PLOTTING THE GRAPH OF A FUNCTION

Summary

By combining the concepts of the first and second derivatives, it is now possible to plot the graph of a function with staggering precision : the first derivative represents the slope of a function and allows us to determine its rate of change; the stationary and critical points allow us to obtain local (or absolute) minima and maxima; the second derivative describes the curvature of the function.

It is crucial to not confuse the characteristics unveiled by the functions f, f', f''.

$f(x)$ → value (height) of the function at point x $(positive, negative)$
$f'(x)$ → slope of the function at point x $(increasing, decreasing)$
$f''^{(x)}$ → curvature of the function at point x $(concave, convex)$

We suggest the following methodology in order to plot the graph of a function.

1. Methodology : how to plot a graph of a function

- Calculate the first derivative ;
- Find all stationary and critical points ;
- Calculate the second derivative ;
- Find all points where the second derivative is zero;
- Create a table of variation by identifying:

1. The value of the function at the stationary and critical points and the points where the second derivative is zero (inflection points) ;
2. All intervals between and around the points mentioned in 1 ;
3. Whether the function is increasing/decreasing between the stationary and critical points ;
4. The concavity/convexity between the points where the second derivative is zero or does not exist ;
5. The local minima and maxima.

- Use the table to plot the graph.

We will use two examples from the previous sections to illustrate the process :

Example 1

Plot the graph of the function $f(x) = 2x^3 - 3x^2 - 12x + 4$.

1. *Calculate the first derivative of $f(x)$* ***;***

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 12 \\ &= 6(x^2 - x - 2) \end{aligned}$$

2. *Find all stationary and critical points ;*

   We obtain a stationary point when $f'(x) = 0$.

$$\begin{aligned} 6(x^2 - x - 2) &= 0 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \Rightarrow x = \{-1, 2\} \end{aligned}$$

   There are thus two stationary points ($x = \{-1, 2\}$). There is however no critical point since the derivative is well defined for all x.

3. *Calculate the second derivative of the function $f(x)$;*

$$\begin{aligned} f''(x) &= (6x^2 - 6x - 12)' \\ &= 12x - 6 \end{aligned}$$

4. *Find all points where the second derivative is zero or does not exist ;*

   The second derivative is zero when

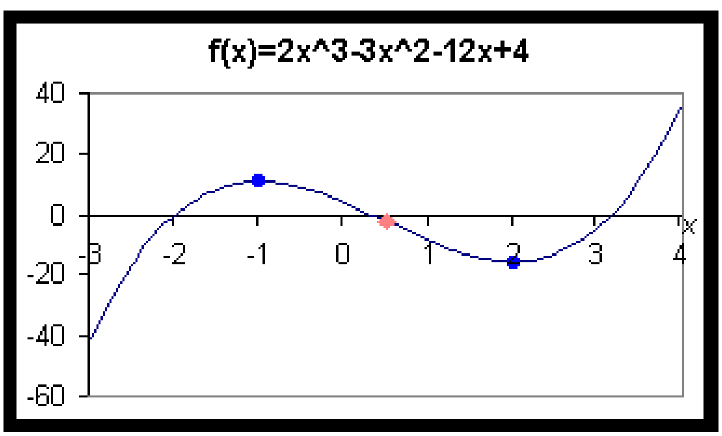
$$\begin{aligned} 12x - 6 &= 0 \\ 12x &= 6 \Longrightarrow x = {}^{1}/_{2} \end{aligned}$$

Create a table of variations by identifying :

1. *The value of the function at the stationary and critical points and the points where the second derivative cancels itself out or does not exist ;*
2. *All intervals between and around the points mentioned in 1 ;*

3. *Whether the function is increasing/decreasing between the stationary and critical points*
4. *The concavity/convexity between the points where the second derivative is zero or does not exist*
5. *The local minima and maxima.*

x	$]-\infty,-1[$	-1	$]-1,{}^{1}/_{2}[$	$1/2$	$]1/2,2[$	2	$]2,\infty[$
$f(x)$		11		- 2,5		-16	
$f'(x)$	+	0	-	-	-	0	+
$f''(x)$	-	-	-	0	+	+	+
↓ or ↑ **∪ or ∩**	↑ ∩	stat.pt. ∩ **Max**	↓ ∩	↓ **change** **of** **curvature**	↓ ∪	stat.pt. ∪ **Min**	↑ ∪



