Liquidity Supply across Multiple Trading Venues [1]

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Abstract

Market fragmentation and technology have given rise to new trading strategies. One of them is to supply liquidity simultaneously across multiple trading venues, which requires multi-venue management of inventory risk. We build an inventory model in which order flow fragments across two venues, and show that multi-venue market-makers might consolidate the fragmented order flow, leading to lower transaction costs. We also show that multi-venue market-making strategies result in interrelated spreads. We empirically investigate the main predictions of our model using Euronext proprietary data that contains member's orders and trades identities for multi-listed firms. We find evidence of cross-venue inventory control, in particular for formally registered market-makers. We also find that bid-ask spreads vary with inventories of multi-venue market-makers and the way order flow fragments across all venues, as uniquely predicted by our model.

Keywords: Market fragmentation, multi-venue market-making, bid-ask spreads

EFM Classification code: 360.

Article 17(4) of MiFID II: A"market making strategy" should be considered when the *"strategy involves posting firm, simultaneous two-way quotes [...] on a single trading venue or across different trading venues, with the result of providing liquidity on a regular and frequent basis to the overall market"*

1 Introduction

In the last decade, falling technology costs and changes in regulation both in the U.S. (RegNMS) and in Europe (MiFID) have fostered the proliferation of alternative trading venues, giving rise to the emergence of multi-venue dealers, that is, intermediaries making the market simultaneously across more than one trading venue. For instance, KCG Holdings Inc., one of the largest U.S. trading firms, trades NYSE-listed securities in ARCA, GETMATCHED, BATS-Z, NYSE, EDGA, NASDAQ, BATS-Y, BX, or LIGHTPOOL. Recent empirical evidence (e.g. Brogaard et al, 2014, Jovanovic and Menkveld, 2011, Menkveld, 2013, or van Kervel, 2014) shows that high frequency traders, namely financial institutions which have invested in high speed capacity, informally undertake this role by engaging in market making across different electronic trading venues.

In this paper, we develop a multi-venue inventory model to analyze how competing dealers strategically supply liquidity across multiple trading venues. We then test the predictions of our model using a proprietary dataset from Euronext on multi-traded stocks, in which we can identify financial institutions involved in multi-venue market-making strategies.

Intuitively, when a dealer has the right to make the market across multiple trading venues, such dealer is able to aggregate orders from the various venues by simultaneously executing them. This form of consolidation might impact her inventory exposure and thus her quoting aggressiveness. Besides, the opportunity to split liquidity supply across venues enables a dealer to specialize in one venue and may lower her incentives to post aggressive prices in other venues. We investigate this intuition using an inventory model based on Ho and Stoll (1983), in which order flow fragments between two trading venues. Two risk averse dealers have to simultaneously post prices in the two venues to absorb the incoming part of the order flow. We introduce an asymmetry by assuming that the venue termed as the dominant market receives a larger portion of the order flow than the alternative venue termed as the satellite market.

We show that the execution of the order flow may remain fragmented if the different parts of the fragmented order flow are executed by different dealers ("fragmentation"). It may also be the case that the total fragmented order flow is executed by a ***single/unique*** dealer ("consolidation"). This result depends on whether order flows sent to the dominant and the satellite market have the same sign or not, and on how divergent the dealers' inventory positions are from each other.

When order flows have the same sign across venues, a dealer faces a "dual liability risk": in case her quotes are hit, the dealer executes, say, a cumulated buy transaction. The premium due to this additional risk could have lead to larger spreads. This is not the case. Actually, the dealer is willing to consolidate the fragmented order flow when her inventory position is extreme relative to her opponent. Attracting the total order flow allows her to reduce her inventory exposure, she thus sets very aggressive quotes across venues. In contrast, when her inventory position is close to her opponent, she may choose to execute only the part of the fragmented order flow which best reduces her relative inventory exposure, thus specializing in one venue.

When order flows have opposite signs, the impact of the cumulated transaction across venues is smaller due to an "offsetting"effect. Counter-intuitively, this might not be desirable for dealers. For instance, when a dealer's inventory position is very long relative to her opponent's, she is reluctant to execute more sell orders that would exacerbate her inventory exposure. She will thus post attractive prices only in the venue receiving buy orders to reduce her inventory risk (specialization). When her position is less long and close to her opponent's, the dealer may be interested in executing orders that offset each other to benefit from the resulting small impact on inventory. She thus may be willing to execute the total fragmented order flow across venues.

Overall we show that our results depend on the possibility of dealers to compete across all venues, or just one of them. When a multi-venue dealer need to consolidate the total order flow to reduce her relative inventory exposure, she is forced to choose very competitive prices in all venues due to the potential specialization in one venue of the opponent. This case of consolidation of the fragmented order flow is thus characterized by a very strong price competition, which yields to lower spreads compared to a batch auction in which the total order flow would have been sent to a single venue (Ho and Stoll, 1983).

We also show that dealers' multi-venue quoting strategies are strategically interdependent, making liquidity, measured by bid-ask spreads, interconnected across venues.

The model implies cross-venues inventory effects. A multi-venue dealer is expected to update her quotes in one venue in response to a transaction in another venue. At the venue level, the model predicts that variations in bid-ask spreads within one venue are related to the way the total order flow fragments and to the relative positioning of dealers' inventories. In particular, a high divergence between dealers' inventories combined with order flows of same sign across venues should be related to more liquidity (tighter bid-ask spreads) in our model due to the higher degree of price competition across venues in this case.

We test these predictions using a proprietary dataset on multi-venue traded stocks from Euronext on a four-month period in 2007. When Euronext was created in 2000 as a result of the merger of three European Stock Exchanges, namely Paris, Brussels and Amsterdam, the stocks which used to be multi-listed in different Exchanges fell into the Euronext jurisdiction. Within Euronext, trading rules in all markets have been harmonized and structured on the Paris Bourse limit order book model, while remaining separated order books with price-time priority enforced within each market, but not across markets, until 2009. Besides, during that period (that is, before the implementation of MiFID in November 2007), Euronext was virtually collecting all the trades.[1] For these reasons, Euronext provides an excellent laboratory, in line with our theoretical framework, to test our predictions. In our dataset, orders and trades sent to or executed in any limit order book are flagged with a unique ID code and the account used by the financial institution. This enables us to identify 46 multi-venue dealers, that is, members acting either as proprietary traders or as exchange-regulated market makers, who post order messages (submission, revision, or cancellation) and trade at least once in each of the two exchanges on which the stock is traded. Due to the supremacy of Euronext during our time period, our reconstitution of dealers' end-of-day positions, that accounts for their trades in all the limit order books of Euronext, is a good proxy for dealers' aggregated ("global") inventories.

Our empirical analysis finds evidence of cross-venue inventory effects. First, the global

[1]For instance, Gresse (2012) or Degryse, De Jong, and van Kervel (2011) report a market share of more than 95% for French and Dutch stocks respectively over our sample period.

inventories of some multi-venue liquidity suppliers in some stocks exhibit mean reversion. Second, using a logit model, we find that multi-venue liquidity suppliers are more likely to submit messages in direction of inventory reduction in a venue when their preexisting orders have been passively hit in the other venue. When they do an active transaction, we find that their activity is more related to arbitrage trading strategy than multi-venue inventory management. Last, our empirical analysis shows that bid-ask spreads within one venue are significantly lower when the divergence of inventory position among dealers is large and when order flows across venues have the same sign, in line with our main prediction.

Our empirical analysis is motivated by a new theoretical approach to multi-market trading. Traditional models including Pagano (1989), Chowdhry and Nanda (1991), Bernhardt and Hughson (1997), Easley, Kiefer and O'Hara (1996), and Foucault and Menkveld (2008) assume that quotes are competitively set by independent pools of market makers in multiple markets to satisfy the zero-profit condition. They focus on the routing or order splitting decisions of strategic liquidity demanders, who can either be informed or not. Naturally, these splitting strategies are anticipated by the liquidity suppliers who adjust their quotes in the different markets. We instead exogenously fix order flows routed towards each market to focus on the inter-dependent quoting strategies of multi-venue market-makers. As Seppi (1997) and Parlour and Seppi (2003), we model competition for order flow based on liquidity provision when liquidity suppliers are not perfectly competitive. Parlour and Seppi (2003) extend the model proposed by Seppi (1997) to analyze the quotes set by a monopolist specialist competing against a competitive order book, and incorporate liquidity demander's optimal splitting. The specialist has a timing advantage over limit orders traders. In our model, market-makers post their quotes simultaneously. We show that risk averse liquidity suppliers using multi-venue strategies make the spreads interrelated across venues, even in the absence of private information.

Few empirical papers focus on the extent to which traders exploit multi-market environments. Menkveld (2008) and Halling, Moulton, and Panayides (2013) focus on how investors adjust their trading strategies to multi-trading. In contrast, we investigate how liquidity suppliers deal with a multi-market environment, and our empirical analysis is most closely related to van Kervel (2014) and Jovanovic and Menkveld (2011). van Kervel (2014) finds that trades on the most active venues for 10 FTSE100 stocks are often followed by immediate cancellations of limit orders on competing venues, which would be expected in the presence of a multi-venue dealers facing a dual liability risk. Jovanovic and Menkveld (2011) statistically identify a multi-venue dealer actively trading across Euronext and Chi-X, and find that the participation of this dealer has an impact on spreads and volumes. Both findings are in line with our theoretical predictions and complement our empirical analysis. Since each institution in our sample is identified by a unique ID across the multiple limit order books we are able to precisely analyze the quoting strategies of all the members who exploit the multi-market environment.

The paper is organized as follows. Section 2 describes the model and investigates the price formation in a two-venue market-making environment. Section 3 describes the data, provides summary statistics and tests the main predictions. Section 4 concludes the paper. All proofs are available in the Appendix.

2 The Model

This section analyzes how the existence of multiple trading venues influences the price-setting strategies of risk-averse dealers.

2.1 The basic setting

We consider the market for a risky asset, whose final cash flow is a normal random variable $\tilde{v}$ characterized by an expected value μ and a variance σ^2. There are two types of market participants: investors who demand liquidity and dealers who supply liquidity.

Dealers' reservation price. Liquidity is supplied by two dealers $i = 1, 2$. Each dealer i is endowed with a different initial inventory position of the risky asset I_i, where I_i is the realization of the random variable $\tilde{I}_i$ uniformly distributed on $[I_d, I_u]$. Dealers are *risk-averse* and have the following common CARA utility function:

$$u(\tilde{w}_i) = -\exp(-\rho\tilde{w}_i), \tag{1}$$

where ρ is the risk aversion, and $\tilde{w}_i$ the terminal wealth of dealer i endowed with an initial position I_i.

As Ho and Stoll (1983) demonstrate, dealer i's reservation price r_i to execute the incoming order flow Q is such that:

$$r_i(Q) = \mu - \rho\sigma^2 I_i + \frac{\rho\sigma^2}{2} Q,\ i = 1, 2. \tag{2}$$

By convention, we denote by $Q > 0$ a buy incoming order flow, and by $Q < 0$ a sell incoming order flow. Note that the marginal valuation of dealer i, $(\mu - \rho\sigma^2 I_i)$, depends on the risk of holding an inventory position. A dealer in a long position is reluctant to increase her exposure to inventory risk by adding more inventory and posts relatively low ask and bid prices to encourage selling operations. The second component of reservation prices $((\rho\sigma^2/2)Q)$ represents the price impact of trades and is thus increasing in trade size: larger buy orders will drive dealer i' selling price more above dealer i's marginal valuation (and vice versa).

For ease of exposition, we consider that dealer 1 is endowed with a longer inventory position, i.e. $I_1 \geq I_2$.[2]

Fragmentation of order flow. Liquidity demanders exogenously split their order flow across two trading venues denoted D and S.[3] We assume that the part sent to venue D, denoted Q_D, is larger than that routed to venue S, i.e. $|Q_D| \geq |Q_S|$. We thus term venue D as the dominant market, and venue S as the satellite market. The analysis provided below is restricted to the case in which the total order flow is net-buying and fragments such that $Q_D \geq 0$, while Q_S might be either buy or sell order flow: $Q_S \geq 0$ or $Q_S \leq 0$. Symmetric results are obtained for a net-selling order flow.

Quoting strategies of multi-venue dealers. We assume that dealers have access to all trading venues. Conditional on observing Q_D and Q_S, multi-venue dealers post their quotes simultaneously in venues D and S. The dealer who posts the lowest ask price in venue D executes Q_D, while the dealer with the most attractive price (lowest ask or highest bid, depending on the sign of Q_S) in venue S executes Q_S.

A quoting strategy for dealer i is a couple of quoted prices (a_i^D, p_i^S) where a_i^D is the ask price posted by dealer i in market D and p_i^S is the price posted by i in market S

[2]...

[3]In the base model, order flows are exogenously fragmented. We address the case of endogenizing order flow splitting by liquidity demanders in section 2.4.

(which is an ask price if $Q_S > 0$ or a bid price if $Q_S < 0$). In the next section, we analyze the Nash equilibria of the quoting game, defined as a vector of the quoting strategies of the two dealers. In equilibrium, dealer i executes the order flow Q_m if $p_i^m Q_m < p_{-i}^m Q_m$ for $m \in \{D, S\}$ (see Preliminary Remarks in Appendix for detailed trading profits). All random variables are independent and their distributions are common knowledge.

In our set-up dealers must manage their risky inventory position by keeping track of orders across all trading venues. Because making the market "globally", i.e. across various venues, affects dealer's total exposure to inventory risk, we also qualify their inventory as "global" inventory as opposed to ordinary inventory that guides a dealer taking risks just in one venue.[4]

Figure 1 shows the extensive form of the trading game. The focus of the paper is to analyze price formation across venues when Q_D and Q_S are simultaneously non-zero, which occurs with probability λ, and to investigate differences between the case in which Q_D and Q_S have same sign (with probability γ), or opposite sign (with probability $1-\gamma$). We find the solution of this game by backward induction.

2.2 Equilibrium quotes in fragmented markets

In this paper, we assume that dealers observe competitors' quotes, as if markets were transparent. Consider the benchmark case where order flow is batched and sent to a single venue, which is the case analyzed by Ho and Stoll (1983). The dealer with the longer inventory position (dealer 1 by assumption) posts the most competitive ask quote, by quoting the second lowest reservation price ($(a^{batch})^* = r_2(Q_D + Q_S) - \varepsilon$, in our case). This section analyzes how order flow fragmentation might alter this result.

2.2.1 Preliminary results

When dealers have access to more than one venue, they can choose to post competitive quotes across all venues, or just one of them, and compete to execute the total fragmented order flow, or just a part of it. The outcome of whether the fragmented order flow might

[4] Our definition of "global" inventory is close to the definition of *equivalent* or total inventory emphasized by Ho and Stoll (1983) and discussed in Naik and Yadav (2002). However, while equivalent inventory is the overall position of a dealer across all stocks, global inventory is the net aggregated inventory position of a dealer in a single stock but across all available trading venues.

be consolidated or not through the execution by a single dealer depends on the conditions described by Lemma 1.

Lemma 1 *Assume that $I_1 > I_2$ and that $Q_D + Q_S > 0$.*

1. If $(I_1 - I_2 - Q_D)Q_S < 0$, and if an equilibrium exists, then it is such that the total order flow remains fragmented: orders submitted to the different venues are executed by different dealers. Conversely, if $(I_1 - I_2 - Q_D)Q_S > 0$, and if an equilibrium exists, then it is characterized by the consolidation of the fragmented order flow, through a multi-venue execution by a single dealer.

