

## MATRIX DETERMINANTS

### Summary

Uses .....	1
1- Reminder - Definition and components of a matrix.....	1
2- The matrix determinant .....	2
3- Calculation of the determinant for a $2 \times 2$ matrix .....	2
4- Exercise.....	3
5- Definition of a minor .....	3
6- Definition of a cofactor.....	4
7- Cofactor expansion – a method to calculate the determinant .....	4
8- Calculate the determinant for a $3 \times 3$ matrix .....	5
9- Alternative method to calculate determinants.....	6
10- Exercise .....	7
11- Determinants of square matrices of dimensions 4x4 and greater .....	8

### Uses

The determinant will be an essential tool to identify the maximum and minimum points or the saddle points of a function with multiple variables.

### 1- Reminder - Definition and components of a matrix

A matrix is a rectangular table of form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

A matrix is said to be of dimension  $m \times n$  when it has  $m$  rows and  $n$  columns. This method of describing the size of a matrix is necessary in order to avoid all confusion

between two matrices containing the same amount of entries. For example, a matrix of dimension  $3 \times 4$  has 3 rows and 4 columns. It would be distinct from a matrix  $4 \times 3$ , that has 4 rows and 3 columns, even if it also has 12 entries. A matrix is said to be **square** when it has the same number of rows and columns.

The elements are matrix entries  $a_{ij}$ , that are identified by their position. The element  $a_{32}$  would be the entry located on the third row and the second column of matrix  $A$ . This notation is essential in order to distinguish the elements of the matrix. The element  $a_{23}$ , distinct from  $a_{32}$ , is situated on the second row and the third column of the matrix  $A$ .

## **2- The matrix determinant**

A value called the determinant of  $A$ , that we denote by

$$\det(A) \text{ or } |A|,$$

corresponds to every square matrix  $A$ . We will avoid the formal definition of the determinant (that implies notions of permutations) for now and we will concentrate instead on its calculation.

## **3- Calculation of the determinant for a $2 \times 2$ matrix**

Let us consider the matrix  $A$  of dimension  $2 \times 2$  :

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The determinant of the matrix  $A$  is defined by the relation

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

The result is obtained by multiplying opposite elements and by calculating the difference between these two products.... a recipe that you will need to remember!

### Example

Given the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$$

The determinant of A is

$$\det(A) = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}$$

### 4- Exercise

Calculate the determinant of the following  $2 \times 2$  matrices :

$$a. \begin{pmatrix} 1 & 3 \\ 5 & -2 \end{pmatrix}$$

$$b. \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$$

$$c. \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$$

$$d. \begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}$$

Solutions : a) -17 b) 0 c) 5 d) 11

Before being able to evaluate the determinant of a  $3 \times 3$  matrix (or all other matrices of a greater dimension), you will first need to learn a few concepts...

### 5- Definition of a minor

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 5 & 2 & 3 \\ 8 & 7 & 3 \end{pmatrix}$$

The minor  $M_{12}$  is the determinant of the matrix obtained by eliminating the first row and the second column of  $A$ , i.e.

$$M_{12} = \begin{vmatrix} 5 & 3 \\ 8 & 3 \end{vmatrix} = 5.3 - 3.8 = 15 - 24 = -9$$

The minor  $M_{22}$  is the determinant of the matrix obtained by eliminating the second row and the second column of  $A$ , i.e.

$$M_{22} = \begin{vmatrix} 2 & 4 \\ 8 & 3 \end{vmatrix} = 2.3 - 4.8 = 6 - 32 = -26$$

## 6- Definition of a cofactor

The cofactor,  $C_{ij}$ , of a matrix  $A$  is defined by the relation

$$C_{ij} = (-1)^{i+j}M_{ij}$$

You will notice that the cofactor and the minor always have the same numerical value, with the possible exception of their sign.

