P(A) = \frac{\#A'}{\#\Omega'} = \frac{Number\ of\ combinations\ to\ obtain\ A}{Total\ number\ of\ combinations}}$$

Example

An urn containing 10 balls, where 3 are green (g), 2 are red (r) and 5 are blue (b). From this urn, we randomly pick a ball. The sample space for this random experiment is

$\boldsymbol{\Omega = \{g, r, b\}}$

The elements certainly do not have the same probability to occur.

Given the event $\boldsymbol{G}$ = "pick a green ball". The probability this event occurs is

$$P(G) = \frac{Number\ of\ green\ balls}{Total\ number\ of\ balls} = \frac{3}{10} = 0.3$$

However, given the event $\boldsymbol{R}$ = "pick a red ball". The probability that this event occurs is

$$P(R) = \frac{Number\ of\ red\ balls}{Total\ number\ of\ balls} = \frac{2}{10} = 0.2$$

Whatever the type of sample space, fundamental or not, it is possible to carry out operations on the events (union, intersection, difference, complement). The following section will discuss the effect these operations have on probabilities.

5. Probability properties

Given $\boldsymbol{A}$ and $\boldsymbol{B},$ any events. Then, the following properties must be satisfied:

1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. $P(\bar{A}) = 1 - P(A)$
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
5. $P(A \setminus B) = P(A) - P(A \cap B)$

We can deduce by combining properties 2 and 3 that the probability of obtaining the empty set is zero:

$$P(\emptyset) = 1 - P(\Omega) = 0.$$

Example

A student estimates at 65% his chances of succeeding his statistics course, at 80% his chances of succeeding his finance course and at 50% his chances of succeeding both subjects.

Let us denote $\boldsymbol{S}$ the event "succeed in Statistics", and $\boldsymbol{F}$ the event "succeed in Finance". Thus, we represent the probability that these events occur as

$$\boldsymbol{P}(\boldsymbol{S}) = 0{,}6$$

$$\boldsymbol{P}(\boldsymbol{F}) = 0{,}8$$

respectively. For the event "succeed both subjects", it is the intersection of events $\boldsymbol{S}$ and $\boldsymbol{F}$… Remember that the intersection represents the fact that both events occur. Thus, according to the student pronouncement

$$\boldsymbol{P}(\boldsymbol{S} \cap \boldsymbol{F}) = 0{,}5.$$

What is the probability that the student succeeds in Statistics but not in Finance?

Here, we must evaluate $\boldsymbol{P}(\boldsymbol{S}\backslash\boldsymbol{F})$. According to property 5, we find

$$\boldsymbol{P}(\boldsymbol{S}\backslash\boldsymbol{F}) = \boldsymbol{P}(\boldsymbol{S}) - \boldsymbol{P}(\boldsymbol{S}\cap\boldsymbol{F}) = 0{,}65 - 0{,}5 = 0{,}15.$$

What is the probability that the student succeeds in Finance but not in Statistics?

We want to evaluate $(\boldsymbol{F}\backslash\boldsymbol{S})$:

$$\boldsymbol{P}(\boldsymbol{F}\backslash\boldsymbol{S}) = \boldsymbol{P}(\boldsymbol{F}) - \boldsymbol{P}(\boldsymbol{F}\cap\boldsymbol{S}) = 0{,}8 - 0{,}5 = 0{,}3.$$

There is thus a 15% probability that the student succeeds in Statistics but fails in Finance. However, the probability that the opposite occurs is 30%.

What is the probability that the student succeeds in Finance or Statistics?

The event "succeed in Finance or Statistics” is represented by $\cup\,\boldsymbol{S}$. We therefore want to evaluate $\boldsymbol{P}(\boldsymbol{F}\cup\boldsymbol{S})$. Property 4 allows us to obtain the desired result:

$$\begin{aligned}\boldsymbol{P}(\boldsymbol{F}\cup\boldsymbol{S}) &= \boldsymbol{P}(\boldsymbol{F}) + \boldsymbol{P}(\boldsymbol{S}) - \boldsymbol{P}(\boldsymbol{F}\cap\boldsymbol{S})\\ &= 0{,}8 + 0{,}65 - 0{,}5\\ &= 0{,}95.\end{aligned}$$

There is therefore a 95% probability that the student succeeds one or the other.

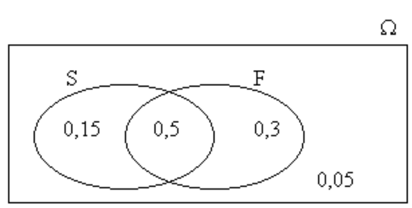
What is the probability that he fails both courses?

