OPTIMIZATION UNDER CONSTRAINTS

Summary

Suppose that in a firm's production plan, it was determined that the level of production that maximizes profits is 100,000 units. Let us also suppose that the firm cannot surpass a production of 75,000 units because of resource constraints, manpower, etc. Finding the optimal level of production must be done while respecting the constraints. The present section will examine this problem.

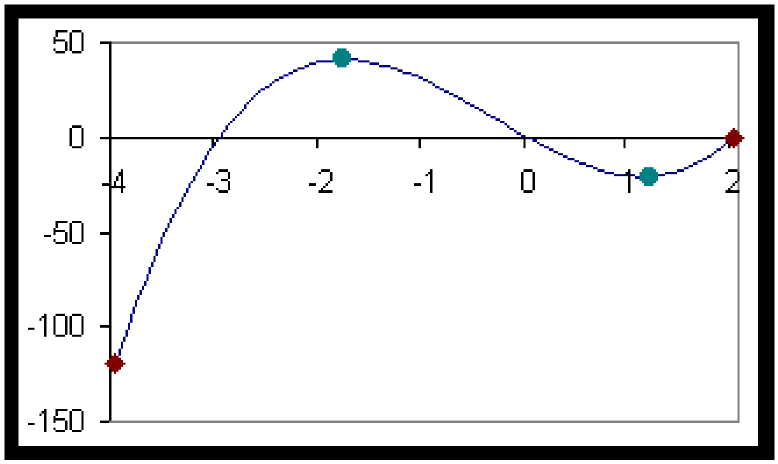
1. Optimization between limits

Theorem

Given a continuous function $f(x)$, defined at a closed interval $[a, b]$. Given x_{min}, the point where f reaches its absolute minimum on $[a, b]$, and x_{max}, the point where f reaches its absolute maximum on $[a, b]$. Thus, x_{min} *and* x_{max} will always be at one of the following points :

- stationary point;
- critical point;
- limit point.

The following graph illustrates the theorem. The described function is limited by the interval $[-4\,,2]$. It has two stationary points *(in green)* : a local maximum at $x\ = -1{,}8$ and a local minimum at $x = 1{,}2$. Note that the local maximum at $x\ = -1{,}8$ also constitutes the absolute maximum, it being the highest point of the function f on $[-4, 2]$. However, the absolute minimum of f on $[-4, 2]$ is not found at $x = 1{,}2$ but at the left limit point $x\ = -\ 4$.



The theorem restricts the possible optima at stationary, critical and limit points. Contrary to problems without constraints, it is not necessary to determine the nature of all stationary and critical points. The study is limited to comparing the value (height) of the function at these points to the value (height) at the function limits, then identifying those that are optimal.

Methodology *Optimization between two boundaries* $[\boldsymbol{a}\,,\boldsymbol{b}]$

- Calculate the first derivative ;
- Find all stationary and critical points of $[a, b]$;
- Evaluate $f(x)$ at stationary, critical and limit points ;
- Identify the absolute minimum and maximum on $[a, b]$.

Example

Find the absolute minimum and maximum of the function

$f(x) =\ x^3 - 12x\ +\ 9$ on the interval $[0, 3]$.

- *Calculate the first derivative;*

$$f'(x) = 3x^2 - 12$$

Find all stationary and critical points on $[0, 3]$*;*

We find the stationary points where

$$\begin{aligned} f'(x) = 0 \rightarrow 3x^2 - 12 &= 0 \\ 3(x^2 - 4) &= 0 \\ 3(x-2)(x+2) &= 0 \\ x &= \{-2,2\}. \end{aligned}$$

Yet since the limits constrain us to the interval $[0, 3]$, only the stationary point $x = 2$ is retained. The function has no critical points since the derivative is defined everywhere on $[0, 3]$.

- *Evaluate* $f(x)$ *at stationary, critical and limit points;*

Stationary point $= 2 : f\ (2) = -7$

Left limit $= 0 : f\ (0) = 9$

Right limit $= 3 : f\ (3) = 0$

- *Identify the absolute minimum and maximum on* $[0, 3]$*.*

Absolute maximum $f\ (0) = 9$

Absolute minimum $f\ (2) = -7$

2. Exercise

A company produces x racing bikes per year. Demand requires that the company makes at least 500 bikes a year. However, the amount of nits produced annually cannot exceed 1000.

a. The unit cost of production is modeled by the function

$$C_{unit}(x) = \frac{200x - \frac{x^2}{9} + 180{,}000}{x}.$$

How many bikes must be produced yearly to minimize the **unit cost**?

b. The sales price depends on the quantity sold, x, according to the function

$$P_{unit}(x) = 700 - \frac{4x}{9}$$

How many bikes must be sold every week to maximize the **total revenue**?

c. How many bikes must be sold weekly to maximize the **total profit (total revenue total cost**)?

3. Optimization under constraints with multiple variables

The function to optimize may often depend on many factors. For example, the profits made may depend on the cost of resources, the number of employees, the sales price. How does one optimize a function with many variables under constraints?

The difficulty resides in the fact that we are confronted with more than one variable. Solving such a problem requires us to use the substitution method, the result being the reduction of the amount of variables.

Substitution method

- Define the different variables $x_1, x_2, x_3, \ldots$;
- Write the objective in function to the variables ;
- Write all the constraints ;
- State, using the constraints, all the variables in terms of one in particular (for example $x_2 = f_1(x_1), x_3 = f_2(x_1), \ldots$);
- Substitute the expressions in the objective ;
- Optimize.

