HEC MONTRÉAL

Institut
d'économie appliquée

Noisy Signaling in Monopoly

Leonard J. MIRMAN
Marc SANTUGINI

Cahier de recherche n° IEA-11-03

Version révisée
Janvier 2012

Institut d'économie appliquée
HEC Montréal
3000 chemin de la Côte-Sainte-Catherine
Montréal (Québec) H3T 2A7
Canada
http://www.hec.ca/iea
iea.info@hec.ca

ISSN : 0825-8643

Noisy Signaling in Monopoly

Leonard J. Mirman* Marc Santugini†

January 6, 2012

Abstract

We provide a closed-form solution of the monopoly problem when the price imperfectly signals quality to the uninformed buyers, as well as expressions for the effects of noise on output, price, and information flows.

Keywords: Asymmetric information, Learning, Monopoly, Noise, Quality, Signaling.
JEL Classifications: D21, D42, D82, D83, D84, L12, L15.

*Department of Economics, University of Virginia. Email: lm8h@virginia.edu.
†Corresponding author. Institute of Applied Economics and CIRPÉE, HEC Montréal. Email: marc.santugini@hec.ca.

1 Introduction

We consider the signaling role of prices in a static monopoly in which a fraction of buyers are uninformed. The signaling role of prices has previously been studied in a static noiseless environment (Bagwell and Riordan, 1991; Daughety and Reinganum, 1995, 2005, 2007, 2008a,b; Janssen and Roy, 2010; Mirman and Santugini, 2011). We provide a closed-form solution of the monopoly problem when the price imperfectly signals quality to the uninformed buyers, as well as expressions for the effects of noise on output, price, and information flows. Noise removes the necessity to define out-of-equilibrium beliefs as shown in Matthews and Mirman (1983) in the context of a limit pricing model. Judd and Riordan (1994) also studies the behavior of a noisy monopoly (using normal distributions) in a dynamic context in which buyers learn from observing the price as well as experience.

2 Model

Consider a market for a homogeneous good of quality θ sold at price p. There are both *informed* and *uninformed* price-taking buyers. The informed buyers know θ and have demand $q_I^d = \theta - p$, The uninformed buyers do not know θ, and have prior beliefs $\tilde{\theta} \sim N\left(\mu_\theta, \sigma_\theta^2\right), \mu_\theta > 0$ with the corresponding p.d.f. $\xi(\theta)$. The uninformed buyers extract information about θ from observing the price, using Bayes' rule to update beliefs. Let $\hat{\xi}(\theta|p)$ be the posterior p.d.f. of $\tilde{\theta}$ given p. Since the only difference between informed and uninformed buyers concerns information, the demand of the uninformed buyers is $q_U^d = \hat{\mu}_\theta(p) - p$, where $\hat{\mu}_\theta(p) = \int_{\mathbb{R}} x\hat{\xi}(x|p)\mathrm{d}x$ is the posterior mean for average quality. Normalizing the mass of buyers to one and letting $\lambda \in [0,1]$ be the fraction of informed buyers, aggregate demand is

$$D(p, \theta, \xi(x|p), \eta) = \lambda(\theta - p) + (1-\lambda)\left(\int_{\mathbb{R}} x\hat{\xi}(x|p)\mathrm{d}x - p\right) + \eta, \qquad (1)$$

where η is a demand shock and a realization of $\tilde{\eta} \sim N\left(0, \sigma_\eta^2\right)$, which is unknown to all buyers.[1]

The monopolist knows the average quality θ, but faces uncertainty in demand due to the demand shock η. The monopolist supplies q units of the good to maximize expected profit

$$\mathbb{E}_{\tilde{\eta}} P\left(\theta, q, \tilde{\eta}, \hat{\xi}(\cdot)\right) q, \tag{2}$$

where $\mathbb{E}_{\tilde{\eta}}$ is the expectation operator for the random variable $\tilde{\eta}$ and $P\left(\theta, q, \tilde{\eta}, \hat{\xi}(\cdot)\right)$ is the inverse random demand function.[2]

3 Equilibrium

Equilibrium consists of the monopolist's strategy $q^*(\theta)$, the distribution of the price signal $\phi^*(p|\theta)$, and the uninformed buyers' posterior beliefs about the average quality upon observing the price, $\hat{\xi}^*(\cdot)$.