2. If there exists an equilibrium such that the total order flow remains fragmented, then the more extreme dealer specializes in the dominant venue, while the less extreme dealer in the satellite venue. If there exists an equilibrium characterized by the consolidation of the fragmented order flow, then the more extreme dealer executes the total order flow across venues.

The outcome of Lemma 1 (consolidation versus fragmentation) depends on two conditions: the price impact of the total fragmented order flow and the relative positioning of dealers' inventory. Regarding the first condition, when Q_D and Q_S have the same sign, the price impact of trading in the two venues is *cumulative*. When order flows have opposite signs, the reverse effect or *offsetting* effect is observed: trading in both venues enables dealers to reduce the impact of a trade executed within a single venue.

Regarding the second condition, dealers' incentives to trade only a part versus the total fragmented order flow depend on their exposure to inventory risk, and in particular on their relative inventory positions. Under our assumption that dealer 1 is endowed with a longer inventory position, she has more incentives to sell than dealer 2. The total order flow $(Q_D + Q_S)$, which is net-buying, is more attractive to her. She is thus willing to post more aggressive selling prices across venues. The price aggressiveness however depends on how divergent her inventory position is to dealer 2' s. When her inventory position is more extreme $(I_1 - I_2 > Q_D)$, dealer 1 would rather execute the largest buy order flow as possible, which is $Q_D + Q_S$ when Q_D and Q_S have the same sign, or only Q_D when Q_S has an opposite sign. Executing the selling order flow Q_S would instead exacerbate her inventory risk exposure. When her inventory position is less extreme and closer to her opponent's position, she finds less desirable to execute order flows with large price impact and thus prefers to execute only Q_D, and, in some cases, the order flow with the smallest possible price impact, which is $Q_D + Q_S$ when Q_D and Q_S have opposite signs.

Note that our results are in line with the outcome of the Vickrey-Clarke-Groves (VCG) mechanism for combinatorial auctions.[5] In particular, order flows across venues can be seen as substitutes when they have the same sign, and as complements when they have opposite signs. Indeed, when Q_D and Q_S have same signs, the marginal gain of trading $Q_D > 0$ when the dealer also trades $Q_S > 0$ is lower than when he does not trade Q_S, while the marginal gain is higher when $Q_S < 0$. Substitutability is also a key determinant of the outcome of the VCG mechanism.[6]

2.2.2 Equilibrium quotes

Lemma 1 shows that dealers' willingness to execute a part or the entire fragmented order flow depends on the divergence of inventories and on whether order flows routed to trading venues have the same sign or not. The interaction of these two characteristics determines the prices posted by dealers at equilibrium as shown in Proposition 1 below.

Proposition 1 *Assume that $I_1 > I_2$ and $Q_D + Q_S > 0$.*
1. If $(I_1 - I_2 - Q_D)Q_S > 0$, there exists a Nash equilibrium, in which dealer 1, the longer dealer, consolidates the fragmented order flow by posting the best prices across venues, while dealer 2 quotes his own reservation prices, that is:

- *1.1. If $Q_S > 0$, dealer 1 chooses the following best selling prices in venue D and venue S:*

$$\left((a_1^D)^*, \quad (a_1^S)^*\right) = (r_2(Q_D) - \varepsilon, \quad r_2(Q_S) - \varepsilon)\,;$$

- *1.2. If $Q_S < 0$, dealer 1 simultaneously posts the best selling price in venue D and the best bid price in venue S, as follows:*

$$(\left(a_1^D\right)^*, \quad \left(b_1^S\right)^*) = (r_2(Q_D) - \rho\sigma^2\left(-Q_S\right) - \varepsilon, \quad r_2\left(Q_S\right) + \varepsilon);$$

[5] In combinatorial auctions, multiple items, which are related but not necessarily identical, are sold simultaneously and bidders may submit bids on packages of items.

[6] To illustrate the VCG mechanism, suppose that there are two items for sale (D and S) and two bidders. Let us denote by $v_i(D)$ bidder i's value for item D, by $v_i(S)$ bidder i's value for item S, and by $v_i(DS)$ bidder i's value for the bundle D and S. In this mechanism, if $v_1(DS) > v_1(D) + v_2(S)$, then the outcome is that bidder 1 wins both items. This condition corresponds to the condition described in Lemma 1 and Proposition 1, which is: $(I_1 - I_2 - Q_D)Q_S > 0$. See Vickrey (1961), Clarke (1971), Groves (1973), and Ausubel and Milgrom (2006) for a discussion of the VCG mechanism.

2. If $(I_1 - I_2 - Q_D)Q_S \leq 0$, there exists a unique Nash equilibrium, in which the longer dealer is the first seller in the dominant market while the shorter dealer quotes the best price in the satellite market, that is:

2.1. If $Q_S > 0$, dealers post the following ask prices:

$$\begin{aligned}
\left((a_1^D)^*, \quad a_1^S\right) &= \left(r_2(Q_D) + \rho\sigma^2 Q_S \left(\frac{Q_D - (I_1 - I_2)}{Q_D}\right) - \varepsilon, \quad r_1(Q_S) + \rho\sigma^2 Q_D\right), \\
\left(a_2^D, \quad (a_2^S)^*\right) &= \left(r_2(Q_D) + \rho\sigma^2 Q_S \left(\frac{Q_D - (I_1 - I_2)}{Q_D}\right), \quad r_1(Q_S) + \rho\sigma^2 Q_D - \varepsilon\right);
\end{aligned}$$

2.2. If $Q_S < 0$, dealers post the following ask prices in venue D and bid prices in venue S:

$$\begin{aligned}
\left((a_1^D)^*, \quad b_1^S\right) &= \left(r_2(Q_D) - \rho\sigma^2(-Q_S) - \varepsilon, \quad r_1(Q_S) + \rho\sigma^2 Q_D\right), \\
\left(a_2^D, \quad (b_2^S)^*\right) &= \left(r_2(Q_D) - \rho\sigma^2(-Q_S), \quad r_1(Q_S) + \rho\sigma^2 Q_D + \varepsilon\right);
\end{aligned}$$

where ε corresponds to one tick.

Note that the simultaneous price formation across more than one venue depends on three characteristics. First, dealers are not constrained to post competitive prices for the total fragmented order flow, and may choose to compete just for a part of it in one venue. Second, order flows may execute at different prices across venues given that dealers post prices reflecting the different impact of order flows which are of different magnitude (second-degree price differentiation).[7] Third, similar to inventory models like Ho and Stoll (1983) or Biais (1993), the relative distance between dealers' inventory position conditions the competitiveness of dealers' quotes across venues. When the divergence is low (resp. high), dealer 1's inventory position is close to (resp. away from) that of dealer 2, and dealers are less (resp. more) able to post competitive prices.

Figure 2 summarizes Proposition 1. Consider first the case in which dealer 1 consolidates the total order flow, i.e. $(I_1 - I_2 - Q_D)Q_S > 0$ (the first row "Consolidation" in the Figure). Suppose that order flows have same sign, and dealer 1's inventory is extreme ($I_1 - I_2 > Q_D$) relative to dealer 2. Dealer 1 is more willing to post very competitive

[7] In Europe, a consolidated tape in which all trades and quotes of all exchanges and multi-trading facilities would be recorded does not exist, orders sometimes execute at prices different that the best existing quoted prices in the market (trade-throughs are allowed).

prices in order to benefit from the *large* impact of $Q_D + Q_S$ to decumulate her inventory. Dealer 2 is however able to post competitive selling prices just in one venue to execute either Q_D or Q_S, forcing dealer 1 to choose very aggressive prices across venues to be sure to attract the total order flow. In this case fragmentation increases intra-market competition. Observe that, relative to the batch auction, (*ex post*) transaction costs are lower: $TC - TC^{batch} = (a^D)^* Q_D + (a^S)^* Q_S - a^{batch}(Q_D + Q_S) = -\rho\sigma^2 Q_D Q_S < 0$. In case order flows have opposite signs and dealer 1' s inventory is closer to her competitor's, antisymmetric effects are observed. Dealer 1 is willing to execute the net order flow $Q_D + Q_S$ to benefit from a *smaller* impact compared to a single trade in venue D or S. The ability of dealer 2 to compete in just one venue forces dealer 1 to post competitive (but not too aggressive) prices due to the closeness of dealers' inventories. In this case, there is a multiplicity of equilibria. We select the equilibrium in which there is price continuity at $I_1 - I_2 = Q_D$ in market D. At any equilibrium though, the weighted averaged price paid by investors is equal to $r_2(Q_D + Q_S)$, that is, the price formed at equilibrium in the batch auction. In this case, *ex post* transaction costs in fragmented markets are thus equal to transaction costs paid in the batch auction.

Consider now the case in which the execution of the total order flow $Q_D + Q_S$ is split among dealers and remains fragmented (the second row termed "Fragmentation" in Figure 2). Suppose that order flows have the same sign, and dealer's 1 inventory is close to dealer 2 ($I_1 - I_2 < Q_D$). Then dealer 1 is less able to post simultaneous aggressive selling prices across venues. It is more profitable to let her opponent execute the smaller part Q_S and post a competitive selling price just in one venue, D. Transaction costs thus vary with the degree of price competition, related to divergence of inventories. When dealers' inventories are very close ($I_1 - I_2 \to 0$), they cannot post very different prices from each other and competition is weak ($TC - TC^{batch} \to \rho\sigma^2 Q_D Q_S > 0$). When dealers' inventories are more divergent ($I_1 - I_2 \to Q_D$), prices are more aggressive, transaction costs are smaller: $TC - TC^{batch} \to -\rho\sigma^2 Q_D Q_S < 0$. Suppose now that Q_D and Q_S have opposite signs, dealer 1's inventory is extreme relative to dealer 2. She is very keen to execute the large buy order Q_D to decumulate her extreme inventory position and reluctant to add more inventory by trading the sell order Q_S, which is anticipated by dealer 2. Dealer 2 therefore posts a non-aggressive price in market S, even less aggressive that dealer 1' inventory is more extreme. *Ex post* transaction costs are thus worse: $TC - TC^{batch} =$

$\rho\sigma^2(I_1 - I_2 - Q_D)(-Q_S) \geq 0$.

It is worth noticing that in case order flows remain fragmented ("Fragmentation"), dealers obtain a better allocation of risk compared to the batch auction, as shown in the following lemma.

Lemma 2 *The fragmented market generates a more efficient outcome in risk sharing among dealers than the batch market in the sense that dealers bear lower aggregate security risk in the fragmented market.*

The better allocation of risk does not however necessarily lead to more competitive prices as detailed above since dealers have less incentives to undercut each other. This result is in the spirit of the one obtained in Biais, Foucault and Salanié (1998).

2.2.3 Expected best offers

In our model, because dealers manage their position globally, they place quotes in one venue taking into account the potential impact of trading in the other venue. Dealers' quoting aggressiveness depends on their eagerness to consolidate or not the total fragmented order flow given their global inventory position. The inter-dependent quoting aggressiveness across venues in turn impacts the magnitude of market spreads in each venue.

Using Proposition 1, we compute the expected best prices in the dominant and satellite markets for any set of inventory positions and any sign for the order flows Q_D and Q_S in Proposition 2 below. For ease of exposition, we denote by q_m the magnitude of the order flow routed to market m: $-q_m = Q_m$ for a net-selling order flow and $q_m = Q_m$ for a net-buying order flow ($m = D, S$).

Proposition 2 *Under the assumption that $q_D < (I_u - I_d)$, the expected (half-) spreads in the dominant and the satellite venues respectively write:*

$$E\left(\underline{s}^D\right) = \frac{\rho\sigma^2}{2}\left(q_D - \frac{2I_d + I_u}{3}\right) + \lambda_S\rho\sigma^2 q_S\left[\gamma\left(\frac{q_D}{(I_u - I_d)} - \frac{(q_D)^2}{3\,(I_u - I_d)^2}\right) - (1-\gamma)\right], \tag{3}$$

$$E\left(\underline{s}^{S}\right)=\frac{\rho\sigma^{2}}{2}\left(q_{S}-\frac{2I_{d}+I_{u}}{3}\right)+\lambda_{D}\rho\sigma^{2}q_{D}\left[\frac{q_{D}}{(I_{u}-I_{d})}-\frac{(q_{D})^{2}}{3\left(I_{u}-I_{d}\right)^{2}}-(1-\gamma)\right], \quad (4)$$

where γ is the probability that order flows routed to D and to S have the same sign and λ_{-m} is the conditional probability to observe simultaneously a non-zero order flow routed to the alternative venue given that the order flow routed to market m is non-zero ($\lambda_{-m} = Pr(q_{-m} \neq 0|q_m \neq 0)$) ($m = D, S$).

In line with the intuitions exposed above, the first component of the expected best offer (Eq. (3) and (4)) is the *direct* price impact of the order flow routed to this venue. It corresponds to the expected best offer that would prevail if q_{-m} is zero (or $\lambda_{-m} = 0$). The second component consists of the *indirect* price impact of trading in another venue (q_{-m}) resulting from the interdependent quoting strategies of multi-venue market-making. In particular, note that the order flow routed to the dominant market has a bigger impact on spreads in the satellite market than the reverse (given that $\lambda_D \geq \lambda_S$, $q_D \geq q_S$, and $(1-\gamma)(q_D/r_d - r_u - (q_D/r_d - r_u)^2) \geq 0$). It is also worth noticing that expected spreads across venues are increasing with the probability that order flows Q_D and Q_S have same signs (γ).

Proposition 2 shows that spreads in one venue are indirectly influenced by orders sent to other venues due to the presence of strategic multi-venue dealers. Multi-venue market-making strategy makes the liquidity (measured by quoted spreads) of different venues interrelated in our model, as stated by the following Corollary:

Proposition 3 *Expected spreads co-vary jointly:*

$$\frac{cov(s^{D},s^{S})}{4\left(\rho\sigma^{2}\right)^{2}\left(I_{u}-I_{d}\right)^{2}}=\frac{1}{18}\left(\frac{3\gamma-1}{2}\right)-\phi_{D}^{2}\left(\frac{1}{6}-\frac{2}{9}\phi_{D}+\gamma\frac{\phi_{D}^{2}}{12}\right)-\gamma\phi_{D}\phi_{S}\left(\frac{\phi_{D}^{4}}{9}-\frac{2\phi_{D}^{3}}{3}+\frac{5\phi_{D}^{2}}{4}-\frac{8\phi_{D}}{9}+\frac{1}{6}\right) \quad (5)$$

where $\phi_D = \frac{q_D}{(I_u - I_d)}$ and $\phi_S = \frac{q_S}{(I_u - I_d)}$, and s^m is the half-spread in venue m ($s^m = (\underline{a}^m - \mu) \times 2$).

Our model proposes a new explanation to the interconnectedness of trading venues. Market interrelations might be explained by arbitrage strategies (Rahi and Zigrand, 2013), duplicate strategies (van Kervel, 2014) or directional trading strategies (Chowdhry and

Nanda, 1991), and also by inventory management strategies of multi-venues market-makers.[8]

2.2.4 Market quality

The previous results raise a natural question: overall, is market performance better or worse when liquidity supply is strategically supplied across multiple venues?