Example

Find all local optima of the function $f(x) = x^{1/3} \cdot (x+1)$

1. *Calculate the first derivative of* $\boldsymbol{f(x)}$ ***;***

$$
\begin{aligned}
f'(x) &= \left(x^{1/3}\right)' \cdot (x+1) + x^{1/3} \cdot (x+1)' && (product\ rule) \\
&= \frac{1}{3} x^{-2/3} \cdot (x+1) + x^{1/3} \cdot 1 \\
&= \frac{x+1}{3x^{2/3}} + x^{1/3} \\
&= \frac{x+1}{3x^{2/3}} + \frac{x^{1/3} \cdot 3x^{2/3}}{3x^{2/3}} && (common\ denominator) \\
&= \frac{x+1+3x}{3x^{2/3}} \\
&= \frac{1+4x}{3x^{2/3}}
\end{aligned}
$$

2. *Find all stationary and critical points ;*

   We obtain a stationary point when $f'(x) = 0$.

   This is obtained when the numerator is zero : $1 + 4x = 0$.

   Therefore, $x = -1/4 = -0{,}25$ is a stationary point.

   A critical point is obtained when $f'(x)$ is not defined. Since the denominator is zero when $x = 0$, this is a critical point.

3. *Calculate the second derivative ;*

$$
\begin{aligned}
f'(x) &= \frac{(1+4x)' \cdot \left(3x^{2/3}\right) - (1+4x) \cdot \left(3x^{2/3}\right)'}{\left(3x^{2/3}\right)^2} && quotient\ rule \\
&= \frac{4 \cdot 3x^{2/3} - (1+4x) \cdot 2x^{-1/3}}{9x^{4/3}} \\
&= \frac{1}{9x^{4/3}} \cdot \left(12x^{2/3} - \frac{2 \cdot (1+4x)}{x^{1/3}}\right)
\end{aligned}
$$

$$= \frac{1}{9x^{4/3}} \cdot \left(\frac{x^{1/3} \cdot 12x^{2/3}}{x^{1/3}} - \frac{2 \cdot (1 + 4x)}{x^{1/3}} \right) \qquad (common\ denominator)$$
$$= \frac{1}{9x^{4/3}} \cdot \left(\frac{12x - (2 + 8x)}{x^{1/3}} \right)$$
$$= \frac{4x - 2}{9x^{5/3}}$$

4. *Find all points where the second derivative is zero or does not exist;*

   The derivative is zero when the denominator is zero:

$$4x - 2 = 0 \rightarrow x = 1/2$$

   The second derivative does not exist when the denominator is zero

$$9x^{5/3} = 0 \rightarrow x = 0$$

   This point had already been identified as a critical point. Beware, the sign of the second derivative will change at that point since the exponents are odd.

Create a table of variations by identifying :

1. *The value of the function at the stationary and critical points and the points where the second derivative is zero or does not exist ;*
2. *All intervals between and around the points mentioned in 1 ;*
3. *Whether the function is increasing/decreasing between the stationary and critical points*
4. *The concavity/convexity between the points where the second derivative is zero or does not exist*

The local minima and maxima.

x	**x□□ 1/4**	**□ 1/4**	**]□ 1/4 ,0[**	**0**	**]0,1/2[**	**1/2**	**] 1/2, ∞ [**
$f(x)$		- 0,4725		0		1,1906	
$f\,'(x)$	-	0	+	Not defined	+	0	+
$f\,''(x)$	+	+	+	Not defined	-	0	+
↓ or ↑ **∪ or ∩**	↓ ∪	***Stat pt.*** ∪ **Min**	↑ ∪	↑ **changes** **curvature**	↑ ∩	↑ **changes** **curvature**	↑ ∪