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Optimal Growth and Uncertainty: Learning*

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Abstract

We introduce learning in a Brock-Mirman environment and study the effect of risk generated by the planner's econometric activity on optimal consumption and investment. Here, learning introduces two sources of risk about future payoffs: structural uncertainty and uncertainty from the anticipation of learning. The latter renders control and learning nonseparable.

We present two sets of results in a learning environment. First, conditions under which the introduction of learning increases or decreases optimal consumption are provided. The effect depends on the strengths and directions of the two sources of risk, which may pull in opposite directions. Second, the effects of changes in the mean and riskiness of the distribution of the signal and initial beliefs on optimal consumption are studied.

1 Introduction

In the early literature on optimal growth, the evolution of output was deterministic, see Cass (1965) and Koopmans (1965). This was a natural place to begin the study of optimal growth since growth had already been studied in a deterministic environment by Ramsey (1928). Brock and Mirman (1972) introduced uncertainty in outcomes in an optimal growth model, which built on earlier studies of stochastic positive growth, see Mirman (1972, 1973). Uncertainty in outcomes is modeled by introducing a random shock in the production function. Hence, the future is riskier than in the deterministic case since future output is random, affecting optimal consumption and investment.

There is, however, another aspect of uncertainty that has yet to be studied in optimal growth: uncertainty in the structure of the economy. Unlike uncertainty in outcomes, structural uncertainty evolves through learning. Indeed, by gathering and analyzing input and output data, the planner becomes an econometrician in order to reduce structural uncertainty, while making consumption and investment decisions. The introduction of learning increases the uncertainty of future payoffs, which affects the expected marginal utility of investment. Here, learning introduces two sources of risk about future payoffs: structural uncertainty and uncertainty from the anticipation of learning.

With structural uncertainty, the planner does not know the value of a specified parameter of the distribution of the production shock, but has beliefs about it. Beliefs are expressed as a nondegenerate prior distribution. Thus, the presence of structural uncertainty characterizes the *beliefs* component of learning.

Moreover, given structural uncertainty, the planner anticipates the updating of prior beliefs to posterior beliefs, i.e., information is gathered from the observation of realized production shocks and processed using Bayesian methods. Because future information is random, the anticipation component of learning generates another source of risk. In other words, the learning process is embedded in the dynamic program. Hence, control and learning are entwined through the anticipation component of learning and cannot be separated.

We introduce learning in a Brock-Mirman environment and study the effect of risk generated by the planner's econometric activity on optimal decisions. Previous work has focused on experimentation.[1] In our model, we assume that the signal is the realization of a random variable affecting production, and it is observed. This allows us to avoid the issue of experimentation and study the effect of risk due to learning.

We focus on the class of optimal stochastic growth models studied by Mirman and Zilcha (1975), with specific utility and production functions. In this class of models, the distributions of the production shock and beliefs are assumed to be general. In particular, prior beliefs need not belong to the conjugate family of the distribution of the production shock.

Two sets of results are provided for the case of learning. The first is the overall effect of introducing learning through both its beliefs and anticipation components. Our results extend the literature on the effect of an increase in risk in future payoffs on optimal consumption and investment to a learning environment. Previous literature has focused only on models in which the planner knows the distributions of stochastic variables. In general, the effect of an increase in risk on optimal policy functions depends on the second derivative of some function.[2] In the case of learning in a growth model, the effect of an increase in risk generated by the planner's econometric activity depends on some second derivatives as well.

The beliefs component affects the expected marginal utility of investment. Here, it is the second derivative of the mean of the production shock with

[1]Experimentation was initially studied in models in which the only link between periods is beliefs. See Prescott (1972), Grossman et al. (1977), Easley and Kiefer (1988, 1989), Kiefer and Nyarko (1989), Balvers and Cosimano (1990), Aghion et al. (1991), Fusselman and Mirman (1993), Mirman et al. (1993), Trefler (1993), Creane (1994), Fishman and Gandal (1994), Keller and Rady (1999), and Wieland (2000). Experimentation in a model with capital accumulation has also been studied. See Freixas (1981), Bertocchi and Spagat (1998), Datta et al. (2002), El-Gamal and Sundaram (1993), Huffman and Kiefer (1994), Beck and Wieland (2002), and Dechert et al. (2007).

[2]See Leland (1968), Hahn (1970), Sandmo (1970), Rothschild and Stiglitz (1971), and Drèze and Modigliani (1972) for a finite-period analysis. For an infinite-horizon setup, see Mirman (1971) in a model with a single agent, and Antoniadou et al. (2007) for the case of a game. Finally, see Huggett (2004) for a detailed review of other issues studied in this literature.

respect to the unknown parameter that determines the effect of an increase in risk due to structural uncertainty. If the mean of the production shock with respect to the unknown parameter is concave, then structural uncertainty increases consumption. In other words, as structural uncertainty is introduced, the marginal utility of investment decreases, inducing less investment. And, the marginal utility of investment increases with convexity, inducing more investment.

The risk generated from the anticipation of learning always increases the marginal utility of investment, leading to a decrease in consumption or precautionary investment. Here, it is the convexity of the marginal utility of investment with respect to the mean of the production shock that leads to precautionary investment.

The total effect of learning depends on the strengths and directions of the beliefs and anticipation components. If the mean of the production shock with respect to the parameter is convex, then both types of risk work in the same direction and consumption decreases. On the other hand, if the mean of the production shock is concave, then both types of risk pull in opposite directions and the effect of learning depends on the strength of each risk.

Second, we perform a comparative analysis of distributions on the learning planner's optimal consumption, using the concepts of first and second-order stochastic dominance. Specifically, the effects of changes in the mean and riskiness of the distributions of the production shock and beliefs on optimal consumption are studied. The effect of riskier distributions on optimal consumption has been studied only in stochastic dynamic models in which the planner knows the distributions of stochastic variables. This analysis is extended to the learning case here.

We show that, while a higher mean of the production shock decreases consumption, a riskier distribution of the production shock has no effect on optimal consumption. The first result is due to the structure of the Mirman-Zilcha model, a higher mean of the production shock makes investment more profitable. The second result follows from the fact that the uncertainty in outcomes due to the random production shock is determined solely through its mean in a Mirman-Zilcha model, so a higher variability of the production shock does not affect behavior. Hence, in this class of models, the learning agent reacts to the anticipation of learning, independent of the amount of learning that takes place. Specifically, the informativeness of the signal has no effect on decisions. In other words, certainty equivalence regarding the random production shock continues to hold in this model with learning. Changes in the mean and riskiness of the distribution of the production shock have, nonetheless, a dynamic effect on optimal consumption in the subsequent period through posterior beliefs.

We also show that more optimistic beliefs decrease consumption if the mean of the production shock is positively related to the unknown parameter. Indeed, more optimistic beliefs increase the expected marginal utility of investment, inducing more investment. Finally, unlike riskier distributions of the production shock, riskier beliefs affect consumption. A riskier distribution of beliefs leads to an increase in uncertainty through both the beliefs and anticipation components. The total effect of riskier beliefs depends on the strengths and directions of these two components.

The paper is organized as follows. In section 2, we introduce learning in a general Brock-Mirman environment. In section 3, optimal consumption and investment are characterized in the class of optimal stochastic growth models studied by Mirman-Zilcha. In section 4, we study the effect of introducing learning on optimal policies. In section 5, we perform a comparative analysis of distributions on the learning planner's optimal consumption. In section 6, the effect of learning on the transition path is briefly discussed. Section 7 presents some final remarks for future research. All proofs are relegated to the appendix.