From Proposition 2, we compute the total expected transaction costs in order to determine whether making the market across multiple venues has a positive or negative impact on investors. The next corollary compares them to expected transaction costs that would prevail in a batch market (our natural benchmark).

Corollary 1 *Expected transaction costs are lower in fragmented markets than in a batch market if $\gamma > \frac{1}{3}$ and q_D is neither too large, nor too small ($r^1_\gamma(I_u - I_d) < q_D < r^2_\gamma(I_u - I_d)$).*

The intuition of the corollary is as follows. For large values of γ, if q_D is large, the probability that dealers' inventory are highly divergent is low, which in turn implies that the probability to observe more aggressive quoting strategies than in the benchmark is also low. Expected transaction costs are thus higher in case of fragmented trading. When q_D is small enough, dealers' prices are more likely to be more competitive, and even more competitive than the benchmark, leading to lower average transaction costs. For small values of γ, if q_D is small, a small divergence between dealers' inventory position is less likely, and the probability to observe quoting strategies as aggressive as in the benchmark is small. If q_D is large, this probability is higher. The higher competitiveness of dealers' quotes is thus obtained in two opposite situations: for large q_D when γ is small and for small q_D when γ is large. The second situation has on average a larger impact, resulting in more competitive spreads when q_D is small enough but not too small (depending on γ).

This ambiguous result of fragmentation on market performance is consistent with the mixed empirical evidence investigating market performance in the context of fragmented markets (see, e.g., the literature review in O'Hara and Ye, 2011). In our model, multi-venue market-makers consolidate the order flow through their inventory management, which may have a positive externality in some cases. Few theoretical models find positive

[8]See Cespa and Foucault (2013) for interconnectedness across different assets.

impacts of fragmentation of trading. Foucault and Menkveld (2008) show that the total depth is larger due to the presence of investors who consolidate the market through their queue jumping strategy across limit order books.

2.3 Discussion

The aim of this section is to assess the impact of relaxing some of the model's assumptions. We analyze two extensions. First, we relax the hypothesis that order flows sent to markets D and S are exogenously split and analyze the impact of endogenizing fragmentation. Second, we investigate whether inventory divergences between dealers are so large that dealers would prefer trading and sharing risks in the inter-dealer market in a first stage, before trading or not in the customer-dealer market in a second stage.

2.3.1 Endogenous fragmentation of the total order flow

This sub-section investigates the case in which investors must trade a given quantity denoted Q and might choose to split orders across venues in order to optimize their execution costs.[9] Note that the strategic decision to spatially split up order flow extends the case in which order flows sent to D and S are exogenously of same sign. As in section 2.1, we suppose that dealer 1 is longer than dealer 2 and that Q is a buy order flow, i.e. $Q > 0$ (results about a sell order flow, or dealer 2 longer than dealer 1 are deduced by symmetry).

We consider that liquidity demanders enjoy some private benefits denoted δ_m to trade in venue m. We assume that $\delta_D > \delta_S$, consistently with the dominant market defined above, and that $\delta_D - \delta_S < \rho\sigma^2 Q$.[10,11] Liquidity demanders choose the proportion α of the order flow routed to market D (and $(1-\alpha)$ to market S) so as to minimize their

[9] See Degryse et al (2013) for an analysis of "order splitting" by liquidity demanders over time rather than over venues.

[10] Numerous studies (Froot and Dabora, 1999, Gagnon and Karolyi, 2004, Foerster and Karolyi, 1999, Shleifer and Vishny, 1997, or Stulz, 2005) document the existence of a domestic bias, due to investment barriers, e.g., regulatory barriers, taxes, or information constraints. Still in 2013 in Europe, brokerage fees charged to a trade in a foreign country or trading venue are 15 to 40% higher than those charged to trade in a national exchange, but the situation was even worse back in 2007 (see documents on Fees and Commissions of various brokers from 2007 to 2013).

[11] When $\delta_D - \delta_S \geq \rho\sigma^2 Q$, the private benefits of trading in venue D are so large that it is never optimal for investors to split the quantity to be traded across trading platforms.

transaction costs.[12] Let us show that there exists an equilibrium such that $\alpha \in (\frac{1}{2}; 1)$, that is, such that investors optimally split order flows across platforms.[13] In this interval, transaction costs write:

$$TC(\alpha) = [(\underline{a}^D(\alpha Q) - \delta_D - \mu)\alpha + (\underline{a}^S((1-\alpha)Q) - \delta_S - \mu)(1-\alpha)] \times Q.$$

In the Appendix, we show the existence and the characterization of an equilibrium α^*. This yields the following proposition.

Proposition 4 *If $I_1 - I_2 > \frac{(\delta_D - \delta_S)}{2\rho\sigma^2}$, there exists an interior equilibrium α^*, such that it is optimal for investors to split their order flow across venues.*

We find that there exists cases in which α^* is strictly lower than 1, which means that liquidity demanders may choose to split order flows across platforms to optimize transaction costs. Liquidity demanders trade-off the benefits of price competition through fragmentation (related to inventories divergence, $I_1 - I_2$) to the private benefits of sending the total quantity in the dominant market ($\delta_D - \delta_S$), that is, when $r_2(Q) - r_1(Q) > (\delta_D - \delta_S)$.

2.3.2 Introduction of an inter-dealer market

In this section, we analyze the sensitivity of our results to the introduction of an inter-dealer market in which dealers are able to optimally share inventory risks (stage 1) before setting quotes in the customer-dealer market (stage 2).

In a conservative approach, we assume that dealers independently and unstrategically maximize their expected profit in the inter-dealer market, then their expected profit in the customer-dealer market (the model is solved sequentially).[14]

In the first stage, we find that at the symmetric equilibrium, dealers perfectly share inventory risk in the inter-dealer market, that is, they trade a quantity $q^* = \frac{I_1 - I_2}{2}$ at price

[12] Following our set up in which markets are transparent, we suppose that liquidity demanders perfectly anticipate what the best bid and ask prices are.

[13] See Supplementary Appendix for a complete proof of the existence and characterization of the equilibria.

[14] When considering the case in which dealers strategically trade in the inter-dealer market *after* observing the realization of the order flows in markets D and S, we find that dealers may find it optimal to reinforce the divergence in their inventory positions. We would like to show that our analysis is robust to non-strategic risk sharing among dealers, that is the reason why our paper focuses on the case in which the two stages are independent.

$p^* = \mu - \rho\sigma^2 \frac{I_1+I_2}{2}$ such that their new inventory positions (I_1', I_2') write $I_1' = I_2' = \frac{I_1+I_2}{2}$. In the second stage, we simply use the equilibrium in the customer-dealer market derived in section 2.2 for the limit case where $I_1' \to I_2'$. Finally, we compute and compare the dealers' expected profits whether they trade or not in the inter-dealer market. This yields the following corollary.

Corollary 2 *The set of parameters for which dealers choose not to trade in the inter-dealer market is non-empty.*

2.4 Testable implications

Our multi-venue inventory model implies two types of testable predictions: at the liquidity supplier level, changes in inventories of multi-venue market-makers might drive their cross-venue quote submission/revision strategies ; at the venue level, risk-aversion and inventories divergence affect the degree of price competition, potentially generating variations in bid-ask spreads.

At the liquidity supplier level, our model predicts that reservation prices depend on dealers' risk aversion, the variance of the risky asset, the total quantity that might be executed across venues and the level of dealers' *global* inventory in the asset across venues. Define a trader's global inventory as her aggregate net volume across all trading venues: $I_{i,t} = I_{i,0} + \sum_{\tau=0}^{\tau=t} Q_\tau^D + \sum_{\tau=0}^{\tau=t} Q_\tau^S$. We can thus formulate the following testable hypothesis:

Hypothesis 1 *If dealers provide liquidity across multiple venues, their global inventories should display mean reversion.*

Our model implies that, after a trade, say in venue S, that increases the inventory exposure, a multi-venue dealer should update quotes in venue S, but also in venue D to elicit inventory-reducing orders. Within-venue inventory effects are tested in Hansh, Naik and Viswanathan (1999) or Reiss and Werner (1998) for dealers markets, and Raman and Yadav (2013) for limit order book markets. We specifically focus on cross-venue inventory effect, that has to be formulated in the context of our experiment, i.e. the limit-order-book environment of Euronext. We thus posit the following hypothesis:

Hypothesis 2 *Multi-venue market-makers should update existing limit orders or submit* *new orders in one venue after a trade in another venue, in a direction that is associated with their inventory changes.*

For instance, after executing a sell order in the satellite venue that increases the total inventory exposure, a multi-venue market-maker should be more likely to cancel an existing buy order in the dominant market, or modify it for a less aggressive price (negative revision), or post a new sell limit order in the dominant market or modify an existing sell order for a more aggressive price (positive revision). We acknowledge that other trading motives could yield to inventory-like orders placement strategies, such as cross-venue arbitrage strategies. In case, say, the bid price in market S jumps above the best ask in market D, an arbitrageur might step in and sell one share in market S, and buy one in market D to reduce the existing price discrepancy. The buy and sell orders submissions from the arbitrageur are empirically similar to inventory-driven strategies. We thus control for arbitrage opportunities in our empirical analysis.

At the trading venue level, Proposition 1 shows that price competition is strong when inventories divergence among dealers is large and when order flows across venues have same signs. We thus expect more liquidity, or tighter spreads, when both are true. Comerton-Forde et al (2010) finds that variation in spreads on the NYSE are related to the aggregate level of the committed capital by market-makers (inventories) and in particular to the tightness of the funding market. We use a measure of differences in inventory positions across dealers to test whether it matters for liquidity variations, while controlling for the signs of order flows across venues:

Hypothesis 3 *Variation is spreads in one venue depends on both the directions of order flows across venues (identical or opposite), and on how extreme dealers' inventory positions are relative to each other.*

This prediction is interesting because it allows us to distinguish our theory from a competing adverse-selection hypothesis: in case an informed trader would split his orders across venues, the adverse selection component of multi-venue market-makers should increase. Liquidity should thus decrease in all venues if order flows across venues have same direction. The impact of the interaction on (lagged) inventories divergence and order flow direction should however have a positive impact on liquidity, as predicted by our model.

3 Empirical analysis

In order to test the predictions of the model, we use a proprietary dataset from Euronext on multi-listed stocks.

3.1 Forming the sample

Euronext was created in 2000 as a result of the merger of three European exchanges, namely Amsterdam, Brussels and Paris. Lisbon joined in 2002. Before the introduction of the Universal Trading Platform (UTP) in 2009, the four exchanges maintained their domestic market. As a result, firms could be multi-listed on several Euronext exchanges; for example, Air France-KLM was traded in Amsterdam and in Paris.

Our sample consists of all multi-traded stocks within Euronext, spanning four months (79 trading days) from January 1, 2007 to April 30, 2007.[15] The data on orders and quotes are provided by Euronext. Euronext files also provide us with the identification of the member participating in each quote or transaction, and whether the member is acting as an agent or as a principal. The data assigns the same code to a member across stocks and across exchanges, enabling us to trace members' inventory changes and quoting behavior across time and across exchanges. Euronext exchanges follow the same market model (same trading hours, and same trading rules), and the payment of membership fees grants access to all Euronext markets. Note also that, during our sample period (pre-MiFID environment), trading was concentrated in Euronext.[16] For all these reasons, Euronext is an excellent environment to test the predictions of our model.

We keep firms that trade in euros using a continuous trading session in at least two exchanges on which they are traded. We also restrict our analysis to members acting in their capacity as a principal (that is, either proprietary traders or exchange-regulated market makers) who post order messages (submission, revision, cancellation) and trade at least once in each of the two exchanges on which the stock is traded. Overall, we follow 46 members, denominated as "multi-venue dealers". Because these dealers do

[15]Three trading days are dropped in January due to missing data about best limits.

[16]Some French stocks were traded on the LSE or the Deutsche Boerse, while some Dutch stocks were traded in Xetra. Gresse (2012) finds a market share of 96.45% for CAC40 stocks and even 99.99% for other SBF120 stocks in October 2007. Degryse, De Jong, and van Kervel (2011) show that Euronext concentrates the trading volume of the 52 AEX Large and Mid cap constituents on our sample period.

not necessarily follow the same stocks, our sample finally consists of 178 couples (stock, dealer), among which 20% involve an exchange-regulated market-maker, called thereafter Designated Market-Maker (DMM) (see Panel C of Table 1).[17]

The final sample contains 20 firms with at least one multi-venue dealer with non-missing data, trading continuously in two Euronext exchanges. Among them, 11 are traded on Euronext Amsterdam, 12 are traded on Euronext Brussels and 17 on Euronext Paris. To determine which is the dominant market (market D in the model) and which is the satellite market (market S in the model), we use the primary market as the (exogenous) dominant platform.

3.1.1 Measuring liquidity

We measure the spread in the market m as the equally-weighted average bid-ask spread for stock j, during a twenty-minutes interval t.[18] We focus on the relative bid-ask spread $RBAS\_m$, and the variation of the relative spread between two consecutive intervals, $\Delta RBAS\_m$, where $m = DOM, SAT$.

3.1.2 Measuring global inventory

As pointed out by Hansch et al. (1998), dealers differ in the amount of capital at risk they commit to their trading activities and/or in their risk aversion. We follow their methodology by building standardized inventory positions to control for these differences. Let $IP_{i,t}^{s}$ denote the inventory position of multi-venue dealer i in stock s at the end of day t. We use the record of all trades executed by each multi-venue dealer in multiple markets as a principal, plus the direction of these trades in both markets to obtain her inventory position at the end of each day. We thus construct a time series for each multi-venue dealer's inventory position in each stock from the start to the end of our sample period. Since more than 95% of the volumes are traded in Euronext during our sample period, our inventory variable is a good proxy for dealers' global inventories. We compute

[17] Our paper does not compare the liquidity provision of exchange-regulated market-makers versus endogenous market-makers, as Anand and Venkatamaran (2013) do using Toronto Stock Exchange data. We however keep trace of their difference in trading behaviors as suggested by the literature.

[18] We compute both equally-weighted and time-weighted averages of the quoted spreads. As the results for the two weighting schemes are virtually identical, we restrict the presentation to the equally-weighted spread measures.

the mean ($\overline{IP}_i^s$) and the standard deviation (S_i^s) for each of these inventory series. The standardized inventory is defined as follows:

$$I_{i,t}^s = \frac{IP_{i,t}^s - \overline{IP}_i^s}{S_i^s}.$$

We then build a measure of the inventory divergence. Let $I_{M,t}^s$ denote the median inventory on day t in stock s, and let $ID_{i,t} = |I_{i,t}^s - I_{M,t}^s|$ denote the member i's inventory position relative to the median inventory. The larger ID_i, the more divergent the inventory position of member i relative to the median is, and the more aggressively she will quote, in order to reduce her inventory exposure (Hansh et al., 1998). We take the mean of the inventory divergence across dealers for each interval t and each stock s, $\overline{RI}_t^s$, to get the degree of quoting aggressiveness induced by inventory management. We use this measure as a proxy of the difference $I_1 - I_2$ in our model.