Definition 3.1. *The tuple* $\left\{q^*(\theta), \phi^*(p|\theta), \hat{\xi}^*(\cdot)\right\}$ *is a noisy signaling equilibrium if, for all* θ*,*

1. *Given* $\hat{\xi}^*(\cdot)$*,*
$$q^*(\theta) = \arg\max_{q \geq 0} \mathbb{E}_{\tilde{\eta}} P\left(\theta, q, \tilde{\eta}, \hat{\xi}^*(\cdot)\right) q. \tag{3}$$

2. *Given* $q^*(\theta)$ *and* $\xi^*(\cdot)$*,*
$$\tilde{p}^*(\theta) = P\left(\theta, q^*(\theta), \tilde{\eta}, \xi^*(\cdot)\right). \tag{4}$$

   is the random price signal with the corresponding p.d.f. $\phi^*(p|\theta)$*.*

[1] An alternative specification yielding (1) is that quality is random, i.e., $\theta + \tilde{\eta}$, where θ is the average quality (unknown to the uninformed buyers), and η is a shock in quality that is unknown to all buyers at the time of purchasing the good.

[2] The same analysis can be performed for a price-setting monopoly, although noise has to be introduced in the cost, i.e., a cost parameter is known to the monopolist, but unknown to the buyers, which renders the price a noisy signal of quality.

3. *Given $\phi^*(p|\theta)$ and prior beliefs, the uninformed buyers' posterior beliefs is*

$$\xi^*(\theta|p) = \frac{\xi(\theta)\phi^*(p|\theta)}{\int_{\mathbb{R}} \xi(x)\phi^*(p|x)\mathrm{d}x}, \tag{5}$$

according to Bayes' rule.

Proposition 3.2 characterizes the noisy signaling equilibrium.

Proposition 3.2. *In equilibrium, the monopolist sells*

$$q^*(\theta) = \frac{\lambda\theta}{2} + \frac{2(1-\lambda)\sigma_\eta^2\mu_\theta}{4\sigma_\eta^2 + \lambda\sigma_\theta^2}. \tag{6}$$

The distribution of the price signal is $\tilde{p}^*(\theta) \sim N(\mu_p, \sigma_p^2)$, where

$$\mu_p = \frac{4\sigma_\eta^2 + (2-\lambda)\lambda\sigma_\theta^2}{4\sigma_\eta^2 + \lambda^2\sigma_\theta^2} q^*(\theta), \tag{7}$$

$$\sigma_p^2 = \frac{\left(4\sigma_\eta^2 + (2-\lambda)\lambda\sigma_\theta^2\right)^2}{\left(4\sigma_\eta^2 + \lambda^2\sigma_\theta^2\right)^2}\sigma_\eta^2. \tag{8}$$

The uninformed buyers' posterior beliefs are $\tilde{\theta}\Big| p \sim N(\hat{\mu}_\theta(p), \hat{\sigma}_p^2)$, where

$$\hat{\mu}_\theta(p) = \frac{4\sigma_\eta^2\mu_\theta}{4\sigma_\eta^2 + \lambda\sigma_\theta^2} + \frac{2\lambda\sigma_\theta^2 p}{4\sigma_\eta^2 + (2-\lambda)\lambda\sigma_\theta^2}, \tag{9}$$

$$\hat{\sigma}_\theta^2 = \frac{4\sigma_\eta^2\sigma_\theta^2}{4\sigma_\eta^2 + \lambda^2\sigma_\theta^2}. \tag{10}$$

Proof. Given (9), plugging $\hat{\mu}_\theta(p) = \int_{\mathbb{R}} x\hat{\xi}^*(x|p)\mathrm{d}x = a + bp$ into (1) and solving for the price as a function of q yields the *revised* random inverse demand

$$P\left(\theta, q, \tilde{\eta}, \hat{\xi}^*(\cdot)\right) = \frac{\lambda\theta + (1-\lambda)a + \tilde{\eta} - q}{1 - (1-\lambda)b}. \tag{11}$$

Here,

$$a = \frac{4\sigma_\eta^2 \mu_\theta}{4\sigma_\eta^2 + \lambda\sigma_\theta^2}, \tag{12}$$

$$b = \frac{2\lambda\sigma_\theta^2}{4\sigma_\eta^2 + (2-\lambda)\lambda\sigma_\theta^2}. \tag{13}$$

Given (11), the first-order condition corresponding to $\max_{q \geq 0} \mathbb{E}_{\tilde{\eta}} P\left(\theta, q, \tilde{\eta}, \hat{\xi}^*(\cdot)\right) q$ is $(\lambda\theta + (1-\lambda)a - 2q)/(1-(1-\lambda)b) = 0$, which yields (6).