2 Model

Brock-Mirman Environment. Consider an economy in which output is determined by the production function $f(k, \eta)$, $f_1 > 0$, $f_{11} < 0$, as introduced in Mirman (1970). Here, k is capital and η is a realization of the random production shock $\tilde{\eta}$. The p.d.f of $\tilde{\eta}$ is $\phi(\eta|\theta^*)$ for $\eta \in H \subset \mathbb{R}$, which depends on a parameter $\theta^* \in \Theta \subset \mathbb{R}^N$ for $N \in \mathbb{N}$. The relationship between the distribution of $\tilde{\eta}$ and the parameter θ^* is strictly monotonic.

Each period, a planner divides output y between consumption c and investment $k = y - c$. Capital k is used for the production of output $\hat{y}$ in the subsequent period, i.e.,

$$\hat{y} = f(y - c, \eta). \tag{1}$$

The objective is to maximize the expected sum of discounted utilities, where the discount factor is $\delta \in (0, 1)$ and the utility function is $u(c)$, $u' > 0$, $u'' < 0$. Expectations are taken with respect to the sequence of future production shocks.

We first recall the informed growth model of Brock and Mirman (1972), where the planner faces no structural uncertainty, i.e., the planner is *informed* because θ^* is known. Given θ^*, the informed planner anticipates the effect of the production shock on future output. The value function is

$$V_I(y; \theta^*) = \max_{c \in [0,y]} \left\{ u(c) + \delta \int_H V_I(f(y - c, \eta); \theta^*)) \phi(\eta|\theta^*) \mathrm{d}\eta \right\}, \tag{2}$$

yielding optimal consumption $g_I(y; \theta^*)$.

Learning Planning. We now relax the assumption of no structural uncertainty. Here, the planner faces structural uncertainty because θ^* is not known. Structural uncertainty is characterized by a priori beliefs about θ^*, expressed as a prior p.d.f. ξ on Θ. That is, the probability that $\theta^* \in S$ is $\int_S \xi(\theta)\mathrm{d}\theta$ for any $S \subset \Theta$.

Structural uncertainty leads to learning and, thus, evolves over time. Indeed, the planner observes η, which yields information, and uses Bayesian methods to learn about θ^*. Formally, given ξ and η, the posterior $\hat{\xi}(\cdot|\eta)$ is

$$\hat{\xi}(\theta|\eta) = \frac{\phi(\eta|\theta)\xi(\theta)}{\int_\Theta \phi(\eta|x)\xi(x)\mathrm{d}x} \tag{3}$$

for $\theta \in \Theta$, by Bayes' Theorem. Bayes' rule (3) characterizes the learning process through the updating of beliefs in light of the information gleaned from observing η. Observing η directly, allows us to focus on an environment with learning but no experimentation. Indeed, (3) is independent of consumption.

The learning planner makes consumption and investment decisions, while learning about θ^*. That is, endowed with initial output and beliefs, consumption and investment are chosen. The production shock η is then realized and the output, in the subsequent period, is determined from (1). Information is gleaned from observing η, which, from (3), affects beliefs about θ^*.

A learning planner's decisions are subject to both (1) and (3). Indeed, the learning planner anticipates the effect of the production shock on both future output and posterior beliefs. The value function of the learning planner is

$$V_L(y,\xi) = \max_{c\in[0,y]} \left\{ u(c) + \delta \int_H V_L\left(f(y-c,\eta), \hat{\xi}(\cdot|\eta)\right) \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta\right] \mathrm{d}\eta \right\} \tag{4}$$

subject to (3), yielding optimal consumption $g_L(y,\xi)$.

Learning increases the uncertainty of future payoffs by introducing two sources of risk: structural uncertainty and uncertainty from the anticipation of learning. In other words, there are two distinct components of learning.

The first is about *beliefs*. While the informed planner's beliefs about θ^* are degenerate, the learning planner's are nondegenerate. There is an increase in uncertainty of future payoffs when knowledge of the distribution of the production shock, $\phi(\eta|\theta^*)$ in (2), is replaced by the expected p.d.f. of $\tilde{\eta}$ with respect to beliefs ξ, $\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta$ in (4).

The second component concerns *anticipation*, i.e., learning is anticipated using Bayesian updating. In a dynamic context, rational expectations imply that the information contained in the future production shock is anticipated. Anticipation of learning is integrated into (4) by anticipating the updated beliefs from ξ to $\hat{\xi}(\cdot|\eta)$ using (3).

Nonseparability of Control and Learning. The anticipation of learning is related to the nonseparability of control and learning since the optimal policy takes account of the change in beliefs which is contained in the subsequent period expected value function. In other words, separation of control and learning occurs if and only if there is no anticipation component.

The anticipation of updated beliefs affects optimal behavior because the dynamics given in (1) and (3) are entwined through the production shock.[3] If the only link between periods were beliefs, i.e., no capital accumulation, then the anticipation of learning would have no effect on optimization.

In order to study the effect of introducing learning, overall and through its two components, we introduce the intermediate case of an adaptive learner.[4] As with the learning planner, the adaptive learning planner does not know θ^*, and has beliefs about it expressed as a p.d.f ξ on Θ. However, unlike the learning planner, the adaptive learning planner does not anticipate learning.

Given beliefs, the adaptive learning planner anticipates the effect of the production shock solely on future output, while beliefs are assumed to remain constant in his objective function. Therefore, the value function of the adaptive learning planner is

$$V_{AL}(y;\xi) = \max_{c\in[0,y]} \left\{ u(c) + \delta \int_H V_{AL}(f(y-c,\eta);\xi) \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta \right] \mathrm{d}\eta \right\}, \tag{7}$$

yielding optimal consumption $g_{AL}(y;\xi)$. The adaptive learning planner does, however, update beliefs in each period. Once information arrives, the adaptive learning planner adapts and updates beliefs, subject to (3). Therefore, the adaptive learning planner reacts to new information, but does not anticipate it.

Note that the informed and adaptive learning planners differ solely in the distribution of the production shock. Indeed, knowledge of the distribution of the production shock, $\phi(\eta|\theta^*)$ in (2), is replaced by the expected p.d.f. of $\tilde{\eta}$ with respect to beliefs ξ, $\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta$ in (7). Thus, the adaptive learning

[3]Formally,

$$\int_H V_L\left(f(y-c,\eta),\hat{\xi}(\cdot|\eta)\right) \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta\right] \mathrm{d}\eta \neq \int_H V_L(f(y-c,\eta),\xi) \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta\right] \mathrm{d}\eta, \tag{5}$$

even though the expectation of the posterior p.d.f is equal to its prior, i.e.,

$$\int_H \hat{\xi}(\cdot|\eta) \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta\right] \mathrm{d}\eta = \xi(\theta) \tag{6}$$

for $\theta \in \Theta$.

[4]See Evans and Honkapohja (2001) for a detailed exposition of adaptive learning. See also Milani (2007).

planner faces a more variable distribution of the production shock than the informed planner.

Comparisons. While comparing informed and learning planners captures the overall effect of introducing learning in growth, the introduction of the intermediate case of an adaptive learning planner allows us to study the beliefs and anticipation components independently. First, comparing (2) and (7) captures the beliefs component, i.e., the risk generated from not knowing θ^*. Second, comparing (4) and (7) captures the anticipation component, i.e., the risk generated from uncertain posterior beliefs.

Remarks. In general, dynamic programs with learning such as (4) are intractable, i.e., they are not solvable either analytically or numerically, when there is no separability of control and learning.[5] The problem is not only whether a solution exists, but if a solution can be characterized and its properties studied. Two aspects of dynamic programming with learning should be noted.