Let us again consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 5 & 2 & 3 \\ 8 & 7 & 3 \end{pmatrix}$$

We have already shown that the minor  $M_{12} = -9$ . Thus the corresponding cofactor,  $C_{12}$ , is

$$C_{12} = (-1)^{1+2}M_{12} = -1.(-9) = 9$$

The minor  $M_{12}$  and the cofactor  $C_{12}$  are of different signs.

The minor  $M_{22} = -26$ . Its corresponding cofactor  $C_{22}$  is

$$C_{22} = (-1)^{2+2}M_{22} = 1.(-26) = -26$$

This time, the minor  $M_{22}$  and the cofactor  $C_{22}$  are identical.

Evaluating the determinant of a  $3 \times 3$  matrix is now possible. We will proceed by reducing it in a series of  $2 \times 2$  determinants, for which the calculation is much easier. This process is called an **cofactor expansion**.

## 7- Cofactor expansion - a method to calculate the determinant

Given a square matrix  $A$  and its cofactors  $C_{ij}$ . The determinant is obtained by cofactor expansion as follows:

- Choose a row or a column of  $A$  (if possible, it is faster to choose the row or column containing the most zeros)...
- Multiply each of the elements  $a_{ij}$  of the row (or column) chosen by its corresponding cofactor,  $C_{ij}$ ...
- Add these results.

## 8- Calculate the determinant for a $3 \times 3$ matrix

For a  $3 \times 3$  matrix, this would mean that by choosing to make an expansion along the first row, the determinant would be

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

If we had chosen to carry out an expansion along the second column, we would have to calculate

$$\det A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

While the choice of row or column may differ, the result of the determinant will be the same, no matter what the choice we have made. Let us verify this with an example.

### Example

What is the determinant of matrix  $A$ ?

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 0 & -2 \end{pmatrix}$$

### Solution

Let us follow the procedure proposed above (cofactor expansion):

- Choose a row or a column of  $A$ ... For now, let us choose the first row.
- Multiply each of the elements of this row by their corresponding cofactors... The elements of the first row are  $a_{11} = 2$ ,  $a_{12} = 1$ , et  $a_{13} = 3$  that we multiply with the corresponding cofactors, i.e.  $C_{11}$ ,  $C_{12}$  et  $C_{13}$ . These are

$$C_{11} = (-1)^{1+1}M_{11} = 1 \begin{vmatrix} 0 & 2 \\ 0 & -2 \end{vmatrix} = 1(0 \cdot (-2) - 2 \cdot 0) = 0$$

$$C_{12} = (-1)^{1+2}M_{12} = (-1) \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 1(1 \times (-2) - 2 \times 2) = 6$$

$$C_{13} = (-1)^{1+3}M_{13} = 1 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 1(1 \times (0) - 2 \times 0) = 0$$

Finally, we need to calculate

$$\det A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$\det A = 2 \times 0 + 6 \times 1 + 3 \times 0 = 6$$

Let us verify if an expansion along the second column coincides with the previous result. Note that the choice of the second column is much more effective since the determinant will be obtained from the calculation

$$\det A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

Two of the three elements of the second column are zero. In effect,  $a_{12} = 1, a_{22} = 0, \text{ and } a_{32} = 0$ . It is thus useless to calculate the cofactors  $C_{22}$  and  $C_{32}$ . The corresponding cofactor for  $a_{12}$  is

$$C_{12} = (-1)^{1+2}M_{12} = (-1) \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 1(1 \times (-2) - 2 \times 2) = 6$$

The determinant of  $A$  is thus

$$\det A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = 1 \times 6 + 0 \times C_{22} + 0 \times C_{32} = 6,$$

which corresponds to the answer obtained by an expansion along the first row.

## **9- Alternative method to calculate determinants**

This second method is in all points equivalent to cofactor expansion but will allow you to avoid the use of cofactors.