3.1.3 Determining the direction of order flows across venues

The model's predictions depend on whether order flows sent across venues have the same or the opposite direction. We define the net order flow in market m (i.e. trade imbalance) in stock s during a twenty-minutes interval t, $TrIMB\_m$, as the number of buyer-initiated trades minus the number of seller-initiated trades. The dummy variable d_POS takes the value of one if the order flows have the same direction ($TrIMB\_DOM \times TrIMB\_SAT > 0$) on a given twenty-minutes interval, and zero otherwise. Note that we exclude the first and last five minutes of trading in order to avoid contamination by specific trading behaviors during the open or close of the markets.[19]

3.1.4 Control variables

In regressions, we control for the existence of arbitrage opportunities given that arbitrageurs by buying the asset in one venue and reselling it in the other venue behave as inventory-driven market-makers. The dummy d_AO takes the value of one if the best bid in one venue exceeds the best ask in the other venue, i.e. $max(Bid\_SAT, Bid\_DOM) > min(Ask\_SAT, Ask\_DOM)$. We also expect arbitrageurs to use more often aggressive

[19] On February 19, 2007, the closing fixing moved from 5:25 pm to 5:30 pm. We therefore drop all observations before 9:05 am and after 5:20 pm.

transactions (marketable orders) than passive transactions (non-aggressive limit orders) to take fast arbitrage opportunities. We thus use the dummy d_AT which takes the value of one if the origin of transaction executed by the dealer is an aggressive order, and zero if it is a limit order hit. In some regressions, we also control for the pending time to the next market close ($TimeClos$), the (log) transaction size in number of shares ($TrSize$), and the number of trades $NbTr$.

3.2 Summary statistics

Table 1 presents summary statistics for our sample. Panel A presents statistics across stocks. The average (median) firm has a stock price of 53.3 (50.09) Euros, a market cap of 31.5 (23.2) billion Euros, and 10 (6) multi-venue dealers trading on the stock. There is an average number of 3 arbitrage opportunities per day, and 60% of order flows across venues have the same direction. Panel B presents statistics computed within each market. Relative (quoted) spreads of the satellite market are five to ten times larger than those of the dominant market, depending if one takes means or medians. The number of trades is much smaller (twenty five times in average) in the satellite market, reflecting lack of trade activity, and transaction size is also much smaller. Surprisingly, the number of best limit updates is only three times less in average in the satellite venue, which attracts also 33% of order messages in average. It seems that the satellite market is not a very active trading place, but it is closely monitored. T-tests of the difference in means between the two markets (not shown) confirm the statistical significance of these differences. Panel C presents statistics computed for each multi-venue dealer. In particular, we test for mean reversion in dealer global inventories, by considering the following standard model of inventory time series,

$$\Delta I_{it} = \alpha + \beta I_{it-1} + \varepsilon_t$$

where ΔI_{it} is the change in inventory from the previous trade. Mean reversion predicts that $\beta < 0$ (if $\beta = 0$, it has a unit root and it is non-stationary).

Across the 178 couples dealers-stock, Panel C shows that the average mean-reversion parameter (β) is -0.073, which means that dealers reduce, in average, inventory by 7.3% during the next trade. Mean reversion might be strong. The null hypothesis of a unit root is rejected at the 1% level by the Phillips-Perron test (Perron, 1988) in 11 cases, at

5% in 11 cases and at 10% in 2 cases.

3.3 Multivariate analysis

3.3.1 Inventory management across venues

The main implication of our multi-venue inventory model is that dealers actively manage their inventory position across venues. The aim of the paper is in particular to investigate cross-venues inventory management strategies. We thus test whether a multi-venue dealer updates his orders in one venue in response to a transaction in another venue (a change in his global inventory). We focus on the use of the dominant more liquid market after a transaction in the satellite market. For example, after a buy in the satellite market, a multi-venue dealer should cancel, negatively revise existing buy orders, or submit new sell orders or positively modify sell orders in the dominant market, and symmetrically after a sell. We thus implement the following Logit regression for each multi-venue dealer:

$$\begin{aligned} Pr(d\_i) &= \alpha + \beta_1 d\_DMM + \beta_2 |I_{i,\tau-1}| + \beta_3 d\_DMM \times |I_{i,\tau-1}| \\ &\quad + \beta_4 d\_AO_{s,\tau} + \beta_5 log(TrSize_{s,\tau}) + \beta_6 TimeClos_{s,\tau} + \varepsilon_{s,\tau} \qquad (6) \end{aligned}$$

where d_i is the dummy variable that takes 1 if dealer i sends an order in the dominant market in direction of inventory following a trade at time τ in the satellite market.[20] The explanatory variables are the lagged absolute inventory position of dealer i, the dummy variable for designated market-makers, and the interaction between both. We also control for the existence of an arbitrage opportunity at the time of the trade, the size of the trade, and the pending time to the close. Our specification also includes firm fixed-effects to control for time-invariant firm heterogeneity. We run the regression both after an active and a passive transaction.

The results of the Logit analysis are presented in Table 2. Panel A reports the results for order submissions after a passive transaction, while Panel B reports the results for order submissions after an active transaction. In both cases, the likelihood that multi-venue dealers manage actively their inventory across venues is larger when they are dedicated market-makers. However cross-venues trading strategies seem different according

[20] Orders are tracked through their first 10 seconds after a trade.

to whether the change in global inventory has been caused by a passive transaction or an active transaction. The probability to post cross-venues management orders is negatively related to the existence of an arbitrage opportunity when the transaction is passive, while it is positively related when it is active. Dealers are thus more likely to submit messages in direction of inventory in dominant market when their preexisting orders have been passively hit in the satellite market, while they might use arbitrage strategies when they cause active transactions. Consistent with this finding, we find that dedicated market-makers are more likely to use cross-inventory management when their inventory position is more extreme and when the transaction is passive. When the transaction is active, the inventory position of dedicated market-makers has no influence, in favor of an arbitrage trading strategy. In summary, these results are consistent with the main implication of our model that multi-venue dealers might implement active inventory management across venues.

3.3.2 Spreads

To test the prediction on spreads of our model, we estimate the relation between the variation in twenty-minute bid-ask spreads in the satellite market and the price competition among multi-venue dealers which is related to their inventory divergence ($\overline{RI}^s$) and to the direction of order flows across venues (i.e., whether the dummy d_POS is equal to one). We run the following panel regression model:

$$\Delta RBAS\_SAT_t^s = \alpha + \beta_1 \overline{RI}_{t-1}^s + \beta_2 d\_POS_t^s + \beta_3 d\_POS_t \times \overline{RI}_{t-1} + \beta_4 NbTr\_SAT_t^s + \varepsilon_t^s \quad (7)$$

Lemma 1 predicts that the sign of the order flows routed across platforms impacts the spreads. More specifically, from Corollary 1, we expect the following sign: $\beta_2 > 0$. Given that the average inventory divergence, $\overline{RI}$, is a proxy for quoting aggressiveness, we expect that in case of extreme inventory divergence and same direction of order flows across venues, dealers compete aggressively to execute both orders, $\beta_3 < 0$. This interaction term allows us to depart from any adverse selection effect which would predict $\beta_3 > 0$. Finally, the number of trades in the satellite market, $NbTr\_SAT$, controls for the impact of trades, we thus expect $\beta_4 > 0$.

All specifications include day dummies and use clustered standard errors by stock to accommodate the possibility that relative spreads are strongly correlated within firms.

Table 3 presents estimation results. We report two specifications: the first with time fixed-effect (Column 1)) and the second with day and firm fixed-effects. The main conclusions from the analysis are as follows. First, spreads in the satellite market vary with the direction of order flows across venues (coeff. 0.108, t-stat. 2.14 in column 1), consistently with our predictions. Second, the variable of interest which is the interaction term between the direction of order flows and inventory divergence has a negative and statistical significant impact on spreads (coeff. -0.12, t-stat. -2.00). Dealers post more aggressive prices when there is a high difference of inventories between them and when order flows have the same direction, as predicted by Proposition 1. Results for other control variables are not statistically significant.

The most important result of Table 3 is that spreads in the satellite market are significantly lower when the divergence of inventory position among dealers is large and when order flows across venues have the same sign, as uniquely predicted by Proposition 1.

4 Conclusion

A better understanding of fragmentation and cross-venues trading strategies is all the rage these days. We develop a multi-venue inventory model in which risk averse market makers quote a single asset in two venues. Counter-intuitively, we find that cross-venue inventory control may yield a consolidation of liquidity. Our model has interesting policy implications as we show that fragmentation may decrease total transaction costs. The intuition for this result is that the opportunity to execute only the portion of the order flow that best reduces inventory exposure, and to post competitive quotes in either venue, obliges competitors to post aggressive quotes across all venues.

Our model also yields unique empirical predictions. In particular, we show that quote aggressiveness depends i. on the way order flow fragments between venues, ii. on the divergence of dealers' inventory positions, and iii. on the interaction between the two. We exploit the co-existence of multiple order books for the same security within Euronext to test these predictions. Our empirical results are as follows. First, we document the presence of multi-venue market makers. Second, we find evidence of cross-venue inventory control effects. Third, our panel regression analysis reveals that bid-ask spreads in a venue vary with the order flow routed to the alternative venue, and to its interaction with the dispersion in inventory positions. These empirical findings are in line with the predictions of the model and cannot be explained by alternative theories, e.g. arbitrage strategies.

Overall, our results complement the existing literature on liquidity interconnectedness across multiple venues. We suggest the existence of an alternative common factor in liquidity that differs from the traditional public or private information channels, related to multi-venue market makers' inventory management. This raises intriguing issues. For practitioners, it calls for new measures of idiosyncratic liquidity that would enable brokers looking for best execution to assess the relative performance of each trading venue. For regulators, it calls for a better understanding of the potential consequences of a volatility shock, since we exhibit the presence of indirect effects of the asset's volatility on bid-ask spreads due to the covariance between spreads.

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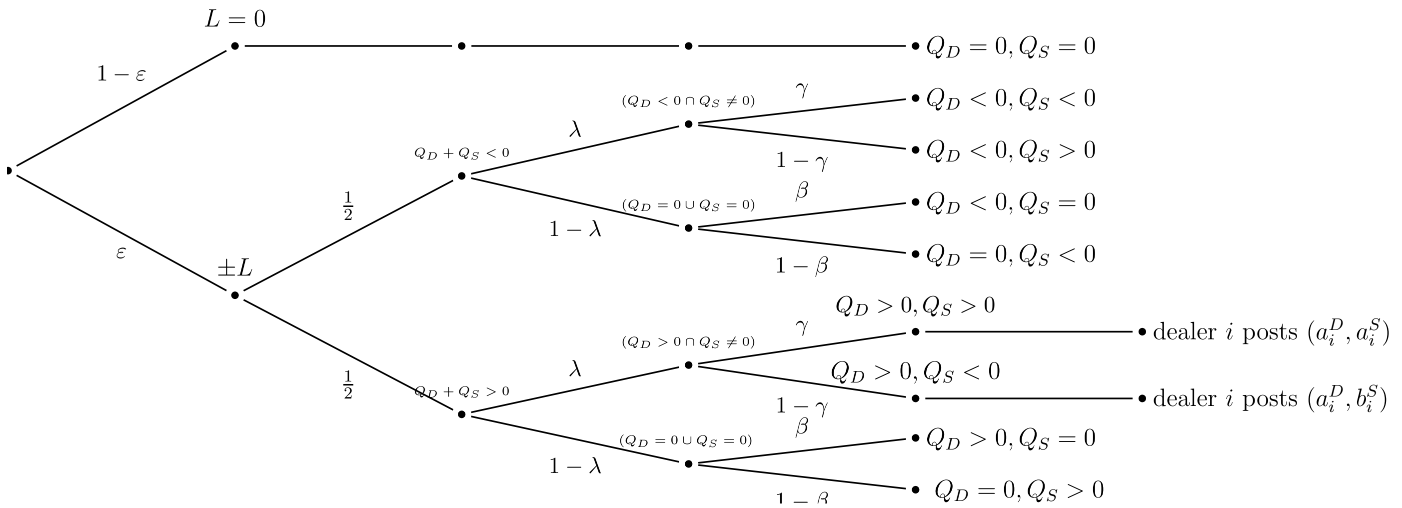

Figure 1: Tree of the quoting game across trading venues

Figure 1 represents the tree of the trading game. At date 1 (not represented on the Figure), dealer i is endowed with an inventory position denoted I_i. At date 2, liquidity demanders are affected by a liquidity shock with probability ε and send a net buying order flow with probability 1/2. This order flow fragments between two trading venues D and S . With probability λ, the part sent to D denoted Q_D and the other part sent to S denoted Q_S are simultaneously different from zero, and are of same sign with probability γ. Dealers observe . At date 3, they post an ask price if $Q_m > 0$ or a bid price if $Q_m < 0$ in market m. Simultaneously, if $Q_{-m} \neq 0$, they post an ask price if $Q_{-m} > 0$ or a bid price if $Q_{-m} < 0$ in market $-m$. In Section 3, we find the equilibrium quotes of the subgames (1+) and (1-) for $m = D$, and (1+) and (3-) for $m = S$.

	$Q_S > 0$	$Q_S < 0$
	Consolidation	
1. $(I_1 - I_2 - Q_D) \times Q_S > 0$	Intense competition - Consolidation of the total fragmented order flow through dealer 1 $TC - TC^{batch} = -\rho\sigma^2 Q_D Q_S < 0$	Normal competition - Consolidation of the total fragmented order flow through dealer 1 $TC - TC^{batch} = 0$
	Fragmentation	
2. $(I_1 - I_2 - Q_D) \times Q_S < 0$	Competition : dealer 1 executes Q_D and dealer 2 executes Q_S $TC - TC^{batch} = \rho\sigma^2 (Q_D - 2(I_1 - I_2)) Q_S$ $TC - TC^{batch} < 0$ if $(I_1 - I_2) \in]Q_D/2, Q_D]$ $TC - TC^{batch} \geq 0$ if $(I_1 - I_2) \in [0, Q_D/2]$	Weak competition: dealer 1 executes Q_D and dealer 2 executes Q_S. $TC - TC^{batch} = \rho\sigma^2 (I_1 - I_2 - Q_D)(-Q_S) > 0$

Figure 2: Outcome of the quoting game - Summary of Proposition 1

Table 1

Summary statistics

This table reports summary statistics for the data used in this study. The sample consists of 20 multi-listed, continuously-traded stocks on Euronext exchanges, from January 1, 2007 through April 30, 2007 (79 trading days). The quotes and trades data comes from Euronext, and other stock-level information comes from Compustat Global.

Panel A reports the daily mean across the 20 stocks for the variables used in this study. Market capitalization is price times shares outstanding, in millions of Euros. Bid-Ask Spread is the equally-weighted average difference between the best bid and the best ask during the day. Time-weighted Bid-Ask Spread is weighted by the number of seconds the quotation was outstanding. Price (or Midpoint) is the average, throughout the trading day, of the average between the best bid and the best ask. Relative Spread, is equal to the equally-weighted average of ratio between the spread and the midpoint. Number of Trades is the number of transactions per day across the total number of trading venues. Number of Messages is the daily total number of orders (submissions, revisions, cancellations) across the total number of trading venues. Trade Size is the daily average size of transactions across trading venues. Number of Arbitrage Opportunities is the daily number of times the best bid in the dominant (resp. satellite) market is greater than the best ask in the satellite (resp. dominant) market. Number of Global dealers is the total number of liquidity suppliers (as defined in Section 4.1). Average Inventory Divergence (RI_m) is the average divergence across dealers' inventory position, where inventory positions are measured each 20 minutes interval. d_POS is a dummy variable that takes the value of one if order flows across venues have the same direction.