First, (4) depends on the variable y and the prior p.d.f ξ on Θ. Unless the space Θ contains a finite number of elements, the state space (y, ξ) is infinitely-dimensioned, yielding the curse of dimensionality.

Second, the evolution of beliefs, according to Bayes' law, does not prevent the prior and posterior p.d.f.'s ξ and $\hat{\xi}(\cdot|\eta)$ from belonging to different families. This makes the solution of an infinite-horizon dynamic programming problem with Bayesian dynamics generally intractable. Indeed, the learning planner makes consumption and investment decisions, anticipating updating beliefs every period. In other words, the value function, $V\left(f(y-c,\eta), \hat{\xi}(\cdot|\eta)\right)$ in (4), encompasses beliefs that have been updated infinitely many times. These updated beliefs may belong to many different families of distributions.

Learning with general functions and distributions has focused on existence as well as limit beliefs and actions.[6] Studies that characterize optimal policies are always in the context of specific functional forms, the space of

[5]When there is separability, the dynamic program becomes a standard growth problem, so that the learning planner is identical to the adaptive learning planner.

[6]See Easley and Kiefer (1988, 1989), Kiefer and Nyarko (1989), Aghion et al. (1991), El-Gamal and Sundaram (1993), among others.

the unknown parameter restricted to two values, finite periods, or the use of conjugate priors, especially the normal distribution.[7] While we specialize the model in order to obtain tractable solutions for (2), (4), and (7) within a well-known class of optimal stochastic growth models, we characterize optimal consumption and investment with results that hold for general distributions.

3 Optimal Consumption and Investment

In order to deal with the complexities of learning in growth, we focus on the class of optimal stochastic growth models studied by Mirman and Zilcha (1975) with the following assumptions.

Assumption 3.1. The utility function is $u(c) = \ln c$.

Assumption 3.2. The production function is Cobb-Douglas, $f(k, \eta) = k^{\eta}$.

Assumption 3.3. The support of $\tilde{\eta}$ is $H = [0, 1]$ and η is observable.

Assumptions 3.1, 3.2, and 3.3 hold for the remainder of the paper. The model with log utility, Cobb-Douglas production, and general distributions of the production shock and beliefs about θ^* yields closed-form solutions for optimal consumptions in the cases of informed, adaptive learning, and learning.

The combination of log utility and Cobb-Douglas production is needed to obtain a tractable characterization of optimal consumption and investment in a learning context. For log utility and the Cobb-Douglas production function $f(k, \eta) = \eta k^{\alpha}$, $\alpha \in [0, 1]$, learning about the distribution of $\tilde{\eta}$ has no effect because the multiplicative shock plays no role. Moreover, making the utility function more general, while keeping a Cobb-Douglas production function leads to intractability.

The Mirman-Zilcha class of models has three features that makes the analysis possible. First, Assumptions 3.1 and 3.2 imply that optimal consumption and investment are linear in output in the no learning case. The

[7] See the rest of the literature on learning in footnote 1.

linearity property remains under learning, although the fraction of output consumed now depends on beliefs and evolves with new information.

Second, from Assumptions 3.1 and 3.2, the uncertainty in outcomes, i.e., the random production shock, enters the optimization problem through its mean. In other words, the Mirman and Zilcha (1975) class of models displays certainty equivalence. This feature is exploited in the learning case since the uncertainty in outcomes is mapped to its mean, implying that the unknown parameter affects optimal consumption solely through $\mu(\theta) = \int_0^1 \eta\phi(\eta|\theta)\mathrm{d}\eta$, the mean of $\tilde{\eta}$ given $\theta \in \Theta$. The relationship between the mean of the production shock and the unknown parameter is the key in determining the effect of learning on the optimal consumption function and comparative analysis.

Third, Assumptions 3.1 and 3.2 imply that no assumption is needed on the production shock, as well as on the distribution of prior beliefs. The Mirman-Zilcha class of models does away with all the difficulties inherent in Bayesian analysis. In particular, the prior need not belong to the conjugate family of the distribution of the production shock. In other words, solutions for optimal consumption and investment are valid for a wide range of priors, even those that are outside of families of distributions that are closed under sampling.

We first state the optimal consumptions of both the informed and adaptive learning planners. We then present and illustrate the optimal consumption of the learning agent.

Benchmark Models. From Mirman-Zilcha, the optimal consumption of the informed planner, corresponding to (2), is

$$g_I(y;\theta^*) = (1 - \delta\mu(\theta^*))y, \tag{8}$$

while the optimal consumption of the adaptive learning planner, corresponding to (7), is

$$g_{AL}(y,\xi) = \left(1 - \delta \int_\Theta \mu(\theta)\xi(\theta)\mathrm{d}\theta\right) y. \tag{9}$$

The presence of structural uncertainty does not affect the optimal consumption function, since the true expectation of $\tilde{\eta}$, $E[\tilde{\eta}|\theta^*] = \mu(\theta^*)$ in (8), is replaced by the expectation of $\tilde{\eta}$ given beliefs, $E[\tilde{\eta}|\xi] = \int_\Theta \mu(\theta)\xi(\theta)\mathrm{d}\theta$ in (9).

Learning Planner. In the appendix, we show that the value function of the learning planner is of the form,

$$V_L(y, \xi) = \kappa_1(\xi)\ln y + \kappa_2(\xi), \tag{10}$$

where $\kappa_1(\xi) = \int_\Theta (1 - \delta\mu(\theta))^{-1}\xi(\theta)\mathrm{d}\theta$ and $\kappa_2(\xi)$ depends on ξ.

Proposition 3.4. *The optimal consumption of the learning planner is*

$$g_L(y, \xi) = \left(\int_\Theta \frac{\xi(\theta)\, d\theta}{1 - \delta\mu(\theta)}\right)^{-1} y. \tag{11}$$

Despite the fact that this class of growth models displays certainty equivalence, certainty equivalence does not imply the separation of control and learning. Indeed, the anticipation of learning changes the optimal consumption function for the learning planner.

We present four examples that show the wide applicability of our model, not only in terms of distributions, but also in terms of general unknown structures. For instance, normal distributions are not needed to get analytic results. In Example 3.5, the case of learning about two unknown parameters is presented. Example 3.6 deals with a uniform distribution for $\tilde{\eta}$ with unknown support. Example 3.7 illustrates the case in which the learning planner does not know to which family $\tilde{\eta}$ belongs, as well as not knowing the parameters characterizing each family. Finally, Example 3.8 shows that the model encompasses the case of an informed planner with degenerate beliefs. That is, in Example 3.8, the planner knows the distribution of $\tilde{\eta}$, as in Mirman-Zilcha.