First, we must clearly identify the event "fail both subjects". We can calculate this using its complement, "succeed at least one of the subjects", that we just presented, and where

$$\begin{aligned}P(\overline{F\cup S}) &= 1 - P(F\cup S)\\ &= 1 - 0.95\\ &= 0.05.\end{aligned}$$

There is thus a 5% chance that the student fails both courses.

Below is a diagram summarizing the probabilities of the events presented above.



6. Conditional probabilities

Conditional probabilities come into play as soon as we are interested in the probability that an event **A** occurs, *knowing that another event $\boldsymbol{B}$ has occurred*. We denote this probability by $\boldsymbol{P(A \mid B)}$. This type of probability obliges us to consider $\boldsymbol{B}$ (rather than $\boldsymbol{\Omega}$ as the sample space in which we study the probability of $\boldsymbol{A}$.

For all $\boldsymbol{A}$ and $\boldsymbol{B}$, elements of $\boldsymbol{\Omega}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example

Let us return to the case presented in the last example.

What is the probability that the student succeeds in Statistics knowing that he has passed Finance?

We have to calculate $\boldsymbol{P(S \mid F)}$ **:**

$$P(S|F) = \frac{P(S \cap F)}{P(F)} = \frac{0.5}{0.8} = 0.625.$$

Note that the probability that $\boldsymbol{S}$ occurs, initially 0.65, is diminished when we presume the realization of $\boldsymbol{F}$. This calculation allows us to conclude that these events are not independent of each other, i.e. that the realization or non-realization of an event modifies the probability that the other is realized.

Example

An urn containing 10 balls of which 3 are green (g), 2 are red (r) and 5 are blue (b). From this urn we randomly pick a ball. The sample space of this experiment is

$$\boldsymbol{\Omega} = \{\boldsymbol{v}, \boldsymbol{r}, \boldsymbol{b}\}$$

Given the events les événements

$\boldsymbol{G}$ = "pick a green ball"

$\boldsymbol{R}$ = "pick a red ball"

$\boldsymbol{B}$ = "pick a blue ball"

Calculate the probability of picking a green ball, if you know that the ball you picked is either green or red.

$$P(G|G \cup R) = \frac{P(G)}{P(G \cup R)} = \frac{3/_{10}}{5/_{10}} = \frac{3}{5} = 0.6$$

7. Independent events

Given two events $\boldsymbol{A}$ and $\boldsymbol{B}$. We say of $\boldsymbol{A}$ and $\boldsymbol{B}$ that they are ***independent events*** if and only if

$$\boldsymbol{P}(\boldsymbol{A} \mid \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A})$$

This definition can be interpreted in the following manner: if the realization of event $\boldsymbol{B}$ does not modify the probability that event $\boldsymbol{A}$ occurs, then $\boldsymbol{A}$ and $\boldsymbol{B}$ are independent.

Example

Consider the experiment that consists of flipping a coin twice and noting the outcome. The sample space is

$$\boldsymbol{\Omega} = \{\mathbf{hh}, \mathbf{ht}, \mathbf{th}, \mathbf{tt}\}$$

What is the probability that the second flip gives the result *tails* knowing that the first flip was *heads*?

Note that by $H1$ the event "first flip gives heads" and by T2, the event "the second flip gives *tails* ". We are asked to evaluate P($T2|H1$):

$$P(T2|H1) = \frac{P(T2 \cap H1)}{P(H1)} = \frac{1/4}{2/4} = \frac{1}{2} = 0.5.$$

The probability of obtaining *tails* during the second flip is

$$P(T2) \ = \ ^2/_4 = 0.5.$$

Thus, $\boldsymbol{P(T2|H1) = P(T2)}$. The events $\boldsymbol{T2}$ and $\boldsymbol{H1}$ are therefore independent. We will leave you the task of verifying that this would not be the case for events $\boldsymbol{T2}$ and $\boldsymbol{H2}$, where $\boldsymbol{H2}$ represents the event "obtain *heads* during the second flip".

From the definition of independence, we obtain the following result concerning the intersection of events: if $\boldsymbol{A}$ and $\boldsymbol{B}$ are independent events, then $\boldsymbol{P(A \cap B) = P(A)P(B)}$**.**

Indeed, since $\boldsymbol{A}$ and $\boldsymbol{B}$ are independent by definition if and only if $\boldsymbol{P(A \mid B) = P(A)}$, we have

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ \Leftrightarrow \ P(A) &= \frac{P(A \cap B)}{P(B)} \\ \Leftrightarrow P(A \cap B) &= P(A)P(B) \end{aligned}$$