Panel B reports summary statistics by market type. It contains news variables. Percentage of Trades is the ratio of the number of transactions executed in the dominant (resp. satellite) market over the total number of transactions. Number of Best Limits Updates is the total number of times there is a change in the best limits. Percentage of Active Trades is the ratio of the number of transactions caused by a market or a marketable order over the total number of transactions in the trading venue. Percentage of Passive Trades is the ratio of the number of limit order hit over the total number of transactions in the trading venue. Percentage of Cancellations (resp. New Submissions) is the ratio of the number of cancellations (resp. new submissions) over the total number of messages in the trading venue. Percentage of Revisions is the ratio of the number of revised orders over the total number of messages in the trading venue.

Panel C reports summary statistics by global dealers. Trade Size in the Dominant Market is the average daily size of transactions executed in the dominant market. Percentage of Trades in the Satellite Market is the ratio of trades executed by global dealer j in the satellite market over the total number of his transactions. Percentage of Messages in Direction of Inventory is the ratio of the number of messages submitted within 10 seconds in the dominant market after a transaction in the satellite market which are in direction of inventory management over the total number of messages submitted within 10 seconds in the dominant market after a transaction in the satellite market. d_DMM is the dummy that take one if the global dealer is a dedicated market-maker is the stock.

Table 1

Summary statistics (cont.)

Panel A. Summary statistics by stock

	N	Mean	Std. Dev.	Q1	Median	Q3
Market Capitalization (in billion)	2628	31491	32882	129	23151	84201
Bid-Ask Spread	3449	0,680	1,723	0,014	0,119	2,413
Time-weighted Daily Bid-Ask Spread	3449	0,610	1,629	0,013	0,108	2,003
Relative Bid-Ask Spread	3449	1,070	2,383	0,048	0,271	3,648
Price (Midpoint)	3449	53,30	36,40	9,58	50,09	113,38
Number of Trades	3476	2656	3352	0	1380	9235
Number of Messages	3476	10622	10379	65	8229	29907
Trade Size	2884	491	576	33	304	1617
Number of Arbitrage Opportunities per day	3476	3	9	0	0	18
Number of Liquidity Suppliers	3476	10	9	1	6	22
Average inventory divergence, RI_m	3470	0,63	0,36	0,07	0,61	1,20
d_POS	2743	0,59	0,29	0,00	0,60	1,00

Panel B. Summary statistics by type of market

B.1 Dominant market

	N	Mean	Std. Dev.	Q1	Median	Q3
Bid-Ask Spread	1580	0,11	0,13	0,01	0,06	0,39
Relative Bid-Ask Spread	1580	0,28	0,37	0,04	0,12	1,08
Number of Best limits Updates	1580	6063	5901	52	4853	16963
Number of Trades	1580	2577	3108	10	1457	8563
Percentage of Messages	1578	64	27	15	66	100
Percentage of Active Trades	1338	44	26	8	39	100
Percentage of Passive Trades	1338	56	26	0	61	92
Percentage of Cancellations	1410	12	13	0	9	39
Percentage of Revisions	1410	33	36	0	16	100
Percentage of New Submissions	1410	22	17	0	25	47
Transaction Size	1578	620	684	52	360	2305

B.2 Satellite market

	N	Mean	Std. Dev.	Q1	Median	Q3
Bid-Ask Spread	1567	1,23	2,37	0,03	0,33	4,91
Relative Bid-Ask Spread	1567	1,87	3,28	0,10	1,00	6,61
Number of Best limits Updates	1554	2624	3797	3	794	10626
Number of Trades	1580	99	402	0	3	534
Percentage of Messages	1578	33	27	0	27	85
Percentage of Active Trades	1109	31	28	0	30	100
Percentage of Passive Trades	1109	69	28	0	70	100
Percentage of Cancellations	1395	8	11	0	4	26
Percentage of Revisions	1395	79	26	0	90	99
Percentage of New Submissions	1395	8	12	0	4	29
Transaction Size	1112	349	369	18	250	1000

Panel C. Summary statistics by global dealer						
	N	Mean	Std. Dev.	Q1	Median	Q3
Percentage of Trades in D	187	17	23	0	8	71
Number of Trades in D	187	70	131	0	8	377
Trade Size in D	187	724	656	92	579	2136
Percentage of Trades in S	178	7	17	0	1	40
Number of Trades in S	178	9	28	0	1	69
Trade Size in S	178	599	679	38	352	1881
Percentage of Messages in Direction of Inventory	110	66	30	0	66	100
Percentage of Passive Transactions in S	178	53	30	0	52	98
Delay to submit a message in Direction of Inv.	110	3	2	0	3	8
Dummy for Dedicated Market-Maker	178	0,19	0,39	0,00	0,00	1,00
Average Mean Reversion of Inventory	178	-0,073	0,150	-0,314	-0,013	0,001

Table 2

Likelihood of Expected Inventory-related Message

following a Transaction in the Satellite Market

This table presents estimates of the relation between the likelihood of a message in the dominant market in direction of inventory after a trade in the satellite market. The left-hand side variable is Indicator of Expected Message, a dummy variable that takes the value 1 if the message has the expected value. Left-hand side variables are described in caption of Table 1. DMM × Standardized Inventory is an interaction term equal to the product of DMM and Standardized Inventory. Panel A shows regression specifications in the subsample of.passive transactions. Panel B shows regression specifications in the subsample of active transactions. All specifications include firm fixed effects and t-statistics are calculated using standard errors clustered by liquidity supplier. The symbols ***, **, * denote significance levels of 1%, 5% and 10%, respectively. for the two-tailed hypothesis test that the coefficient equals zero.

Table 2
Likelihood of Expected Inventory-related Message following a Transaction in the Satellite Market (cont.)

Panel A. Passive Transactions

Dependent variable:	Indicator of Expected Message			
	(1)		(2)	
Log Trade Size	0.032		0.032	
	(1.05)		(1.05)	
Standardized Inventory	0.018		-0.02	
	(0.56)		(-0.55)	
DMM	1.522	***	1.377	***
	(3.70)		(3.42)	
Arbitrage Opportunity	-0.310	***	-0.309	***
	(-3.31)		(-3.33)	
Time to close	0.025		0.025	
	(1.38)		(1.36)	
DMM × Standardized Inventory			0.187	**
			(2.33)	
Intercept	0.217		0.243	
	(0.66)		(0.74)	
Firm FEs	Yes		Yes	
N	18,022		18,022	
Pseudo R^2	0.06		0.06	

Panel B. Active Transactions

Dependent variable:	Indicator of Expected Message			
	(1)		(2)	
Log Trade Size	-0.015		-0.014	
	(-0.45)		(-0.45)	
Standardized Inventory	-0.005		0.043	
	(-0.08)		(0.59)	
DMM	0.646	**	0.733	***
	(2.44)		(3.76)	
Arbitrage Opportunity	0.597	***	0.603	***
	(4.46)		(4.58)	
Time to close	0.013		0.014	
	(0.80)		(0.81)	
DMM × Standardized Inventory			-0.125	
			(-0.67)	
Intercept	1.402	**	1.348	**
	(2.30)		(2.10)	
Firm FEs	Yes		Yes	
N	9,100		9,100	
Pseudo R^2	0.06		0.06	

Table 3

Determinants of Relative Spreads in the Satellite Market

This table presents estimates of the relation between the change in relative bid-ask spreads in the satellite market and the divergence of inventory position across dealers and direction of order flows across platforms. The left-hand side variable is the Change in Relative Spread of the Satellite market in the 20-minutes interval. The right-hand-side variables are defined in caption of Table 1. Same Direction × Lag Absolute RI is an interaction term equal to the product of Same Direction and Lag Absolute RI. t-statistics are calculated using standard errors clustered by firm. The symbols ***, **, * denote significance levels of 1%, 5% and 10%, respectively. for the two-tailed hypothesis test that the coefficient equals zero.

Dependent variable:	Change in Relative Spread of Market S			
	(1)		(2)	
Same Direction	0.108	**	0.105	**
	(2.14)		(2.13)	
Lag Absolute RI	0.087		0.076	
	(1.14)		(1.34)	
Same Direction × Lag Absolute RI	-0.12	**	-0.119	**
	(-2.00)		(-2.01)	
Number of Trades in Market S	-0.050		0.004	
	(-1.30)		(0.12)	
Intercept	-0.078		-0.065	
	(-0.93)		(-1.03)	
Time FEs	Yes		Yes	
Firm FEs	No		Yes	
N	11,172		11,172	
Adjusted R^2	0.01		0.03	

5 Appendix – Proofs

Preliminary elements used in the proofs: Trading Profits

When $Q_S > 0$, dealer i's trading profit is simply given by:

$$v_i\left(a_1^D, a_2^D, a_1^S, a_2^S\right) = \begin{cases} a_i^D Q_D + a_i^S Q_S - r_i\left(Q_D + Q_S\right)\left(Q_D + Q_S\right) & \text{if } a_i^D < a_{-i}^D \text{ and } a_i^S < a_{-i}^S, \\ \left(a_i^D - r_i\left(Q_D\right)\right) Q_D & \text{if } a_i^D < a_{-i}^D \text{ and } a_i^S > a_{-i}^S, \\ \left(a_i^S - r_i\left(Q_S\right)\right) Q_S & \text{if } a_i^D > a_{-i}^D \text{ and } a_i^S < a_{-i}^S, \\ 0 & \text{if } a_i^D > a_{-i}^D \text{ and } a_i^S > a_{-i}^S. \end{cases}$$

When $Q_S < 0$, dealer i's trading profit is given by:

$$v_i\left(a_1^D, a_2^D, b_1^S, b_2^S\right) = \begin{cases} a_i^D Q_D - b_i^S\left|Q_S\right| - r_i\left(Q_D + Q_S\right)\left(Q_D + Q_S\right) & \text{if } a_i^D < a_{-i}^D \text{ and } b_i^S > b_{-i}^S, \\ \left(a_i^D - r_i\left(Q_D\right)\right) Q_D & \text{if } a_i^D < a_{-i}^D \text{ and } b_i^S < b_{-i}^S, \\ \left(r_i\left(Q_S\right) - b_i^S\right)\left|Q_S\right| & \text{if } a_i^D > a_{-i}^D \text{ and } b_i^S > b_{-i}^S, \\ 0 & \text{if } a_i^D > a_{-i}^D \text{ and } b_i^S < b_{-i}^S. \end{cases}$$

where a_i^m (resp. b_i^m) denote the ask (resp. bid) price set by dealer $i \in \{1; 2\}$ in market $m \in \{D; S\}$.[21]

Proof of Lemma 1

Case 1. We first look for the necessary conditions that must be simultaneously fulfilled to guarantee the existence of an equilibrium characterized by the *consolidation* of the order flow.

Dealer $i \in \{1, 2\}$ trades the total fragmented order flow in equilibrium if and only if the ask price a_D prevailing in market D (in which $Q_D > 0$), and the ask or the bid price p_S prevailing in market S (in which $Q_S > 0$ or $Q_S < 0$) are the maximum (resp. minimum in market S if $Q_S < 0$) prices such that: (i) trading $Q_D + Q_S$ is profitable for dealer i (i.e. $v_i(Q_D + Q_S) \geq 0$), but not for dealer $-i$ (i.e. $v_{-i}(Q_D + Q_S) < 0$); (ii) for $m = \{D, S\}$, even if he expects to trade Q_m, dealer i is willing to trade Q_{-m} (i.e. $v_i(Q_D + Q_S) \geq v_i(Q_S)$ and $v_i(Q_D + Q_S) \geq v_i(Q_D)$); (iii) dealer $-i$ is not willing to undercut dealer i neither in

[21] As in Biais (1993), the utility function of dealers given in (1) is linearized, under the assumption $Q_D < I_u - I_d$. Note that, in our centralized setting, the criticism on the linear approximation used by Biais (1993) for the fragmented market raised by de Frutos and Manzano (2002) does not apply. The assumption $Q_D < I_u - I_d$ also guarantees that *ex ante*, dealer i (for $i = 1, 2$) has a probability to post the best price in market m ($m = D, S$) which is strictly greater than 0 and strictly lower than 1.

market D nor in market S (i.e. $v_{-i}(Q_D) < 0$ and $v_{-i}(Q_S) < 0$). Using the definition of dealers' reservation prices and trading profits, these conditions rewrite as follows:

$$\begin{aligned}
\text{i} &: a_D Q_D + p_S Q_S \geq r_i(Q_D + Q_S)(Q_D + Q_S), \\
\text{i'} &: a_D Q_D + p_S Q_S < r_{-i}(Q_D + Q_S)(Q_D + Q_S); \\
\text{ii} &: a_D \geq r_i(Q_D) + \rho\sigma^2 Q_S \\
\text{ii'} &: p_S Q_S \geq \left(r_i(Q_S) + \rho\sigma^2 Q_D\right) Q_S \\
\text{iii} &: a_D < r_{-i}(Q_D) \\
\text{iii'} &: p_S Q_S < r_{-i}(Q_S) Q_S
\end{aligned}$$

Suppose that dealer 1 trades Q_D+Q_S. If $(I_1-I_2-Q_D)Q_S \leq 0$, then $(r_1(Q_S) + \rho\sigma^2 Q_D)\, Q_S \geq r_2(Q_S)Q_S$. Thus conditions (ii') and (iii') cannot hold simultaneously. A necessary condition for such an equilibrium to exist is thus $(I_1 - I_2 - Q_D)Q_S > 0$, i.e., either $I_1 - I_2 > Q_D$ if $Q_S > 0$ or $I_1 - I_2 < Q_D$ if $Q_S < 0$.

Suppose that dealer 2 trades $Q_D + Q_S$. Recall that by assumption $I_1 > I_2$ (implying that $r_1(Q_D + Q_S) < r_2(Q_D + Q_S)$) and $Q_D + Q_S > 0$. Thus conditions (i) and (i') cannot simultaneously hold for $i = 2$. Therefore, there exists no equilibrium such that dealer 2 trades the total order flow.

Case 2. We now look for the necessary conditions that must be simultaneously fulfilled to guarantee the existence of an equilibrium characterized by the *fragmentation* of the order flow.

There exists an equilibrium such that dealer $i \in \{1, 2\}$ trades Q_D and dealer $-i$ trades Q_S if and only if the ask price a_D prevailing in market D (where $Q_D > 0$), and the ask or bid price p_S prevailing in market S (where Q_S is either positive or negative) are the maximum (resp. minimum in market S if $Q_S < 0$) prices such that: (I) dealer i is better off trading Q_D rather than Q_S (i.e. $v_i(Q_D) > v_i(Q_S)$) and dealer $-i$ is better off trading Q_S rather than Q_D (i.e. $v_{-i}(Q_S) > v_{-i}(Q_D)$); (II) dealer $-i$ is better off trading Q_S only rather than $Q_D + Q_S$ (i.e. $v_{-i}(Q_D + Q_S) < v_{-i}(Q_S)$) and dealer i is better off trading Q_D only rather than $Q_D + Q_S$ (i.e. $v_i(Q_D + Q_S) < v_i(Q_D)$) ; (III) trading Q_D is profitable for dealer i (i.e. $v_i(Q_D) \geq 0$) and trading Q_S is profitable for dealer $-i$ (i.e. $v_i(Q_S) \geq 0$).