Example 3.5. Let $\tilde{\eta}$ have a beta distribution with unknown parameters $\theta = (\alpha, \beta)$, and beliefs $\xi(\alpha, \beta)$, $\alpha, \beta > 0$. Then, $\mu(\theta) = \alpha/(\alpha + \beta)$ and

$$g_L(y, \xi) = \left(\int_{\mathbb{R}^2_{++}} \frac{\xi(\alpha, \beta)\mathrm{d}\alpha\mathrm{d}\beta}{1 - \delta\alpha/(\alpha + \beta)}\right)^{-1} y. \tag{12}$$

Example 3.6. Let $\tilde{\eta}$ have a uniform distribution with unknown support $[0, \theta]$, and beliefs $\xi(\theta)$, $\theta \in [0, 1]$. Then, $\mu(\theta) = \theta/2$ and

$$g_L(y, \xi) = \left(\int_0^1 \frac{\xi(\theta)\mathrm{d}\theta}{1 - \delta\theta/2} \right)^{-1} y. \tag{13}$$

Example 3.7. Let $\Theta = \{\theta_1, \theta_2\}$, where θ_1 represents a beta distribution with unknown parameters (α, β), and beliefs $\xi_B(\alpha, \beta)$, $\alpha, \beta > 0$, while θ_2 represents a truncated normal distribution with support $[0, 1]$, unknown parameters (m, σ^2), and beliefs $\xi_N(m, \sigma^2)$, $m > 0, \sigma^2 \in \mathbb{R}_{++}$. If $0 \leq \rho \leq 1$ is the prior probability that the production shock is beta distributed, then

$$g_L(y, \rho, \xi_B, \xi_N) = \left(\rho \int_{\mathbb{R}^2_{++}} \frac{\xi_B(\alpha, \beta)\mathrm{d}\alpha\mathrm{d}\beta}{1 - \delta\mu_1(\alpha, \beta)} + (1 - \rho) \int_{\mathbb{R}_{++}} \int_{\mathbb{R}} \frac{\xi_N(m, \sigma^2)\mathrm{d}m\mathrm{d}\sigma^2}{1 - \delta\mu_2(m, \sigma^2)} \right)^{-1} y, \tag{14}$$

where $\mu_1(\alpha, \beta)$ is the mean of a beta random variable with parameters (α, β) and $\mu_2(m, \sigma^2)$ is the mean of a truncated normal random variable with parameters (m, σ^2).

Example 3.8. Let the beliefs be denoted as ξ^* be degenerate at θ^*, i.e., $\xi^*(\theta) = 1$ for $\theta = \theta^*$ and $\xi^*(\theta) = 0$ for $\theta \in \Theta \backslash \{\theta^*\}$. Then, $g_L(y, \xi^*) = (1 - \delta\mu(\theta^*))y$, which is identical to (8).

4 The Effect of Learning on Optimal Policies

Learning increases the uncertainty of future payoffs, which affects the expected marginal utility of investment. As noted previously, learning introduces two sources of risk about future payoffs: structural uncertainty and uncertainty from the anticipation of learning. Structural uncertainty is the beliefs component, while uncertainty from the anticipation of learning is the anticipation component. In this section, we study the overall effect of introducing learning through both its beliefs and anticipation components.

Our results extend the literature on the effect of an increase in risk on optimal consumption and investment to a learning environment. Previous literature has focused on models in which the planner knows the distributions of stochastic variables. In this literature, the effect of an increase in risk on optimal policy functions depends on the second derivative of some functions of the random variable being studied.[8]

In the learning case, the effect of an increase in risk also depends on the second derivatives of some functions of the appropriate random variable. To see this, consider the first-order conditions of the informed planner,

$$\frac{1}{c} = \frac{\delta R(\mu(\theta^*))}{y-c}, \tag{15}$$

the adaptive learning planner,

$$\frac{1}{c} = \frac{\delta R\left(\int_\Theta \mu(\theta)\xi(\theta)\mathrm{d}\theta\right)}{y-c}, \tag{16}$$

and the learning planner,

$$\frac{1}{c} = \frac{\delta \int_\Theta R(\mu(\theta))\xi(\theta)\mathrm{d}\theta}{y-c}. \tag{17}$$

Here, $R(x) = x(1-\delta x)^{-1}$, $R', R'' > 0$, for $x \in [0,1]$ characterizes the effect of uncertainty in outcomes due to the random production shock $\tilde{\eta}$ on the expected marginal utility of investment.

From (15) and (16), structural uncertainty affects the expected marginal utility of investment. Here, it is the second derivative of the mean of the production shock, with respect to the unknown parameter, that determines the effect of an increase in risk due to structural uncertainty.

Moreover, from (16) and (17), the anticipation of learning affects the expected marginal utility of investment. Here, it is the convexity of R that determines the effect of an increase in risk due to the anticipation of learning.

Finally, from (15) and (17), the overall effect of learning on optimal consumption is characterized by the expectation of R with respect to beliefs ξ. Here, it is the second derivative of R with respect to the unknown parameter

[8] Consider a two-period model in which the planner maximizes $u(c) + \delta E[u(f(y-c, \tilde{\eta}))]$ over c. If $\hat{y} = f(y-c) + \eta$, then it is the convexity of the marginal utility of consumption that leads to precautionary investment. If $\hat{y} = \eta f(y-c)$, then it is the convexity of $\eta f'(y-c)u'(\eta f(y-c))$ with respect to η that leads to precautionary investment.

that determines the overall effect of introducing learning through both its beliefs and anticipation components.

Nonseparability of Control and Learning. Since the dynamic effect of output $(y-c)^{-1}$ is entwined with the term R and its expectation with respect to ξ, Thus, control and learning are entwined and cannot be separated.

The nonseparability of control and learning is revealed by comparing (16) and (17), which also determines the impact of the anticipation component of learning. Proposition 4.1 states that the anticipation of learning always decreases optimal consumption. Formally,

Proposition 4.1. $g_{AL}(y;\xi) > g_L(y,\xi)$.

Proposition 4.1 is due to the convexity of R, and, thus, the expected marginal utility of investment, and the use of Jensen's inequality on the right-hand sides of (16) and (17). The risk generated from the anticipation of learning increases the expected marginal utility of investment, leading to a decrease in consumption or precautionary investment.

Beliefs and Anticipation Components. Next, the effect of introducing learning in an optimal growth model, when beliefs are unbiased, is studied. First, we consider beliefs about the mean of the production shock that are unbiased, i.e., $\mu(\theta^*) = \int_\Theta \mu(\theta)\xi(\theta)\mathrm{d}\theta$. Second, we focus on beliefs about the parameter θ^* that are unbiased, i.e., $\theta^* = \int_\Theta \theta\xi(\theta)\mathrm{d}\theta$. In both case, conditions are established under which the introduction of learning, overall and through each of its components, increases or decreases optimal consumption using (15), (16) and (17). In other words, $g_I(y;\theta^*)$, $g_{AL}(y;\xi)$, and $g_L(y,\xi)$ are ordered.

In Proposition 4.2, the effect of learning when beliefs are unbiased about the mean of the production shock is studied. From (15) and (16), risk from structural uncertainty does not change the expected marginal utility of investment, since the uncertainty in outcomes is characterized only through its mean. Since the true mean of the production shock and unbiased beliefs about the true mean of the production shock have the same effect on behavior, there is certainty equivalence. Therefore, the total effect of learning is due to the anticipation of learning. As established in Proposition 4.1, the risk generated from the anticipation component increases the expected marginal utility of investment, leading to precautionary investment. Formally,

Proposition 4.2. *Suppose beliefs are unbiased about the mean of the production shock, $\mu(\theta^*) = \int_\Theta \mu(\theta)\xi(\theta)d\theta$. Then, learning decreases optimal consumption, and $g_I(y;\theta^*) = g_{AL}(y;\xi) > g_L(y,\xi)$.*

In Proposition 4.3, the effect of learning when beliefs are unbiased about the unknown parameter, $\theta^* = \int_\Theta \theta\xi(\theta)\mathrm{d}\theta$, is studied. The effect of learning in this case is not as simple as in Proposition 4.2. The reasons is that both sources of risk due to learning are at work here. Indeed, the effect of structural uncertainty depends on the second derivative of the mean of the production shock with respect to θ. If the mean of the production shock with respect to θ is concave, then structural uncertainty increases optimal consumption. In other words, as structural uncertainty is introduced, with θ^* replaced by unbiased beliefs about θ^*, the marginal utility of investment decreases, inducing less investment. And, the marginal utility of investment increases with convexity, inducing more investment.