Plugging (6), (12), and (13) into (11) yields the price signal

$$\tilde{p}^*(\theta) = \frac{\lambda\theta + (1-\lambda)a + \tilde{\eta} - q^*(\theta)}{1-(1-\lambda)b}, \tag{14}$$

$$= \frac{4\sigma_\eta^2 + (2-\lambda)\lambda\sigma_\theta^2}{4\sigma_\eta^2 + \lambda^2\sigma_\theta^2}(q^*(\theta) + \tilde{\eta}). \tag{15}$$

Since $\tilde{\eta} \sim N(0, \sigma_\eta^2)$, the distribution of the price signal is normal with mean and variance (7) and (8), respectively.

Given the normal distribution of the price with mean and variance (7) and (8), respectively, and prior beliefs $\tilde{\theta} \sim N(\mu_\theta, \sigma_\theta^2)$, the uninformed buyers' posterior beliefs is $\tilde{\theta}|p \sim N(\hat{\mu}_\theta(p), \hat{\sigma}_\theta^2)$, where $\hat{\mu}_\theta(p)$ and $\hat{\sigma}_\theta^2$ are defined by (9) and (10), respectively.[3] $\square$

[3] Note that expression (15) can be rewritten as $\tilde{p} = \frac{\lambda\theta/2 + (1-\lambda)a/2 + \tilde{\eta}}{1-(1-\lambda)b}$. Let $\tilde{z} \equiv 2(1-(1-\lambda)b)\tilde{p}/\lambda - (1-\lambda)a/\lambda = \tilde{\theta} + 2\eta/\lambda$, so that $\tilde{z}|\,\tilde{\theta} \sim N(\tilde{\theta}, 4\sigma_\eta^2/\lambda^2)$. Given that $\tilde{\theta} \sim N(\mu_\theta, \sigma_\theta^2)$, the posterior distribution upon observing $z = 2(1-(1-\lambda)b)p/\lambda - (1-\lambda)a/\lambda$ is

$$\tilde{\theta}\Big|\, z \sim N\left(\frac{\frac{4\sigma_\eta^2}{\lambda^2}\mu_\theta + \sigma_\theta^2(2(1-(1-\lambda)b)p/\lambda - (1-\lambda)a/\lambda)}{\frac{4\sigma_\eta^2}{\lambda^2} + \sigma_\theta^2}, \frac{1}{\frac{1}{\frac{4\sigma_\eta^2}{\lambda^2}} + \frac{1}{\sigma_\theta^2}}\right). \tag{16}$$

Equating the posterior mean defined by (16) to $a + bp$ and solving for a and b confirms (12) and (13).

4 The Effect of Noise

We next study the effect of noise on information flows by comparing the noisy and noiseless outcomes. Let

$$\psi_q = q^*(\theta)\big|_{\sigma_\eta^2>0} - q^*(\theta)\big|_{\sigma_\eta^2=0}\,, \tag{17}$$

$$\psi_{\mu_p} = \mu_p\big|_{\sigma_\eta^2>0} - \mu_p\big|_{\sigma_\eta^2=0}\,, \tag{18}$$

$$\psi_{\sigma_p^2} = \sigma_p^2\big|_{\sigma_\eta^2>0} - \sigma_p^2\big|_{\sigma_\eta^2=0}\,, \tag{19}$$

$$\psi_{\hat{\mu}_\theta^*} = \int_{\mathbb{R}} \hat{\mu}_\theta(p)\phi^*(p)\mathrm{d}p\bigg|_{\sigma_\eta^2>0} - \int_{\mathbb{R}} \hat{\mu}_\theta(p)\phi^*(p)\mathrm{d}p\bigg|_{\sigma_\eta^2=0}\,, \tag{20}$$

$$\psi_{\hat{\sigma}_\theta^2} = \hat{\sigma}_\theta^2\big|_{\sigma_\eta^2>0} - \hat{\sigma}_\theta^2\big|_{\sigma_\eta^2=0} \tag{21}$$

be the effects of noise on output, the mean and variance of the equilibrium price, and the posterior mean and variance of quality (evaluated at the equilibrium mean price), respectively.