These latter conditions may be rewritten as follows:

$$\begin{aligned}
\text{I} &: a_D > r_i(Q_D) + (p_S - r_i(Q_S))\,\frac{Q_S}{Q_D} \\
\text{I '} &: p_S Q_S > r_{-i}(Q_S)Q_S + (a_D - r_{-i}(Q_D))\,Q_D \\
\text{II} &: a_D < r_{-i}(Q_D) + \rho\sigma^2 Q_S \\
\text{II'} &: p_S Q_S < \left(r_i(Q_S) + \rho\sigma^2 Q_D\right) Q_S \\
\text{III} &: a_D \geq r_i(Q_D) \\
\text{III'} &: p_S Q_S \geq r_{-i}(Q_S)Q_S
\end{aligned}$$

Suppose that dealer 1 trades Q_D and dealer 2 trades Q_S. If $(I_1 - I_2 - Q_D)Q_S \geq 0$, then conditions II' and III' cannot hold simultaneously. A necessary condition for such an equilibrium to exist is thus $(I_1 - I_2 - Q_D)Q_S < 0$, that is, either $I_1 - I_2 < Q_D$ if $Q_S > 0$ or $I_1 - I_2 > Q_D$ if $Q_S < 0$.

Suppose that dealer 1 trades Q_S and dealer 2 trades Q_D. If $Q_S < 0$, then conditions II and III cannot hold simultaneously, since $r_1(Q_D) + \rho\sigma^2 Q_S < r_2(Q_D)$. If $Q_S > 0$, a necessary condition for conditions I and I' to hold simultaneously is

$$r_1(Q_S) + (a_D - r_1(Q_D))\,\frac{Q_D}{Q_S} < r_2(Q_S) + (a_D - r_2(Q_D))\,\frac{Q_D}{Q_S},$$

which is never true since $I_1 > I_2$ and $|Q_D| > |Q_S|$. Consequently, there exists no equilibrium in which the longer dealer (here, dealer 1) would be the first buyer in market S while the shorter dealer 2 would be the first seller in market D.

Case 3. Suppose finally that $(I_1 - I_2 - Q_D)Q_S = 0$, then $(r_1(Q_S) + \rho\sigma^2 Q_D) = r_2(Q_S)$. If dealer 1 expects to trade Q_D in market D, dealers 1 and 2 have the same reservation prices for Q_S in market S. We show below that in equilibrium, dealer 1 trades Q_D, and both dealers are equally likely to execute Q_S and post the same price in market S, that is, the price at which they both are indifferent between trading or not: $(p_1^S)^* = (p_2^S)^* = (r_1(Q_S) + \rho\sigma^2 Q_D) = r_2(Q_S)$. ■

Proof of Proposition 1

From Lemma 1, there are various cases to consider, depending on the signs of $I_1 - I_2 - Q_D$ and Q_S.

Case 1.1.: $Q_S > 0$ and $I_1 - I_2 > Q_D$, i.e., $(I_1 - I_2 - Q_D)Q_S > 0$.

From Lemma 1, we know that dealer 1 consolidates the total fragmented order flow by posting the best ask prices in both market D and S. The ask prices a_D and a_S are the maximum prices that satisfy the set of conditions i to iii' (Case 1 in the proof of Lemma 1).

$$\begin{aligned}
\text{ii and iii} &: r_1(Q_D) + \rho\sigma^2 Q_S \leq a_D < r_2(Q_D) \\
\text{ii' and iii'} &: r_1(Q_S) + \rho\sigma^2 Q_D \leq a_S < r_2(Q_S) \\
\text{i} &: r_1(Q_D + Q_S)(Q_D + Q_S) \leq a_D Q_D + a_S Q_S \\
\text{i'} &: a_D Q_D + a_S Q_S < r_2(Q_D + Q_S)(Q_D + Q_S)
\end{aligned}$$

From the two first inequalities, $(a_D)^* = r_2(Q_D) - \varepsilon$ and $(a_S)^* = r_2(Q_S) - \varepsilon$ are natural candidates for the equilibrium, as they are the maximum prices that satisfy conditions ii, ii', iii and iii'. Straightforward computations show that they also satisfy conditions i and i' (details are omitted for brevity).

Case 1.2: $Q_S < 0$ and $I_1 - I_2 < Q_D$, i.e., $(I_1 - I_2 - Q_D)Q_S > 0$.

From Lemma 1, we know that dealer 1 consolidates the total fragmented order flow by posting the best ask price in market D and the best bid price in market S. The ask price a_D in market D and the bid price b_S in market S are respectively the maximum and the minimum prices that satisfy the set of conditions i to iii' (Case 1 in the proof of Lemma 1).

$$\begin{aligned}
\text{ii and iii} &: r_1(Q_D) - \rho\sigma^2(-Q_S) \leq a_D < r_2(Q_D) \\
\text{ii' and iii'} &: r_2(Q_S) < b_S \leq r_1(Q_S) + \rho\sigma^2 Q_D \\
\text{i and i'} &: r_1(Q_D + Q_S)(Q_D + Q_S) \leq a_D Q_D + b_S Q_S < r_2(Q_D + Q_S)(Q_D + Q_S)
\end{aligned}$$

The natural candidates for the equilibrium $a_D = r_2(Q_D) - \varepsilon$ and $b_S = r_2(Q_S) + \varepsilon$ do not satisfy condition i'. Consequently, the constraint i' is binding at equilibrium, and equilibrium prices must be such that:

$$(a_D)^* = r_2(Q_D + Q_S) + (b_S^* - r_2(Q_D + Q_S))\frac{(-Q_S)}{Q_D} - \varepsilon$$

First, we use the expression of $(a_D)^*$ in dealer 1's trading profit (conditional on the fact that she executes Q_D and Q_S): $v_1^{-L}(Q_D + Q_S) = \rho\sigma^2(I_1 - I_2)(Q_D + Q_S)$. This trading profit does not depend on equilibrium prices. Consequently, there may exist a continuum of prices that may sustain the equilibrium. Second, inputing $(a_D)^*$ into conditions ii to iii', the equilibrium price in market S must satisfy:

$$\text{ii and iii} : (r_1 - r_2)\frac{Q_D}{-Q_S} + r_2(Q_S) \leq (b_S)^* < (r_2 - r_1)\frac{Q_D}{-Q_S} + r_2(Q_S) - \rho\sigma^2 Q_D$$

$$\text{ii' and iii'} : r_2(Q_S) < (b_S)^* \leq r_1(Q_S) + \rho\sigma^2 Q_D$$

Obviously, since $I_1 > I_2$, we have $(r_1 - r_2)\frac{Q_D}{-Q_S} < r_2(Q_S)$ and $r_1(Q_S) + \rho\sigma^2 Q_D < (r_2 - r_1)\frac{Q_D}{-Q_S} + r_2(Q_S) - \rho\sigma^2 Q_D$. Thus the second inequality is constraining both the minimum and the maximum possible price in market S. Within all equilibria defined by:

$$(a_D)^* = r_2(Q_D + Q_S)\frac{Q_D + Q_S}{Q_D} + (b_S)^*\frac{(-Q_S)}{Q_D} - \varepsilon,$$

$$(b_S)^* \in (r_2(Q_S) + \epsilon, r_1(Q_S) + \rho\sigma^2 Q_D + \epsilon],$$

we select the only equilibrium that is continuous at $I_1 - I_2 = Q_D$, that is, $(a_D)^* = r_2(Q_D) - \rho\sigma^2(-Q_S) - \epsilon$, from which we deduce that $(b_s)^* = r_2(Q_S) + \epsilon$.

Case 2.1.: $Q_S > 0$ and $I_1 - I_2 < Q_D$, i.e., $(I_1 - I_2 - Q_D)Q_S < 0$.
From Lemma 1, we know that dealer 1 executes Q_D while dealer 2 executes Q_S. The ask prices a_D and a_S are the maximum prices that satisfy the set of conditions I to III' (Case

2 in the proof of Lemma 1).

$$\begin{aligned}
\text{II and III} &: r_1(Q_D) \leq a_D < r_2(Q_D) + \rho\sigma^2 Q_S \\
\text{II' and III'} &: r_2(Q_S) \leq a_S < r_1(Q_S) + \rho\sigma^2 Q_D \\
\text{I} &: a_D > r_1(Q_D) + (a_S - r_1(Q_S))\,\frac{Q_S}{Q_D} \\
\text{I'} &: a_S > r_2(Q_S) + (a_D - r_2(Q_D))\,\frac{Q_D}{Q_S}
\end{aligned}$$

The candidates for the equilibrium $a_D = r_2(Q_D) + \rho\sigma^2 Q_S - \varepsilon$ and $a_S = r_1(Q_S) + \rho\sigma^2 Q_D - \varepsilon$ from the two first inequalities do not satisfy condition I'. Consequently, the constraint I' is binding at equilibrium, and equilibrium prices must be such that:

$$(a_D)^* = r_2(Q_D) + ((a_S)^* - r_2(Q_S))\frac{Q_S}{Q_D} - \varepsilon \tag{8}$$

First, notice that under the latter condition, condition I always holds (given that $(I_1 - I_2)(Q_D - Q_S) > 0$). Second, inputting $(a_D)^*$ defined in Eq.8 into conditions (II and III) and (II' and III') yields the following restrictions on $(a_S)^*$:

$$\begin{aligned}
\text{II and III:}\,& r_2(Q_S) + (r_1(Q_D) - r_2(Q_D))\,\frac{Q_D}{Q_S} \leq (a_S)^* < r_2(Q_S) + \rho\sigma^2 Q_D \\
\text{II' and III':}\,& r_2(Q_S) \leq (a_S)^* < r_1(Q_S) + \rho\sigma^2 Q_D
\end{aligned}$$

Third, we input $(a_D)^*$ (defined in Eq.(8)) in both the trading profit of dealer 1, conditional on the fact that she executes Q_D, and the trading profit of dealer 2, conditional on the fact that he executes Q_S:

$$\begin{aligned}
v_1(Q_D) &= \left(r_2(Q_D) + ((a_S)^* - r_2(Q_S))\frac{Q_S}{Q_D} - r_1(Q_D)\right) Q_D, \\
v_2(Q_S) &= ((a_S)^* - r_2(Q_S))Q_S.
\end{aligned}$$

We observe that dealers' profits are strictly increasing in a_S. Consequently, dealers' reaction functions are such that the best ask price in market S is the highest possible one. From conditions (II and III) and (II' and III'), and under the hypothesis that $I_1 - I_2 < Q_D$, we deduce that condition (II' and III') is binding and that $(a_S)^*$ is such that:

$$(a_S)^* = r_1(Q_S) + \rho\sigma^2 Q_D - \varepsilon,$$

from which we deduce that

$$(a_D)^* = r_2(Q_D) + \rho\sigma^2(I_2 - I_1 + Q_D)\frac{Q_S}{Q_D} - \varepsilon$$

Consequently, there exists a unique equilibrium such that dealer 1 post $(a_D)^*$ and trades Q_D while dealer 2 posts the best ask price equal to $(a_S)^*$ and trades Q_S.

Case 2.2.: $Q_S < 0$ and $I_1 - I_2 > Q_D$, i.e., $(I_1 - I_2 - Q_D)Q_S < 0$.
From Lemma 1, we know that dealer 1 executes Q_D while dealer 2 executes Q_S. The ask price a_D in market D and the bid price b_S in market S are respectively the maximum and the minimum prices that satisfy the set of conditions I to III' (Case 2 in the proof of Lemma 1).

$$\begin{aligned}
\text{II and III} &: r_1(Q_D) \leq a_D < r_2(Q_D) + \rho\sigma^2 Q_S \\
\text{II' and III'} &: r_1(Q_S) + \rho\sigma^2 Q_D < b_S \leq r_2(Q_S) \\
\text{I } &: a_D > r_1(Q_D) + (b_S - r_1(Q_S))\,\frac{Q_S}{Q_D} \\
\text{I '} &: b_S < r_2(Q_S) + (r_2(Q_D) - a_D)\,\frac{Q_D}{-Q_S}
\end{aligned}$$

From the two first inequalities, $a_D = r_2(Q_D) - \rho\sigma^2(-Q_S) - \varepsilon$ and $b_S = r_1(Q_S) + \rho\sigma^2 Q_D + \varepsilon$ are natural candidates for the equilibrium. It is easily shown that they also satisfy conditions I and I'. Therefore, there exists a unique equilibrium such that dealer 1 posts the best ask price in market D, equal to $(a_D)^* = r_2(Q_D) - \rho\sigma^2(-Q_S) - \varepsilon$, while dealer 2 posts the best ask in market S equal to $(b_S)^* = r_1(Q_S) + \rho\sigma^2 Q_D + \varepsilon$.

Case 3: $I_1 - I_2 = Q_D$, i.e., $(I_1 - I_2 - Q_D)Q_S > 0$.
Notice that if $I_1 - I_2 = Q_D$, then the equilibrium described in 1.1. cannot be sustained because conditions ii' and iii' (i.e. $r_1(Q_S) + \rho\sigma^2 Q_D \leq a_S = r_2(Q_S)$) cannot hold simultane-

ously due to the strict inequality, which contradicts the equality $r_1(Q_S)+\rho\sigma^2 Q_D = r_2(Q_S)$. If $(a_D)^* = r_2(Q_D) - \varepsilon$ and $(a_S)^* = r_2(Q_S) = r_1(Q_S) + \rho\sigma^2 Q_D$ however, conditions i, i', ii, ii' and iii hold. Thus at these prices, dealer 2 becomes indifferent between trading Q_S or not, and so is dealer 1.

■

Proof of Proposition 2

We decompose the proof into two results, depending on the sign of Q_S.

Result 1 *Suppose that order flows have the same sign (with probability γ). Then, the expected ask prices in the dominant (D) and the satellite (S) markets are equal to:*

$$E\left(\underline{a}^{m,+}\right) = \frac{2r_d\left(q_m\right) + r_u\left(q_m\right)}{3} + \rho\sigma^2 q_{-m}\left(\frac{q_D}{I_u - I_d} - \frac{1}{3}\left(\frac{q_D}{I_u - I_d}\right)^2\right), m = S, D. \quad (9)$$

Proof. We first compute the expected ask that prevails in market D. For sake of brevity, let us define $r_d\left(q_D\right) = r_d$, $r_u\left(q_D\right) = r_u$, $r_1\left(q_D\right) = x$ and $r_2\left(q_D\right) = y$. The support of the uniform distribution function of x and y simplifies to $[r_u, r_d]$.