This point is illustrated in Example 3.5 in which $\tilde{\eta}$ has a beta distribution with parameters $\alpha, \beta > 0$. If $\alpha \equiv \theta$ is unknown and β is known, then $\mu''(\theta) = -2\beta/(\theta+\beta)^3 < 0$, and structural uncertainty increases consumption. However, if α is known and $\beta \equiv \theta$ is unknown, then $\mu''(\theta) = 2\alpha/(\alpha+\theta)^3 > 0$, and structural uncertainty decreases consumption.

The total effect depends on the strengths and directions of the beliefs and anticipation components. If the mean of the production shock with respect to the parameter is convex, then both types of risk work in the same direction and optimal consumption decreases. However, if the mean of the production shock is concave, then both types of risk pull in opposite directions and the effect of learning depends on the strength of each risk. Mathematically, it is the second derivative of R with respect to θ that determines the strength of the overall effect, i.e.,

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\theta^2} = \frac{\mu''(\theta)(1-\delta\mu(\theta)) + 2\delta\mu'(\theta)^2}{(1-\delta\mu(\theta))^3}, \tag{18}$$

for $\theta \in \Theta$. The sign of (18) is determined by the sign of μ'' and the relationship $\mu'' \gtreqless -2\delta\mu'^2/(1-\delta\mu)$. Formally,

Proposition 4.3. *Suppose that beliefs are unbiased about the parameter,* $\theta^* = \int_\Theta \theta\xi(\theta)\,d\theta$.

1. *If* $\mu'' > 0$, *then* $g_I(y;\theta^*) > g_{AL}(y;\xi) > g_L(y,\xi)$.
2. *If* $\mu'' = 0$, *then* $g_I(y;\theta^*) = g_{AL}(y;\xi) > g_L(y,\xi)$.
3. *If* $-2\delta mu'^2/(1-\delta\mu) < \mu'' < 0$, *then* $g_L(y,\xi) < g_I(y;\theta^*) < g_{AL}(y;\xi)$.
4. *If* $\mu'' = -2\delta mu'^2/(1-\delta\mu)$, *then* $g_L(y,\xi) = g_I(y;\theta^*) < g_{AL}(y;\xi)$.
5. *If* $\mu'' < -2\delta mu'^2/(1-\delta\mu)$, *then* $g_I(y;\theta^*) < g_L(y,\xi) < g_{AL}(y;\xi)$.

In case 1, the convexity of the mean of the production shock implies that structural uncertainty decreases consumption, as does the anticipation component. In other words, the two types of risk work in the same direction. In case 2, the mean of the production shock is linear in θ, so that structural uncertainty has no effect on the expected marginal utility of investment. Here, consumption decreases solely due to the anticipation component. In case 3, the mean of the production shock is concave. Here, the beliefs and anticipation components pull in opposite directions. The beliefs component increases, while the anticipation component, as is always the case, decreases consumption. But the mean of the production shock is not concave enough for the beliefs component to be dominant, and the overall effect of learning is to decrease consumption. Case 4 is a knife-edge case in which beliefs and anticipation components pull in opposite directions in equal strength. In case 5, the beliefs and anticipation components pull in opposite directions, but the mean of the production shock is concave enough to overwhelm the anticipation component. Thus, consumption increases.

5 Comparative Analysis

In this section, the effect of different properties of the signal and initial beliefs on the learning planner's optimal consumption is studied. Specifically, we study the effect of changes in the mean and riskiness of the distribution ϕ of the production shock $\tilde{\eta}$ as well as beliefs ξ about θ^* using the concepts of first-order and second-order stochastic dominance. Note that the concept of second-order stochastic dominance has been used implicitly in Section 4 where the effect of an increase in risk due to the introduction of learning is studied. The effect of riskier distributions on optimal consumption has been studied only in stochastic dynamic models in which the planner knows the distributions of stochastic variables. This analysis is extended to the learning case.

To facilitate the discussion, let $g_L^j(y, \xi)$ denote optimal consumption and $\mu^j(\theta) = \int_0^1 \eta\phi^j(\eta|\theta)\mathrm{d}\eta$, for the distribution ϕ^j. Moreover, let $g_L(y, \xi^j)$ denote optimal consumption with respect to ξ^j, $j = 1, 2$.

First, the definitions of first-order stochastic dominance and second-order stochastic dominance are stated.

Definition 5.1. The p.d.f. φ^1 first-order stochastically dominates the p.d.f. φ^2, $\varphi^1 \succ_1 \varphi^2$, if, for every nondecreasing function $\lambda : \mathbb{R} \to \mathbb{R}$, $\int_{\mathbb{R}} \lambda(x)\varphi^1(x)\mathrm{d}x \geq \int_{\mathbb{R}} \lambda(x)\varphi^2(x)\mathrm{d}x$.

Definition 5.2. For any two p.d.f.'s φ^1 and φ^2, φ^1 second-order stochastically dominates the p.d.f. φ^2, $\varphi^1 \succ_2 \varphi^2$, i.e., φ_1 is less risky than φ_2, if, for every concave function $\lambda : \mathbb{R} \to \mathbb{R}$, $\int_{\mathbb{R}} \lambda(x)\varphi^1(x)\mathrm{d}x \geq \int_{\mathbb{R}} \lambda(x)\varphi^2(x)\mathrm{d}x$.

5.1 Properties of the Signal

Proposition 5.3 shows that a higher mean of the production shock $\tilde{\eta}$ decreases consumption.

Proposition 5.3. *If $\phi^1 \succ_1 \phi^2$, then $g_L^1(y, \xi) \leq g_L^2(y, \xi)$.*

From (17), the expected marginal utility of investment is greater under ϕ^1 than under ϕ^2 for $\phi^1 \succ_1 \phi^2$, inducing more investment and less consumption.[9]

[9] This result is in contrast to the literature that analyzes the effect of signals on future productivity on the business cycle, where it is established that the anticipation of a higher expected production shock in the long-run reduces current investment and increases current consumption. The reason for the different result is that, in that literature, investment

Proposition 5.4 shows that an increase in the riskiness of the distribution of the production shock $\tilde{\eta}$ has no effect on consumption.

Proposition 5.4. *If $\phi^1 \succ_2 \phi^2$, then $g_L^1(y, \xi) = g_L^2(y, \xi)$.*

From (17), only $\mu(\theta)$ affects the expected marginal utility of investment. Hence, in this class of models, the learning agent reacts to the anticipation of learning, independent of the amount of learning that takes place. Specifically, the informativeness of the signal has no effect on decisions. In other words, certainty equivalence regarding the random production shock continues to hold in this model with the introduction of learning.

Finally, changes in the mean and riskiness of the distribution of the production shock have a dynamic effect on consumption in the subsequent period through updating beliefs. Indeed, if $\phi^1 \succ_1 \phi^2$ or $\phi^1 \succ_2 \phi^2$, then $\hat{\xi}^1 \neq \hat{\xi}^2$, for the same η. Hence, $g_L^1(\hat{y}, \hat{\xi}^1) \neq g_L^2(\hat{y}, \hat{\xi}^2)$.

5.2 Properties of Prior Beliefs

Proposition 5.5 shows the effect of more optimistic beliefs about θ^* on consumption. The effect of more optimistic beliefs depends on the first derivative of $\mu(\theta)$.

Proposition 5.5. *Suppose that $\xi^1 \succ_1 \xi^2$.*

1. *If $\mu' > 0$, then $g_L(y, \xi^1) \leq g_L(y, \xi^2)$.*

2. *If $\mu' < 0$, then $g_L(y, \xi^1) \geq g_L(y, \xi^2)$.*

3. *If $\mu' = 0$, then $g_L(y, \xi^1) = g_L(y, \xi^2)$.*

is delayed until the increase in the expected production shock occurs. Then, it is more profitable to invest. See Beaudry and Portier (2004), Beaudry and Portier (2006), Beaudry and Portier (2007), Christiano et al. (2006), and Jaimovich and Rebelo (2006). The opposite result is established here because the planner faces a higher expected production shock for the next period, hence current investment is more profitable and, thus, increases immediately.