- Allocate a sign  $+/-$  to each element by following the rule: we associate a positive sign to the position  $a_{11}$ , then we alternate the signs by moving horizontally or vertically.
- Choose a row or column of  $A$  (if possible, it is faster to choose the row or column of  $A$  containing the most number of zeros)...
- Multiply each element of  $a_{ij}$  of the row (or column) chosen by its corresponding minor, i.e. the remaining determinant when we eliminate the row and column in which  $a_{ij}$  is.
- Add or subtract these results according to the sign allocated to the elements during the first step.

Let us verify that this method will produce the same result as in the previous example:

### Example

Given the matrix  $A$  to which we allocate a sign  $+/-$  according to the rule stated above.

$$A = \begin{pmatrix} 2^+ & 1^- & 3^+ \\ 1^- & 0^+ & 2^- \\ 2^+ & 0^- & -2^+ \end{pmatrix}$$

- Let us choose the third column (it is certainly not the best choice since the second row has the most zeros, but...)
- We then multiply each element by its corresponding minor:

$$3 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 3 \times 0 = 0$$

$$2 \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2 \times (-2) = -4$$

$$-2 \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -2 \times (-1) = 2$$

- Finally, the respective signs of the elements of the third column tell us the operations to carry out between these values to obtain the determinant:

$$\det A = +0 - (-4) + 2 = 6$$

## 10- Exercise

Calculate the determinant of the following matrices:

$$a) \begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 2 & 0 \end{pmatrix}$$

$$b) \begin{pmatrix} 1 & 0 & 2 \\ 1 & 3 & 4 \\ 0 & 6 & 0 \end{pmatrix}$$

$$c) \begin{pmatrix} 3 & -2 & 4 \\ 2 & -4 & 5 \\ 1 & 8 & 2 \end{pmatrix}$$

$$d) \begin{pmatrix} 8 & -1 & 9 \\ 3 & 1 & 8 \\ 11 & 0 & 17 \end{pmatrix}$$

Solutions : a) 24    b) -12    c) -66    d) 0

## 11- Determinants of square matrices of dimensions 4x4 and greater

The methods presented for the case of  $3 \times 3$  matrices remain valid for all greater dimensions. You must again follow the steps for cofactor expansion:

Given a square matrix  $A$  and its cofactors  $C_{ij}$ , the determinant is obtained by following a cofactor expansion as follows:

- Chose a row or column of  $A$  (if possible, it is faster to choose the row or column that contains the most zeros) ...
- Multiply each of the elements  $a_{ij}$  of the row (or column) chosen, by the corresponding cofactor  $C_{ij}$ ...
- Add the results.

We must however mention a distinction. The cofactor associated to the element  $a_{ij}$  of a  $4 \times 4$  matrix is the determinant of a  $3 \times 3$  matrix, since it is obtained by eliminating the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column of  $A$ .

### **Example**

Calculate the determinant of matrix  $A$

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ -1 & 0 & 3 & 1 \\ 3 & 1 & 2 & 0 \end{pmatrix}$$

It is essential, to reduce the amount of calculations, to choose the row or column that contains the most zeros (here, the fourth column). We will proceed to a cofactor expansion along the fourth column, which means that

$$\det A = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44}$$

As  $a_{14}$  and  $a_{44}$  are zero, it is useless to find  $C_{14}$  and  $C_{44}$ . The cofactors  $C_{24}$  and  $C_{34}$  will be necessary...

$$C_{24} = (-1)^{2+4}M_{24} = 1 \begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$



$$C_{34} = (-1)^{3+4}M_{34} = -1 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

We let the reader verify that  $C_{24} = 18$  et  $C_{34} = -2$ . Consequently, the determinant of  $A$  is

$$\det A = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44}$$

$$\det A = 0 \times C_{14} + 1 \times 18 + 1 \times (-2) + 0 \times C_{44} = 16$$

### ***Exercise***

Show that the determinant of  $A$  in the previous example is 16 by a cofactor expansion along

- a) The first row
- b) The third column