By definition,

$$E\left(\underline{a}^{D,+}\right) = E\left(\min\left(a_1^{D,+}, a_2^{D,+}\right)\right).$$

Given Proposition **??** and the symmetry of our hypotheses, the latter equation writes:

$$E\left(\underline{a}^{D,+}\right) = 2 \times E\left(y\mathbb{1}_{x+\rho\sigma^2 Q_D<y} + \left(y + \rho\sigma^2 Q_S \frac{\rho\sigma^2 Q_D - (y-x)}{\rho\sigma^2 Q_D}\right)\mathbb{1}_{x+\rho\sigma^2 Q_D>y}\right)$$

The latter expression rewrites

$$\begin{aligned} E\left(\underline{a}^{D,+}\right) &= \frac{2}{\left(r_d - r_u\right)^2}\left[\int_{r_u}^{r_d-\rho\sigma^2 Q_D}\int_{x+\rho\sigma^2 Q_D}^{r_d} y dy dx + \int_{r_u}^{r_d}\int_{x}^{r_d}\left(y + \rho\sigma^2 Q_S\left(\frac{\rho\sigma^2 Q_D - (y-x)}{\rho\sigma^2 Q_D}\right)\right) dy\right. \\ &- \left.\int_{r_u}^{r_d-\rho\sigma^2 Q_D}\int_{x+\rho\sigma^2 Q_D}^{r_d}\left(y + \rho\sigma^2 Q_S\left(\frac{\rho\sigma^2 Q_D - (y-x)}{\rho\sigma^2 Q_D}\right)\right) dy dx\right]. \end{aligned}$$

After straightforward calculations, we get:

$$E\left(\underline{a}^{D,+}\right) = \frac{2r_d\left(q_D\right) + r_u\left(q_D\right)}{3} + \rho\sigma^2 q_S\left(\frac{q_D}{\left(I_u - I_d\right)} - \frac{\left(q_D\right)^2}{3\left(I_u - I_d\right)^2}\right).$$

We now turn to the expected ask prevailing in market S using a similar reasoning. The expression writes:

$$\begin{aligned} E\left(\underline{a}^{S,+}\right) &= E\left(\min\left(a_1^{S,+}, a_2^{S,+}\right)\right) \\ &= 2 \times E\left(y \mathbb{1}_{x+\rho\sigma^2 q_D < y} + \left(x + \rho\sigma^2 q_D\right) \mathbb{1}_{x+\rho\sigma^2 q_D > y}\right) \end{aligned}$$

Using similar computations to those used in the previous case, the latter expression rewrites:

$$E\left(\underline{a}^{S,+}\right) = \frac{2r_d(q_S) + r_u(q_S)}{3} + \rho\sigma^2 q_D \left(\frac{q_D}{I_u - I_d} - \frac{(q_D)^2}{3\left(I_u - I_d\right)^2}\right).$$

Q.E.D.

Result 2 *(Opposite signs) Suppose that order flows have opposite signs (with probability* $1-\gamma$*). The expected asks in markets* D *and* S *respectively write:*

$$\begin{aligned} E\left(\underline{a}^{D,-}\right) &= \frac{2r_d\left(q_D\right) + r_u\left(q_D\right)}{3} - \rho\sigma^2 q_S, \\ E\left(\underline{a}^{S,-}\right) &= \frac{2r_d\left(q_S\right) + r_u\left(q_S\right)}{3} - \rho\sigma^2 q_D + \frac{\left(\rho\sigma^2 q_D\right)^2}{\left(r_d - r_u\right)} - \frac{\left(\rho\sigma^2 q_D\right)^3}{3\left(r_d - r_u\right)^2}. \end{aligned} \tag{10}$$

Proof. We first compute the expected ask prevailing in market D (for $Q_D > 0$ and $Q_S < 0$).

$$E\left(\underline{a}^{D,-}\right) = E\left(\min\left(a_1^{D,-}, a_2^{D,-}\right)\right)$$

Straightforward computations yield:

$$E\left(\underline{a}^{D,-}\right) = \frac{2r_d\left(q_D\right) + r_u\left(q_D\right)}{3} - \rho\sigma^2\left(q_S\right).$$

Symmetrically, the expected ask prevailing in market S (considering that $Q_D < 0$ and $Q_S > 0$) writes:

$$\begin{aligned} E\left(\underline{a}^{S,-}\right) &= E\left(\min\left(a_1^{S,-}, a_2^{S,-}\right)\right) \\ &= E\left(\begin{array}{c} r_2\left(Q_S\right) \mathbb{1}_{r_2(Q_S) > r_1(Q_S) - \rho\sigma^2(-Q_D)} + \left(r_1\left(Q_S\right) - \rho\sigma^2\left(-Q_D\right)\right) \mathbb{1}_{r_2(Q_S) < r_1(Q_S) - \rho\sigma^2(-Q_D)} \\ + r_1\left(Q_S\right) \mathbb{1}_{r_1(Q_S) > r_2(Q_S) - \rho\sigma^2(-Q_D)} + \left(r_2\left(Q_S\right) - \rho\sigma^2\left(-Q_D\right)\right) \mathbb{1}_{r_1(Q_S) < r_2(Q_S) - \rho\sigma^2(-Q_D)} \end{array}\right) \end{aligned}$$

This can be developed as follows:

$$E\left(\underline{a}^{S,-}\right) = \frac{1}{(r_d - r_u)^2}\Bigg[\int_{r_u+\rho\sigma^2(-Q_D)}^{r_d}\int_{r_u}^{x-\rho\sigma^2(-Q_D)}\left(x-\rho\sigma^2\left(-Q_D\right)\right)dydx$$
$$+\int_{r_u}^{r_d-\rho\sigma^2(-Q_D)}\int_{x+\rho\sigma^2(-Q_D)}^{r_d}\left(y-\rho\sigma^2\left(-Q_D\right)\right)dydx$$
$$+\left(\int_{r_u}^{r_d}\int_{x}^{r_d}(x)\,dydx-\int_{r_u}^{r_d-\rho\sigma^2 Q_D}\int_{x+\rho\sigma^2 Q_D}^{r_d} ydydx\right)$$
$$+\left(\int_{r_u}^{r_d}\int_{r_u}^{x}\left(y+\rho\sigma^2 Q_D\right)dydx-\int_{r_u+\rho\sigma^2 Q_D}^{r_d}\int_{r_u}^{x-\rho\sigma^2 Q_D}(y)\,dydx\right)\Bigg].$$

This finally yields:

$$E\left(\underline{a}^{S,-}\right) = \frac{2r_d\left(Q_S\right)+r_u\left(Q_S\right)}{3}-\rho\sigma^2\left(-Q_D\right)+\frac{\left(\rho\sigma^2\left(-Q_D\right)\right)^2}{(r_d-r_u)}-\frac{\left(\rho\sigma^2\left(-Q_D\right)\right)^3}{3\left(r_d-r_u\right)^2}.$$

Q.E.D.

Proposition **??** is obtained from Results 1 and 2 considering the extensive form of the game represented in Figure 1. Note that we change slightly notations $q_m = Q_m$ for a net-buying order flow and $q_m = -Q_m$ for a net-selling order flow ($m = S, D$) in order to ease computations of the Corollaries. ■

Proof of Proposition 3

By definition:

$$covar(s^D, s^S) = 2covar(\underline{a}^D, \underline{a}^S) = 2\left(\gamma covar(\underline{a}^{D,+}, \underline{a}^{S,+}) + (1-\gamma)covar(\underline{a}^{D,-}, \underline{a}^{S,-})\right)$$

We decompose the proof into two results, depending on the sign of Q_S, that is, on whether $\gamma = 1$ or $\gamma = 0$.

Result 3 *(Same signs) Suppose that $\lambda_S = \lambda_D = 1$ and that order flows have same signs ($\gamma = 1$). The covariance between the ask prices in the dominant (D) and the satellite (S) markets is equal to:*

$$covar\left(\underline{a}^{D,+}, \underline{a}^{S,+}\right) = \frac{\rho\sigma^4}{18}(I_u - I_d)^2$$
$$-\ \rho\sigma^4 q_D \times \left(\ -\tfrac{q_D-q_S}{6}+\tfrac{2}{9}(q_S-q_D)\tfrac{q_D}{(I_u-I_d)}+\tfrac{15q_S-q_D}{12}\tfrac{q_D^2}{(I_u-I_d)^2}+\tfrac{2}{3}q_S\tfrac{q_D^3}{(I_u-I_d)^3}+\tfrac{q_S}{9}\tfrac{q_D^4}{(I_u-I_d)^4}\ \right)$$

Proof.
By definition,

$$E\left(\underline{a}^{D,+}\underline{a}^{S,+}\right)=E\left(\min\left(a_1^{D,+},a_2^{D,+}\right)\times\min\left(a_1^{S,+},a_2^{S,+}\right)\right)$$

We develop the expectation below, considering equilibrium prices for each couple of realizations of (I_1,I_2) as a function of the reservation prices of the dealers, as given in Proposition **??**.

$$\begin{aligned}
&E\left(\underline{a}^{D,+}\underline{a}^{S,+}\right)\\
=&\int_{I_d+q_D}^{I_u-q_D}\begin{pmatrix}\int_{I_d}^{I_1-q_D}\left(r_2(q_D)\right)\left(r_2(q_S)\right)f(I_1)f(I_2)dI_2\\ +\int_{I_1-q_D}^{I_1}\left(r_2(q_D)+\rho\sigma^2 q_S\frac{q_D-(I_1-I_2)}{q_D}\right)\left(r_1(q_S)+\rho\sigma^2 q_D\right)f(I_1)f(I_2)dI_2\\ +\int_{I_1}^{I_1+q_D}\left(r_1(q_D)+\rho\sigma^2 q_S\frac{q_D-(I_2-I_1)}{q_D}\right)\left(r_2(q_S)+\rho\sigma^2 q_D\right)f(I_1)f(I_2)dI_2\\ +\int_{I_1+q_D}^{I_u}\left(r_1(q_D)\right)\left(r_1(q_S)\right)f(I_1)f(I_2)dI_2\end{pmatrix}dI_1\\
+&\int_{I_d}^{I_d+q_D}\begin{pmatrix}\int_{I_d}^{I_1}\left(r_2(q_D)+\rho\sigma^2 q_S\frac{q_D-(I_1-I_2)}{q_D}\right)\left(r_1(q_S)+\rho\sigma^2 q_D\right)f(I_1)f(I_2)dI_2\\ +\int_{I_1}^{I_1+q_D}\left(r_1(q_D)+\rho\sigma^2 q_S\frac{q_D-(I_2-I_1)}{q_D}\right)\left(r_2(q_S)+\rho\sigma^2 q_D\right)f(I_1)f(I_2)dI_2\\ +\int_{I_1+q_D}^{I_u}\left(r_1(q_D)\right)\left(r_1(q_S)\right)f(I_1)f(I_2)dI_2\end{pmatrix}dI_1\\
+&\int_{I_u-q_D}^{I_u}\begin{pmatrix}\int_{I_d}^{I_1-q_D}\left(r_2(q_D)\right)\left(r_2(q_S)\right)f(I_1)f(I_2)dI_2\\ +\int_{I_1-q_D}^{I_1}\left(r_2(q_D)+\rho\sigma^2 q_S\frac{q_D-(I_1-I_2)}{q_D}\right)\left(r_1(q_S)+\rho\sigma^2 q_D\right)f(I_1)f(I_2)dI_2\\ +\int_{I_1}^{I_u}\left(r_1(q_D)+\rho\sigma^2 q_S\frac{q_D-(I_2-I_1)}{q_D}\right)\left(r_2(q_S)+\rho\sigma^2 q_D\right)f(I_1)f(I_2)dI_2\end{pmatrix}dI_1.
\end{aligned}$$

Computation yields:

$$\begin{aligned}
E\left(\underline{a}^{D,+}\underline{a}^{S,+}\right)\quad&=\frac{\rho\sigma^4}{12(I_u-I_d)^2}\\
&\times\begin{pmatrix}-3q_D^4+\\ -q_D^3(6I_d+3q_S-10I_u)\\ -2q_D^2(-3I_d^2+4I_dq_S+q_S^2-6q_SI_u+3I_u^2)\\ -q_D(I_d-I_u)(4I_d^2-9I_dq_S+6q_S^2-2I_dI_u-3q_SI_u-2I_u^2)\\ +2(I_d-I_u)^2(3I_d^2-2I_dq_S+2I_dI_u-q_SI_u+I_u^2)\end{pmatrix}
\end{aligned}$$

Besides, $E\left(\underline{a}^{D,+}\right)$ and $E\left(\underline{a}^{S,+}\right)$ are given in Proposition **??**, which yields:

$$\begin{aligned}
E\left(\underline{a}^{D,+}\right)E(\underline{a}^{S,+}) &= \frac{\rho\sigma^4}{36(I_u-I_d)^4} \\
&\times \left(2q_D^2 q_S - 3q_D(I_d-I_u)(I_d-2q_S-I_u) + 2(I_d-I_u)^2(2I_d+I_u)\right) \\
&\times \left(2q_D^3 + 6q_D^2(I_d-I_u) + (I_d-I_u)^2(4I_d+2I_u-3q_S)\right)
\end{aligned}$$

Result 3 is finally obtained from the definition of the covariance:

$$covar\left(\underline{a}^{D,+},\underline{a}^{S,+}\right) = E\left(\underline{a}^{D,+}\underline{a}^{S,+}\right) - E\left(\underline{a}^{D,+}\right)E\left(\underline{a}^{S,+}\right)$$

Q.E.D.

Result 4 *(Opposite signs) Suppose that $\lambda_S=\lambda_D=1$ and that order flows have opposite signs ($\gamma=0$). The covariance between ask prices in markets D and S writes:*

$$\begin{aligned}
covar\left(\underline{a}^{D,-},\underline{a}^{S,-}\right) &= \frac{\rho\sigma^4}{18}(I_u-I_d)^2 \\
&- \frac{\rho\sigma^4}{36}q_D^2 \times \left(3\frac{q_D^2}{(I_d-I_u)^2} + 8\frac{q_D}{(I_d-I_u)} + 6\right)
\end{aligned}$$

Proof. By definition,

$$E\left(\underline{a}^{D,-}\underline{a}^{S,-}\right) = E\left(\min\left(a_1^{D,-},a_2^{D,-}\right)\times\min\left(a_1^{S,-},a_2^{S,-}\right)\right).$$

We consider equilibrium prices for each couple of realizations of (I_1,I_2) as a function of the reservation prices of the dealers. According to Proposition **??**:

$$\begin{aligned}
(a_{-H}^D)^* &= r_2(q_D) - \rho\sigma^2 q_S \\
(a_{-L}^D)^* &= r_2(q_D) - \rho\sigma^2 q_S
\end{aligned}$$

and by symmetry:

$$\begin{aligned}
(a_{-H}^S)^* &= r_2(q_S) - \rho\sigma^2 q_D \\
(a_{-L}^S)^* &= r_1(q_S)
\end{aligned}$$