From (17), if $\mu' > 0$, then $\xi^1 \succ_1 \xi^2$ implies that the expected marginal utility of investment is greater under ξ^1 than under ξ^2. Here, more optimistic beliefs about the production shock induces more investment and less consumption.

While, as stated in Proposition 5.4, a riskier distribution of $\tilde{\eta}$ does not affect consumption, Proposition 5.6 shows that riskier beliefs about θ^* does affect consumption. Proposition 5.6 generalizes Proposition 4.3. Recall that in Proposition 4.3, informed and learning planners are compared by increasing risk around θ^*. Here, two learning planners, one with riskier beliefs about θ^* than the other are compared.

As in Proposition 4.3, the effect of a riskier prior on consumption is determined by the sign of (18). Formally,

Proposition 5.6. *Suppose that* $\xi^1 \succ_2 \xi^2$.

1. *If* $\mu'' < -2\delta m u'^2/(1-\delta\mu)$, *then* $g_L(y, \xi^1) \leq g_L(y, \xi^2)$.
2. *If* $\mu'' > -2\delta m u'^2/(1-\delta\mu)$, *then* $g_L(y, \xi^1) \geq g_L(y, \xi^2)$.
3. *If* $\mu'' = -2\delta m u'^2/(1-\delta\mu)$, *then* $g_L(y, \xi^1) = g_L(y, \xi^2)$.

The discussion is similar to the one for Proposition 4.3. In case 1, the beliefs and anticipation components pull in opposite directions, but the mean of the production shock is concave enough to overwhelm the anticipation component. Thus, as beliefs become riskier, consumption decreases. In case 2, the anticipation component is dominant, implying that the expected marginal utility of investment is convex in θ, leading to precautionary investment as beliefs become riskier. In case 3, both beliefs and anticipation components pull in opposite directions in equal strength, implying that the expected marginal utility of investment is linear in θ. There is no reaction to riskier beliefs.[10]

Finally, changes in the mean and riskiness of beliefs have a dynamic effect on consumption in the subsequent period through updating beliefs.

[10] It is possible to extend Proposition 4.2 by comparing two learning planners, one with riskier beliefs about $\mu(\theta^*)$ than the other one. As in Proposition 4.2, consumption always decreases as beliefs about $\mu(\theta^*)$ become riskier.

Indeed, if $\xi^1 \succ_1 \xi^2$ or $\xi^1 \succ_2 \xi^2$, then $\hat{\xi}^1 \neq \hat{\xi}^2$, for the same η. Hence, $g_L(\hat{y}, \hat{\xi}^1) \neq g_L(\hat{y}, \hat{\xi}^2)$.

6 Transition Path

While the anticipation of learning increases the uncertainty of future payoffs, learning itself decreases structural uncertainty along the transition path. Therefore, the learning planner generally converges to the informed planner. In this section, we briefly discuss the effect of learning along the transition path. The rate of convergence of the learning planner depends upon how quickly and accurately information can be gleaned from observations. In other words, it is the flow and processing of information that determines the difference between the informed and learning planners' transition paths.

Flow of Information. The flow of information depends on the properties of the distribution of the signal. For example, a more diffuse signal decreases the flow of information, which slows down learning. Indeed, Figure 1 shows that the fraction of output consumed by a learning planner with a diffuse signal converges more slowly than a learning planner with a tight signal. Figure 1 reports simulations from a simplified version of Example 3.5 with $\delta = 0.99$. The learning planner with a diffuse signal knows that $\alpha = 0.4$, but not that $\beta = 0.4$ and has flat initial beliefs about β on $\{0.2, 0.4, 0.6\}$. The learning planner with a tight signal knows that $\alpha = 1.4$, but not that $\beta = 1.4$, and has flat initial beliefs about β on $\{1.2, 1.4, 1.6\}$.

Processing of Information. The processing of information through Bayesian updating depends in part on initial beliefs. The more biased prior beliefs are, the slower is the convergence of the learning planner. Consider again a simplified version of example 3.5 with $\delta = 0.99$. The learning planner knows that $\alpha = 0.4$, but not that $\beta = 0.4$. Beliefs have support $\{0.2, 0.4, 0.6\}$. Figure 2 shows that the fraction of output consumed by a learning planner with biased beliefs, i.e., $\rho_{0.2} = 0.9, \rho_{0.4} = 0.05, \rho_{0.6} = 0.05$, converges slower than the one with initial unbiased beliefs, i.e., $\rho_{0.2} = \rho_{0.4} = \rho_{0.6} = 1/3$.