Example

The company Kola produces and distributes pop. The containers (cans) have a cylindrical form of height *h* and radius *r*. In order to reduce costs, Kola wants to minimize the aluminum surface necessary for the construction of the containers. However, they must ensure that a container has a volume of 128 cm^3. What are the dimensions of the container that minimize the objective while satisfying the constraint ?

- Define the different variables $x_1, x_2, x_3, \ldots$;

  r : radius of the cylinder ($r > 0$)

  h : height of the cylinder ($h > 0$)

- Write the objective in function in function of the unknowns (variables);

  Objective : minimize the surface of the container

$$Min\ S(r,h) = 2\pi r^2 + 2\pi rh$$

- Write all the constraints ;

  Constraint : volume = 128π

$$\pi r^2 h = 128\pi$$

- State, using the constraints, all the variables in terms of one variable (for example $x_2 = f_1(x_1), x_3 = f_2(x_1)$, ...);

  From the expression $\pi r^2 h = 128\pi$, it is easy to isolate the variable h:

$$h = \frac{128\pi}{\pi r^2} \Longrightarrow h = \frac{128}{r^2}$$

- Substitute these expressions in the objective ;

$$S(r,h) = 2\pi r^2 + 2\pi r h$$
$$S(r) = 2\pi r^2 + 2\pi r\left(\frac{128}{r^2}\right)$$
$$S(r) = 2\pi r^2 + \frac{256\pi}{r}$$

- Optimize.

  To optimize, we use the methods studied in the previous sections:

$$S'(r) = \left(2\pi r^2 + \frac{256\pi}{r}\right)' \Longrightarrow S'(r) = 4\pi r - \frac{256\pi}{r^2}$$

A stationary point is obtained when $'(r) = 0$:

$$4\pi r - \frac{256\pi}{r^2} = 0$$
$$4\pi r = \frac{256\pi}{r^2}$$
$$r^3 = \frac{256\pi}{4\pi}$$
$$r^3 = 64 \Longrightarrow r = 4$$

We verify that we to obtain a minimum at that point :

$$S''(r) = \left(4\pi r - \frac{256\pi}{r^2}\right) \Longrightarrow S''(r) = 4\pi + \frac{512\pi}{r^3}$$

For $r > 0$, the second derivative is always positive. The function S is therefore always convex, implying that the stationary point $r = 4$ constitutes an absolute minimum. The corresponding value of h is obtained with

$$h = \frac{128}{r^2} = \frac{128}{4^2} = \frac{128}{16} = 8$$

The surface is minimized when $r = 4$ and $h = 8$.

The resolution by substitution remains a very efficient method, even in problems with more than two variables.

Example

With exactly 2700 cm^2 of cardboard, we wish to construct a box (width x, depth y, height z) that can contain a volume V. We require the width to be double its depth. We would like to maximize the volume the box can hold. Which values of x, y, z fulfill our objective.

- Define the different variables $x_1, x_2, x_3, \dots$;

  x : width of box ($x > 0$)

  y : depth of box ($y > 0$)

  z : height of box ($z > 0$)

- Write the objective in terms of the variables ;

  Objective : maximize the box volume

$$Max\, V(x, y, z) = xyz$$

- Write all the constraints ;

1. Material surface available 2700 m^2

$$2xy + 2yz + 2zx = 2700$$

2. Required dimensions

$$x = 2y$$

- State, using the constraints, all the variables in terms of one in particular (for example $x_2 = f_1(x_1), x_3 = f_2(x_1), \dots$);

  The variable x is stated in terms of y in the constraint 2 : $x = 2y$

  By substituting this expression in constraint 1, we can also state z in terms of :

$$2xy + 2yz + 2zx = 2700$$
$$\rightarrow 2(2y)y + 2yz + 2z(2y) = 2700$$

- $4y^2 + 6zy = 2700$

$$\rightarrow z = \frac{2700 - 4y^2}{6y}$$

  Substitute these expressions in the objective function;

$$V(x, y, z) = xyz$$
$$V(y) = 2y \cdot y \cdot \left(\frac{2700 - 4y^2}{6y}\right)$$
$$V(y) = \frac{1}{3}y(2700 - 4y^2)$$
$$V(y) = 900y - \frac{4}{3}y^3$$

- Optimize.

  Let us find the stationary points :

$$V'(y) = \left(900y - \frac{4}{3}y^3\right)' = 900 - 4y^2$$

A stationary point is obtained when $V'(y) = 0$:

$$900 - 4y^2 = 0$$

$$(30 - 2y)(30 + 2y) = 0 \qquad (diffenrece\ of\ squares)$$

$$\Longrightarrow y = \{-15, 15\}$$

Yet, only the value $y = 15$ is acceptable since the dimensions must be positive. We must be sure to find a maximum at this point:

$$V''(y) = (900 - 4y^2)' = -8y$$

On the domain $y > 0$, the second derivative is always negative. The function V is therefore always concave, implying that the stationary point $y = 15$ constitutes an absolute maximum. The corresponding values of x and z are

$$x = 2y = 2(15) = 30$$
$$z = \frac{2700 - 4y^2}{6y} = \frac{2700 - 4(15)^2}{6(15)} = \frac{2700 - 900}{90} = 20$$

The volume of the box is therefore maximized when the dimensions are

$$x = 30, y = 15, z = 20.$$