We develop the expectation below:

$$
\begin{aligned}
& E\left(\underline{a}^{D,-}\underline{a}^{S,-}\right) \\
= & \int_{I_d+q_D}^{I_u-q_D}\left(\begin{array}{c}
\int_{I_d}^{I_1-q_D}\left(r_2(q_D)-\rho\sigma^2 q_S\right)\left(r_2(q_S)-\rho\sigma^2 q_D\right)f(I_1)f(I_2)dI_2 \\
+\int_{I_1-q_D}^{I_1}\left(r_2(q_D)-\rho\sigma^2 q_S\right)\left(r_1(q_S)\right)f(I_1)f(I_2)dI_2 \\
+\int_{I_1}^{I_1+q_D}\left(r_1(q_D)-\rho\sigma^2 q_S\right)\left(r_2(q_S)\right)f(I_1)f(I_2)dI_2 \\
+\int_{I_1+q_D}^{I_u}\left(r_1(q_D)-\rho\sigma^2 q_S\right)\left(r_1(q_S)-\rho\sigma^2 q_D\right)f(I_1)f(I_2)dI_2
\end{array}\right)dI_1 \\
+ & \int_{I_d}^{I_d+q_D}\left(\begin{array}{c}
\int_{I_d}^{I_1}\left(r_2(q_D)-\rho\sigma^2 q_S\right)\left(r_1(q_S)\right)f(I_1)f(I_2)dI_2 \\
+\int_{I_1}^{I_1+q_D}\left(r_1(q_D)-\rho\sigma^2 q_S\right)\left(r_2(q_S)\right)f(I_1)f(I_2)dI_2 \\
+\int_{I_1+q_D}^{I_u}\left(r_1(q_D)-\rho\sigma^2 q_S\right)\left(r_1(q_S)-\rho\sigma^2 q_D\right)f(I_1)f(I_2)dI_2
\end{array}\right)dI_1 \\
+ & \int_{I_u-q_D}^{I_u}\left(\begin{array}{c}
\int_{I_d}^{I_1-q_D}\left(r_2(q_D)-\rho\sigma^2 q_S\right)\left(r_2(q_S)-\rho\sigma^2 q_D\right)f(I_1)f(I_2)dI_2 \\
+\int_{I_1-q_D}^{I_1}\left(r_2(q_D)-\rho\sigma^2 q_S\right)\left(r_1(q_S)\right)f(I_1)f(I_2)dI_2 \\
+\int_{I_1}^{I_u}\left(r_1(q_D)-\rho\sigma^2 q_S\right)\left(r_2(q_S)\right)f(I_1)f(I_2)dI_2
\end{array}\right)dI_1.
\end{aligned}
$$

Computation yields:

$$
\begin{aligned}
E\left(\underline{a}^{D,-}\underline{a}^{S,-}\right) & = \frac{\rho\sigma^4}{12(I_u-I_d)^2} \\
& \times\left(\begin{array}{l}
-3q_D^4+2q_D^3(-3I_d+2q_S+5I_u) \\
+q_D(I_d-I_u)^2(4I_d+15q_S+2I_u) \\
+12q_D^2(I_d-I_u)(q_S+I_u) \\
+2(I_d-I_u)^2(3I_d^2-3q_S^2+q_S I_u+I_u^2+2I_d q_S+2I_d I_u)
\end{array}\right)
\end{aligned}
$$

Besides, $E\left(\underline{a}^{D,-}\right)$ and $E\left(\underline{a}^{S,-}\right)$ are given in Proposition **??**, which yields:

$$
\begin{aligned}
E\left(\underline{a}^{D,-}\right)E(\underline{a}^{S,-}) & = \frac{\rho\sigma^4}{36(I_u-I_d)^2} \\
& \times\ (-3q_D+4I_d+6q_S+2I_u) \\
& \times\ \left(2q_D^3-6q_D^2(I_u-I_d)+(I_u-I_d)^2(6q_D+4I_d-3q_S+2I_u)\right)
\end{aligned}
$$

Result 4 is finally obtained from the definition of the covariance:

$$
covar\left(\underline{a}^{D,-},\underline{a}^{S,-}\right)=E\left(\underline{a}^{D,-}\underline{a}^{S,-}\right)-E\left(\underline{a}^{D,-}\right)E\left(\underline{a}^{S,-}\right)
$$

Q.E.D.

Corollary **??** is obtained from Results 3 and 4 considering the extensive form of the game represented in Figure 1. ■

Proof of Corollary 1

Remind that $\underline{a}^{batch}$ denotes the minimum ask price in the benchmark model in which the total order flow is consolidated. From Ho and Stoll (1983), we know that:

$$E\left(\ \underline{a}^{batch}\right) = \frac{2r_d\left(q_m + q_{-m}\right) + r_u\left(q_m + q_{-m}\right)}{3}.$$

Using Eq. (9) and (10) and the symmetry of the game, we deduce that the difference in transactions costs between a fragmented or a consolidated order flow is:

$$\begin{aligned} \Delta TC &= \gamma\left(E\left(\underline{a}^{D,+}\right)q_D + E\left(\underline{a}^{S,+}\right)q_S - E\left(\underline{a}^{batch}\right)\left(q_D + q_S\right)\right) \\ &+ (1-\gamma)\left(E\left(\underline{a}^{D,-}\right)q_D - E\left(\overline{b}^{S,-}\right)q_S - E\left(\underline{a}^{batch}\right)\left(q_D - q_S\right)\right). \end{aligned}$$

After straightforward computations the latter expression is equal to:

$$\Delta TC = \rho\sigma^2 q_S\left(I_u - I_d\right)\left(-\frac{(\gamma+1)}{3}\right)P_\gamma\left(\frac{q_D}{I_u - I_d}\right)$$

where

$$P_\gamma(x) = x^3 - 3x^2 + \frac{3}{(\gamma+1)}x + \frac{(\gamma-1)}{(\gamma+1)}$$

for $x \in [0,1]$.

To investigate whether transaction costs are larger or smaller in the batch auction, let us analyze the sign of this cubic polynomial. First, note that:

$$P'_\gamma(x) = 3x^2 - 6x + \frac{3}{(1+\gamma)} = 3\left(x - \left(1 - \sqrt{\frac{\gamma}{1+\gamma}}\right)\right)\left(x - \left(1 + \sqrt{\frac{\gamma}{1+\gamma}}\right)\right).$$

Given that $x \in [0,1]$, then $x - \left(1 + \sqrt{\frac{\gamma}{1+\gamma}}\right) < 0$, and the sign of $P'_\gamma(x)$ only depends on the sign of $\left(x - \left(1 - \sqrt{\frac{\gamma}{1+\gamma}}\right)\right)$. P_γ is increasing if $x < \left(1 - \sqrt{\frac{\gamma}{1+\gamma}}\right)$ and is decreasing if $x > \left(1 - \sqrt{\frac{\gamma}{1+\gamma}}\right)$. Thus, the local maximum is $P_\gamma\left(1 - \sqrt{\frac{\gamma}{1+\gamma}}\right) = \frac{\gamma\left(-1+2\sqrt{\frac{\gamma}{1+\gamma}}\right)}{1+\gamma}$.

Consider the case where $\gamma \leq \frac{1}{3}$. Straightforward computations show that $P_\gamma\left(1 - \sqrt{\frac{\gamma}{1+\gamma}}\right) \leq$

0 (with $P_\gamma(1-\sqrt{\frac{\gamma}{1+\gamma}})=0$ if $\gamma=\frac{1}{3}$). We therefore deduce that $P_\gamma \leq 0$, i.e., $\Delta TC \geqslant 0$ if $\gamma \leq \frac{1}{3}$.

Consider now the case where $\gamma > \frac{1}{3}$. Then $P_\gamma > 0$, or, equivalently, $\Delta TC < 0$ if $x \in [r_\gamma^1, r_\gamma^2]$ where $P_\gamma(r_\gamma^1) = 0 = P_\gamma(r_\gamma^2)$. Note that if $\gamma = 1$, then it is direct to show that $P_1 > 0$ if $x \in [0, \frac{(3-\sqrt{3})}{2}]$, or equivalently, $\Delta TC < 0$ if $q_D < \frac{(3-\sqrt{3})}{2}(I_u - I_d)$. ■

Proof of Lemma 2

In our set up (identical risk aversion and identical pre-trade inventory distribution), we can measure the dealers' aggregate posttrade risk by the sum of the variance of their posttrade assets (Yin, 2005). In the batch auction, the longer dealer executes the total order flow, thus the aggregate posttrade risk is equal to:

$$(\sigma^2_{agg})^{batch} = Var((I_1 - Q_D - Q_S)\tilde{v}) + Var((I_2)\tilde{v})$$

In the fragmented market, the posttrade allocations depend on the sign of $(I_1 - I_2 - Q_D)\,Q_S$.

- If $(I_1 - I_2 - Q_D) > 0$ and $Q_S > 0$, or if $(I_1 - I_2 - Q_D) < 0$ and $Q_S < 0$, the aggregate posttrade risk is similar to that in the batch auction, because the longer dealer consolidates the total order flow:

$$(\sigma^2_{agg})^{cons} = Var((I_1 - Q_D - Q_S)\tilde{v}) + Var((I_2)\tilde{v}) = (\sigma^2_{agg})^{batch}$$

- If $(I_1 - I_2 - Q_D) < 0$ and $Q_S > 0$, or if $(I_1 - I_2 - Q_D) > 0$ and $Q_S < 0$, dealers share the order flow and the aggregate posttrade risk is equal to:

$$(\sigma^2_{agg})^{share} = (\sigma^2_{agg})^{H-} = Var((I_1 - Q_D)\tilde{v}) + Var((I_2 - Q_S)\tilde{v})$$

In the latter situation, the difference of posttrade risk between the fragmented market and the batch auction is equal to:

$$\begin{aligned}(\sigma^2_{agg})^{share} - (\sigma^2_{agg})^{batch} &= \begin{pmatrix} (Var((I_1 - Q_D)\tilde{v}) + Var((I_2 - Q_S)\tilde{v})) \\ -(Var((I_1 - Q_D - Q_S)\tilde{v}) + Var((I_2)\tilde{v})) \end{pmatrix} \\ &= 2Q_S(I_1 - I_2 - Q_D),\end{aligned}$$

which is strictly negative in the case we considered.

■

Proof of Proposition 4

Given the equilibrium prices $\left((a^D)^*, (a^S)^*\right)$ derived in Proposition 1, transaction costs write:

$$TC(\alpha) = [(\underline{a}^D(\alpha Q) - \delta_D - \mu)\alpha + (\underline{a}^S((1-\alpha)Q) - \delta_S - \mu)(1-\alpha)] \times Q \tag{11}$$

We want to show that there exists an interior equilibrium, that is, an $\alpha^* \in [\frac{1}{2}, 1)$ that minimizes transaction costs $TC(.)$. We first conjecture that there exists an equilibrium characterized by a high divergence in dealers position, i.e., $\frac{1}{2} \leq \alpha < \frac{I_1 - I_2}{Q}$. The first order condition yields:

$$\alpha^H = \frac{1}{2} + \frac{\delta_D - \delta_S}{2\rho\sigma^2 Q}$$

The two conditions for an interior equilibrium $\alpha \in [\frac{1}{2}, 1)$ to exist are thus: i. a condition ensuring that our conjecture holds, i.e., $\alpha^H < \frac{I_1 - I_2}{Q}$, and ii. a condition ensuring that the equilibrium is interior, i.e., $\alpha^H < 1$. The latter always holds under our assumption $\delta_D - \delta_S < \rho\sigma^2 Q$. Condition i. thus translates into a condition on the divergence of inventory positions, i.e.

$$I_1 - I_2 > \frac{1}{2}\left(Q + \frac{\delta_D - \delta_S}{\rho\sigma^2}\right). \tag{12}$$

We now conjecture that there exists an equilibrium characterized by a low divergence in dealers position, i.e., $\alpha \geq \frac{I_1 - I_2}{Q}$. The first order condition yields:

$$\alpha^L = \frac{1}{2} - \frac{\delta_D - \delta_S}{2\rho\sigma^2 Q} + \frac{(I_1 - I_2)}{Q}$$

The three conditions for an interior equilibrium to exist are thus: i. a condition ensuring that our conjecture holds, i.e., $\alpha^L \geq \frac{I_1 - I_2}{Q}$, ii. a condition ensuring that the equilibrium is interior, i.e., $\alpha^L < 1$, and iii. a condition ensuring that $\alpha^L \geq \frac{1}{2}$. The first condition always holds under our assumption $\delta_D - \delta_S < \rho\sigma^2 Q$. The second condition translates into $I_1 - I_2 < \frac{Q}{2} + \frac{\delta_D - \delta_S}{2\rho\sigma^2}$, which is the complement of the condition (12) above. Notice that if $I_1 - I_2 = \frac{Q}{2} + \frac{\delta_D - \delta_S}{2\rho\sigma^2}$, then there exists an equilibrium such that $\alpha^* = 1$. The third condition imposes $I_1 - I_2 \geq \frac{\delta_D - \delta_S}{2\rho\sigma^2}$.

If $I_1 - I_2 < \frac{\delta_D - \delta_S}{2\rho\sigma^2}$, then $\alpha^L < \frac{1}{2}$, which contradicts our first result that $\alpha \geq \frac{1}{2}$. The investor would like to send a larger fraction to market S even though $\delta_D > \delta_S$, in order to benefit from a lower price in S. However, if he did, this would switch the dominant and satellite markets, thus prices. There is no solution to the FOC in$[\frac{1}{2}, 1)$. There is a corner equilibrium: $\alpha^* = 1$. ■

Proof of Corollary 2

First stage: the inter-dealer market. If dealer 1 sells a quantity x at price p to dealer 2 in the inter-dealer market, dealers' profits write:

$$\left(v_1^{ID} = \left[p - \mu - \frac{\rho\sigma^2}{2}(x - 2I_1)\right] x; v_2^{ID} = \left[\mu - \frac{\rho\sigma^2}{2}(x + 2I_2) - p\right]\right).$$

We maximize dealers' profit with respect to x to find their supply and demand functions respectively. The crossing of the supply and demand curves yields the following symmetric equilibrium in the inter-dealer market:

$$\left(x_{ID}^* = \frac{I_1 - I_2}{2}; p_{ID}^* = \mu - \rho\sigma^2 \frac{I_1 + I_2}{2}\right).$$

Dealers' equilibrium profits in the inter-dealer market are thus $v_1^{*ID} = v_2^{*ID} = \frac{\rho\sigma^2}{8}(I_1 - I_2)^2$.

Second stage: the customer-dealer market. Notice that dealers find it optimally to perfectly share risk: after trading in the inter-dealer market, dealers 1 and 2 end up with the same inventory position, $I_1' = I_2'$. From the equilibrium in the customer-dealer market derived in section 2.2 for $I_1' = I_2'$, we get the following profits: $v_1^{*CD|ID} = v_2^{*CD|ID} = \rho\sigma^2 Q_D Q_S \mathbb{1}_{Q_S > 0}$.

Comparison. We now compute dealers' expected profits in the presence of an inter-dealer market, namely $V^{*CD+ID} = E(v_i^{*CD|ID} + v_i^{*ID})$, and compare them with the expected profits they obtain in the absence of an inter-dealer market, namely $V^{*CD} = E(v_i^{*CD})$. Computations yield:

$$V^{*CD+ID} = \frac{\rho\sigma^2}{48}(I_u - I_d)^2 + \gamma\rho\sigma^2 q_D q_S,$$

and

$$
\begin{aligned}
V^{*CD} &= \frac{\rho\sigma^2}{6}(I_u - I_d)\left(q_D + (2\gamma - 1)q_S\right) \\
&+ \frac{\rho\sigma^2 q_S}{(I_u - I_d)^2} \times \left[\begin{array}{l} (1-\gamma)(I_u - I_d)^3 - \left(3(1-\gamma)q_D + \frac{1}{2}\gamma q_S\right)(I_u - I_d)^2 \\ + \left\{(1-\gamma)q_D + \frac{1}{2}\gamma q_S\right\} q_D \left(3(I_u - I_d) - q_D\right) \end{array} \right].
\end{aligned}
$$

The inequality $V^{*BD} - V^{BD+ID} > 0$ could be solved in closed-form but the complexity of the solutions makes it difficult to interpret. We therefore compare expected profits numerically for various sets of parameters' values. ■