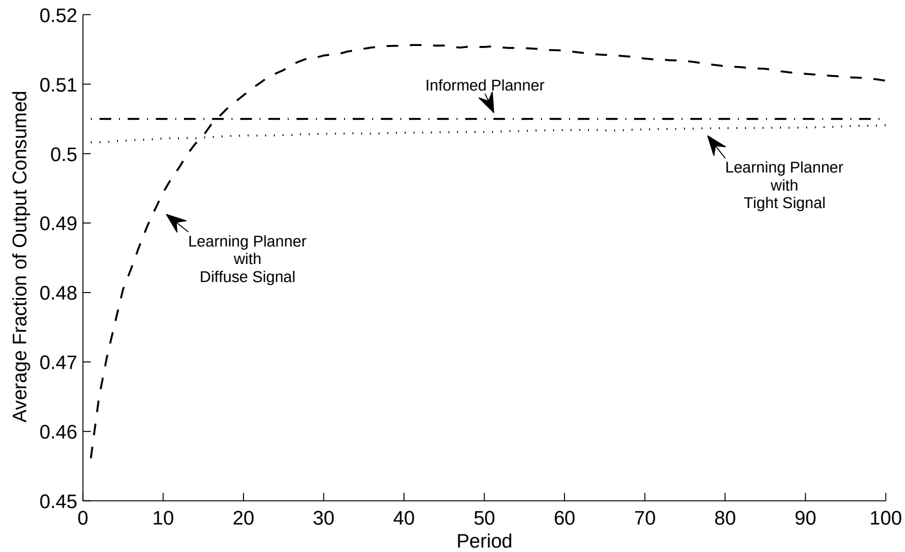

Figure 1: Tight vs. Diffuse Signal

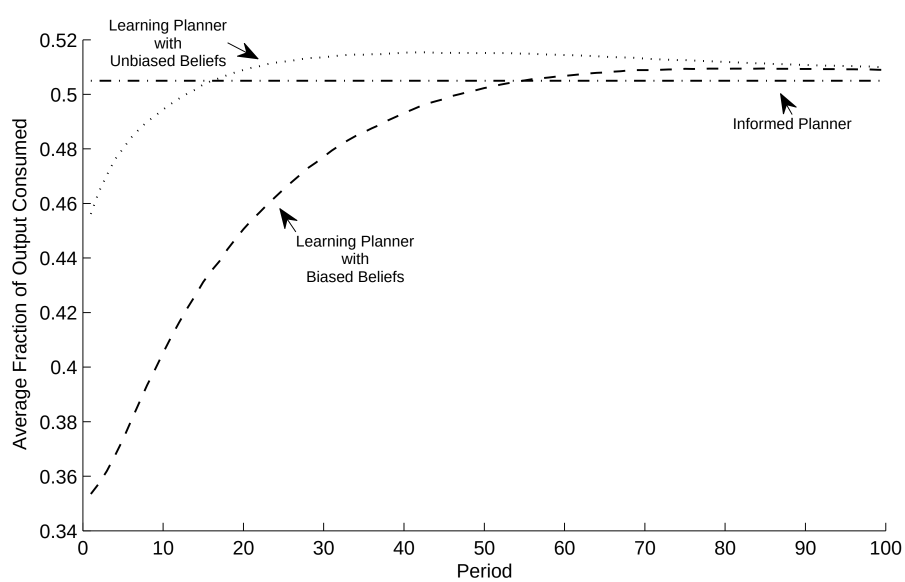

Figure 2: Biased vs. Unbiased Beliefs

7 Final Remarks

In our model, the planner observes the production shock directly. There are situations in which it is reasonable to assume that η is not observable. In this case, output $\hat{y}$ is the signal used to update beliefs about θ^*.[11] This formulation leads to experimentation if the relationship between $\hat{y}$ and η is not strictly monotonic, i.e., η cannot be inferred from observing $\hat{y}$. Therefore, the value function with capital accumulation and experimentation is, in general, no longer concave. Future research should focus on characterizing optimal policies under experimentation to understand how the planner affects the flow of information in reaction to the anticipation of learning.[12]

[11] Note that observing output and not the production shock might lead to incomplete learning. See Appendix B.

[12] Previous work in growth has only characterized optimal policies in a setup with three periods and a two-value support of the unknown parameter. See Bertocchi and Spagat (1998) and Datta et al. (2002).

A Proofs

Proof of Proposition 3.4. We conjecture that the value function of the learning planner is of the form $V_L(y,\xi) = \kappa_1(\xi)\ln y + \kappa_2(\xi)$, where κ_1 and κ_2 depend on ξ. From (4),

$$V_L(y,\xi) = \max_{c\in(0,y)} \left\{ \ln c + \delta \ln(y-c) \int_0^1 \kappa_1\left(\hat{\xi}(\cdot|\eta)\right) \eta \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta\right] \mathrm{d}\eta \right.$$
$$\left. + \delta \int_0^1 \kappa_2\left(\hat{\xi}(\cdot|\eta)\right) \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta\right] \mathrm{d}\eta \right\}. \tag{19}$$

The first-order condition is

$$\frac{1}{c} = \frac{\delta \int_0^1 \kappa_1\left(\hat{\xi}(\cdot|\eta)\right) \eta \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta\right] \mathrm{d}\eta}{y-c}, \tag{20}$$

evaluated at $c = g_L(y,\xi)$, so that

$$g_L(y,\xi) = \frac{y}{1 + \delta \int_0^1 \kappa_1\left(\hat{\xi}(\cdot|\eta)\right) \eta \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta\right] \mathrm{d}\eta}. \tag{21}$$

Plugging (21) into (19) yields

$$V_L(y,\xi) = \left(1 + \delta \int_0^1 \kappa_1\left(\hat{\xi}(\cdot|\eta)\right) \eta \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta\right] \mathrm{d}\eta\right) \ln y + \kappa_3(\xi), \tag{22}$$

$$\equiv \kappa_1(\xi)\ln y + \kappa_2(\xi), \tag{23}$$

where

$$\kappa_3(\xi) = \delta \left(\int_0^1 \kappa_1 \left(\hat{\xi}(\cdot|\eta) \right) \eta \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta \right] \mathrm{d}\eta \right) \ln \left(\delta \int_0^1 \kappa_1 \left(\hat{\xi}(\cdot|\eta) \right) \eta \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta \right] \mathrm{d}\eta \right)$$
$$- \left(1 + \delta \int_0^1 \kappa_1 \left(\hat{\xi}(\cdot|\eta) \right) \eta \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta \right] \mathrm{d}\eta \right)$$
$$\cdot \ln \left(1 + \delta \int_0^1 \kappa_1 \left(\hat{\xi}(\cdot|\eta) \right) \eta \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta \right] \mathrm{d}\eta \right)$$
$$+ \delta \int_0^1 \kappa_2 \left(\hat{\xi}(\cdot|\eta) \right) \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta \right] \mathrm{d}\eta. \tag{24}$$

Therefore,

$$\kappa_1(\xi) \equiv 1 + \delta \int_0^1 \kappa_1 \left(\hat{\xi}(\cdot|\eta) \right) \eta \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta \right] \mathrm{d}\eta. \tag{25}$$

The solution to (25) is

$$\kappa_1(\xi) = \int_\Theta \frac{\xi(\theta)\mathrm{d}\theta}{1 - \delta\mu(\theta)}, \tag{26}$$

where $\mu(\theta) = \int_0^1 \eta\phi(\eta|\theta)\mathrm{d}\eta$. To verify that (26) is the solution to (25), updating (26) to the next period yields

$$\kappa_1 \left(\hat{\xi}(\cdot|\eta) \right) = \int_\Theta \frac{\hat{\xi}(\theta|\eta)\mathrm{d}\theta}{1 - \delta\mu(\theta)}, \tag{27}$$
$$= \int_\Theta \frac{1}{1 - \delta\mu(\theta)} \frac{\phi(\eta|\theta)\xi(\theta)}{\int_\Theta \phi(\eta|x)\xi(x)\mathrm{d}x} \mathrm{d}\theta. \tag{28}$$

Then, plugging (28) into (25) yields

$$\kappa_1(\xi) = 1 + \delta \int_0^1 \left(\int_\Theta \frac{1}{1-\delta\mu(\theta)} \frac{\phi(\eta|\theta)\xi(\theta)}{\int_\Theta \phi(\eta|x)\xi(x)\mathrm{d}x} \mathrm{d}\theta \right) \eta \cdot \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta \right] \mathrm{d}\eta, \tag{29}$$

$$= 1 + \delta \int_0^1 \left(\int_\Theta \frac{\phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta}{1-\delta\mu(\theta)} \right) \eta \mathrm{d}\eta, \tag{30}$$

$$= 1 + \delta \int_\Theta \frac{1}{1-\delta\mu(\theta)} \left[\int_0^1 \eta\phi(\eta|\theta)\mathrm{d}\eta \right] \xi(\theta)\mathrm{d}\theta, \tag{31}$$

$$= 1 + \int_\Theta \frac{\delta\mu(\theta)}{1-\delta\mu(\theta)} \xi(\theta)\mathrm{d}\theta, \tag{32}$$

$$= \int_\Theta \frac{\xi(\theta)\mathrm{d}\theta}{1-\delta\mu(\theta)}, \tag{33}$$

verifying (26) and the conjecture of the value function. Combining (25) and (26) yields

$$g_L(y,\xi) = \left(1 + \delta \int_0^1 \kappa_1 \left(\hat{\xi}(\cdot|\eta) \right) \eta \left[\int_\Theta \phi(\eta|\theta)\xi(\theta)\mathrm{d}\theta \right] \mathrm{d}\eta \right)^{-1} y, \tag{34}$$

$$= y/\kappa_1(\xi), \tag{35}$$

$$= \left(\int_\Theta \frac{\xi(\theta)\mathrm{d}\theta}{1-\delta\mu(\theta)} \right)^{-1} y. \tag{36}$$

Since both the utility and production functions are strictly concave in c, (36) is the unique maximizer corresponding to (19).

Proof of Proposition 4.1. Since $R''(x) = 2\delta/(1-\delta x)^3 > 0$, the right-hand side of (16) is less than the right-hand side of (17) for any c, by Jensen's inequality. Therefore, $g_{AL}(y;\xi) > g_L(y,\xi)$.