Proposition 4.1 provides the effects of noise on output and the price distribution. The effect of noise is to unambiguously increase output. However, noise may increase or decrease the mean price depending on the bias of the prior relative to the true quality. In particular, if the prior mean is much greater than the true quality, then noise increases the mean price. But, if prior beliefs are unbiased, i.e., $\mu_\theta = \theta$, then the mean price decreases with noise.

Proposition 4.1. *From Proposition 3.2, and using* (17)*,* (18)*, and* (19)*,*

$$\psi_q = \frac{2(1-\lambda)\sigma_\eta^2\mu_\theta}{4\sigma_\eta^2 + \lambda\sigma_\theta^2} > 0. \tag{22}$$

Moreover,

$$\psi_{\mu_p} = \frac{4\sigma_\eta^2 + (2-\lambda)\lambda\sigma_\theta^2}{4\sigma_\eta^2 + \lambda^2\sigma_\theta^2}\left(\frac{\lambda\theta}{2} + \frac{2(1-\lambda)\sigma_\eta^2\mu_\theta}{4\sigma_\eta^2 + \lambda\sigma_\theta^2}\right) - \frac{(2-\lambda)\theta}{2} > 0, \tag{23}$$

if and only if $\mu_\theta > \frac{2\sigma_\eta^2+\lambda\sigma_\theta^2/2}{\sigma_\eta^2+(2-\lambda)\lambda\sigma_\theta^2/4}\theta > \theta$*, and*

$$\psi_{\sigma_p^2} = \frac{\left(4\sigma_\eta^2 + (2-\lambda)\lambda\sigma_\theta^2\right)^2}{\left(4\sigma_\eta^2 + \lambda^2\sigma_\theta^2\right)^2 \sigma_\eta^2} > 0. \tag{24}$$

Proposition 4.2 states the effect of noise on posterior beliefs. The posterior mean evaluated at the equilibrium mean price may increase or decrease with noise depending on the bias of the prior mean. For instance, if prior beliefs are biased upward, i.e., $\mu_\theta > \theta$, then posterior beliefs remain biased upward, although the bias is reduced. The reduction in the bias depends on the composition of buyers, the variance of the demand shock, and the variance of the prior beliefs.

Proposition 4.2. *From Proposition 3.2, and using* (20) *and* (21)*,*

$$\psi_{\hat{\mu}_\theta^*} = \frac{4\sigma_\eta^2(\mu_\theta - \theta)}{4\sigma_\eta^2 + \lambda^2\sigma_\theta^2} > 0, \tag{25}$$

if and only if $\mu_\theta > \theta$*. Moreover,*

$$\psi_{\hat{\sigma}_\theta^2} = \frac{4\sigma_\eta^2\sigma_\theta^2}{4\sigma_\eta^2 + \lambda^2\sigma_\theta^2} > 0. \tag{26}$$

References

K. Bagwell and M.H. Riordan. High and Declining Prices Signal Product Quality. *Amer. Econ. Rev.*, 81(1):224–239, 1991.

A.F. Daughety and J.F. Reinganum. Product Safety: Liability, R&D, and Signaling. *Amer. Econ. Rev.*, 85(5):1187–1206, 1995.

A.F. Daughety and J.F. Reinganum. Secrecy and Safety. *Amer. Econ. Rev.*, 95(4):1074–1091, 2005.

A.F. Daughety and J.F. Reinganum. Competition and Confidentiality: Signaling Quality in a Duopoly when there is Universal Private Information. *Games Econ. Behav.*, 58(1):94–120, 2007.

A.F. Daughety and J.F. Reinganum. Communicating Quality: A Unified Model of Disclosure and Signalling. *RAND J. Econ.*, 39(4):973–989, 2008a.

A.F. Daughety and J.F. Reinganum. Imperfect Competition and Quality Signalling. *RAND J. Econ.*, 39(1):163–183, 2008b.

M.C.W. Janssen and S. Roy. Signaling Quality through Prices in an Oligopoly. *Games Econ. Behav.*, 68(1):192–207, 2010.

K.L. Judd and M.H. Riordan. Price and Quality in a New Product Monopoly. *Rev. Econ. Stud.*, 61(4):773–789, 1994.

S.A. Matthews and L.J. Mirman. Equilibrium Limit Pricing: The Effects of Private Information and Stochastic Demand. *Econometrica*, 51(4):981–996, 1983.

L.J. Mirman and M. Santugini. Monopoly Signaling: Existence and Non-Existence. Cahiers de Recherche 08-09, HEC Montréal, Institut d'économie appliquée, 2011.