Proof of Proposition 4.2. Suppose $\mu(\theta^*) = \int_\Theta \mu(\theta)\xi(\theta)\mathrm{d}\theta$. First, $g_I(y;\theta^*) = g_{AL}(y;\xi)$ from (15) and (16). Second, $g_{AL}(y;\xi) > g_L(y,\xi)$ from Proposition 4.1. Therefore, $g_I(y;\theta^*) = g_{AL}(y;\xi) > g_L(y,\xi)$.

Proof of Proposition 4.3. Suppose $\theta^* = \int_\Theta \theta\xi(\theta)\mathrm{d}\theta$. First, if $\mu'' < -2\delta mu'^2/(1-\delta\mu)$, then, for any c, the right-hand side of (15) is greater than the right-hand side of (17) by Jensen's inequality. Therefore, $g_I(y;\theta^*) <$

$g_L(y,\xi)$. The proofs for $\mu'' > -2\delta\mu'^2/(1-\delta\mu)$ and $\mu'' = -2\delta\mu'^2/(1-\delta\mu)$ are identical. Second, if $\mu'' < 0$, then, for any c, the right-hand side of (15) is greater than the right-hand side of (16), since $\mu(\theta^*) > \int_\Theta \mu(\theta)\xi(\theta)\mathrm{d}\theta$ by Jensen's inequality. Therefore, $g_I(y;\theta^*) < g_{AL}(y;\xi)$. The proofs for $\mu'' > 0$ and $\mu'' = 0$ are identical. Third, $g_{AL}(y;\xi) > g_L(y,\xi)$ from Proposition 4.1. Combining these three points yields Proposition 4.2.

Proof of Proposition 5.3. From (11), if $\phi^1 \succ_1 \phi^2$, then $\mu^1(\theta) = \int_0^1 \eta\phi^1(\eta|\theta)\mathrm{d}\eta \geq \int_0^1 \eta\phi^2(\eta|\theta)\mathrm{d}\eta = \mu^2(\theta)$ implying that $g_L^1(y,\xi) \leq g_L^2(y,\xi)$.

Proof of Proposition 5.4. From (11), if $\phi^1 \succ_2 \phi^2$, then $\mu^1(\theta) = \int_0^1 \eta\phi^1(\eta|\theta)\mathrm{d}\eta = \int_0^1 \eta\phi^2(\eta|\theta)\mathrm{d}\eta = \mu^2(\theta)$ implying that $g_L^1(y,\xi) = g_L^2(y,\xi)$.

Proof of Proposition 5.5. Suppose that $\xi^1 \succ_1 \xi^2$. If $\mu' > 0$, then, for every c, the expected marginal return on investment in (17) is greater under ξ^1 than under ξ^2. Therefore, $g_L(y,\xi^1) \leq g_L(y,\xi^2)$. The proofs for $\mu' < 0$ and $\mu' = 0$ are identical.

Proof of Proposition 5.6. Suppose that $\xi^1 \succ_2 \xi^2$. If $\mu'' < -2\delta mu'^2/(1-\delta\mu)$, then, for every c, the expected marginal return on investment in (17) is greater under ξ^1 than under ξ^2, by Jensen's inequality. Therefore, $g_L(y,\xi^1) \leq g_L(y,\xi^2)$. The proofs for $\mu'' > -2\delta mu'^2/(1-\delta\mu)$ and $\mu'' = -2\delta mu'^2/(1-\delta\mu)$ are identical.

B Complete and Incomplete Learning

Let $\hat{y} = f(y - g(y,\xi), \eta)$, where $g(y,\xi)$ is optimal consumption and η is an unobserved realization of $\tilde{\eta}$. Suppose that the support of $\tilde{\eta}$ is $H = [\alpha, \beta]$ with $0 < \alpha < \beta < 1$ and

$$\min\{f(k,\alpha), f(k,\beta)\} \leq f(k,\eta) \leq \max\{f(k,\alpha), f(k,\beta)\}$$

for $k = y - g(y,\xi)$ and $\eta \in [\alpha, \beta]$.

Figure 3 illustrates the case in which complete learning occurs under positive consumption. In the case of no consumption, the stochastic steady state is degenerate at y_3, i.e., $\hat{y} = y_3$ for any θ^* and $\eta \in [\alpha, \beta]$. If the agent is endowed with y_3, complete learning does not occur with zero consumption.

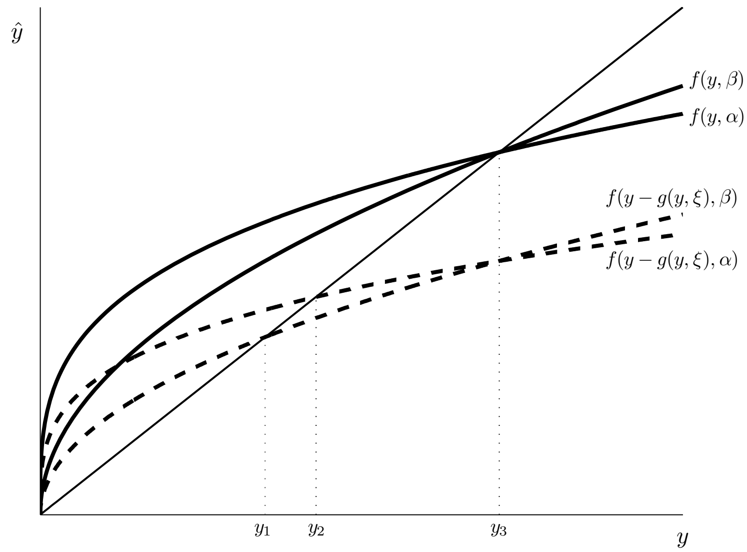

Figure 3: Complete Learning

However, under positive consumption, the stochastic steady state is nondegenerate with support $[y_1, y_2]$, so that the agent eventually learns, from any initial output.

There is a case in which complete learning cannot occur under positive consumption. This is illustrated in Figure 4. In the case of no consumption, the stochastic steady state is nondegenerate with support $[y_2, y_3]$. Here, the agent with zero consumption eventually learns. However, under positive consumption, the stochastic steady state is degenerate at y_1. Here, $\hat{y} = y_1$ for any θ^* and $\eta \in [\alpha, \beta]$. If the agent is endowed with y_1, complete learning does not occur.

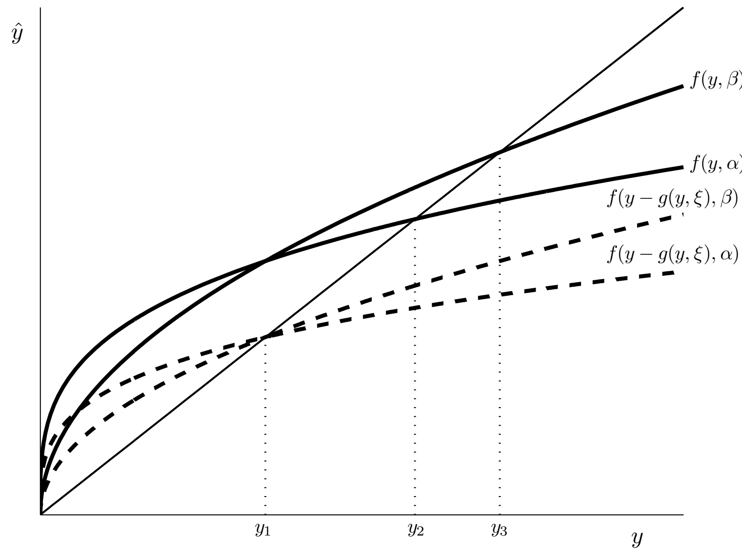

Figure 4: Incomplete Learning

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