

# Modifying Real Options' Probability of Exercise

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## Abstract

A critical difference between real options and financial options lies in that real option holders, writers, and even external stakeholders can modify the characteristics of real options. Furthermore, exercising a real option can have repercussions onto multiple stakeholders. To modify the exercise probability of real options, one may adjust the strike price, or the level and uncertainty of the underlying process. For a typical project modeled by a real option, the strike can be modified by offering a land lot, and the level of the underlying asset can be modified by offering future tax credits. A portfolio manager with a performance based contract can change the variance of the underlying process by adjusting the risk of the portfolio. This paper proposes general model-free conditions under which a modification of real options qualitatively increases or decreases the physical exercise probability of real options. Our results are useful to determine an optimally parsimonious modification of a real option to increase or decrease its exercise probability. We show our results apply in real situations and explain previously obtained empirical results. In the second part of the paper, to quantify the exercise probabilities, we propose a general computational framework using the Least-Squares Monte Carlo method for American-style real options. Findings show that an incentive required to promote the exercise of an American-style option before or at maturity is less than that for a European.

**JEL classification:** G12, G31, G32

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# 1 Introduction

Real options can model various real-life decisions. In a classical project valuation activity, one compares the present value of future cash flows (underlying process) to the capital investment (strike) and models the value of waiting for a decision by a call option (see, e.g., McDonald and Siegel 1985, Brennan and Schwartz 1985, and Dixit and Pindyck 1994.) Capital structure and debt valuation can be studied under a real options approach, as originally proposed by Merton (1974). Under Merton's (1974) model, equity is a call on the firm's assets, and creditors are short a put on the assets. The field of merger and acquisitions takes advantage of the real options approach (see, e.g., Lambrecht (2004), and Bhagwat, Dam, and Harford (2016)). The real options approach is also applied in studies of management compensation when managers have performance-based incentives (see, e.g., Grinblatt and Titman (1989)). It is also useful to study government policies and the effects on businesses (see, e.g., Blyth, Bradley, Bunn, Clarke, Wilson, and Yang (2007)). In fact, "... *real options research has [...] branched out into areas as diverse as management science, strategy, insurance, taxation, environmental economics and engineering*" (Lambrecht (2017)).

Real options share similarities with financial options but differ on some critical aspects. For a financial option, writers and buyers are mostly concerned with its value and sensitivities (also known as the Greeks). External stakeholders might use information implied from financial options but will have little interest in the execution of financial options which are zero-sum games. Real options are different because they can lead to the creation of value or can have externalities. Exercising a real option can have repercussions onto multiple stakeholders. In many situations, these stakeholders are more concerned with the exercise of the option than its value. More importantly, real options are different from financial options because their characteristics can be modified. Both the strike and the dynamics of the underlying process can be modified. Owners, writers, or external stakeholders such as the government can modify an option specification. If a stakeholder has some control over an option specification, she will be interested in better understanding the relationship between a specification and the exercise probability of the real option. Though there are some case-specific papers on real option modifications, we feel academic research has not paid enough attention to

this significant topic.

The objective of this paper is to characterize real options' modifications and the effects on exercise probability. We aim at answering important questions that remain in the literature. For example, in a classical project valuation activity, conventional wisdom says that more risk leads to higher holding value of a real option, which leads to delays in investments. However, Sarkar (2000) finds a non-monotone relation between risk and real option exercise, where more risk may lead to faster or slower real option exercise in some circumstances, though the results are found under a Black-Scholes' like model. Meanwhile, Cunningham (2006) tests and finds empirical support for delays in investments when there is more risk. The relation between risk and investment behavior is still unclear, as highlighted in Byun and Jo (2018), which finds that uncertainty would reduce investments.

In the field of real estate and project valuation in general, several authors study the involvement of the government in promoting or impeding projects. Among many others, Capozza and Sick (1994) study the effect of urban planning on vacant-lot value and the decision to build. Under a CAPM and real options approach, they find a non-monotone relation between risk and real option exercise. An important question arises: to promote projects, should a government change a real option's strike, e.g., by giving free land, or should they modify a real option's underlying process, e.g., by offering tax credits? An equivalent question can be asked on how to impede projects with externalities.

The literature on performance-based incentives raised several important questions related to the real option exercise. Grinblatt and Titman (1989) highlights the potential agency problem related to performance-based contracts for portfolio managers, where the later can modify its real option's underlying process. Carpenter (2000) solves for a manager's optimal policy under a HARA utility function and finds that the manager may increase or decrease the risk of the assets under management to increase the exercise probability of its incentive. Brown, Goetzmann, and Park (2001) finds empirical support for this manager's behavior. Though there is a large body of literature on management compensation, there is still research being conducted on the optimal

compensation contract (see, e.g., Ju, Leland, and Senbet (2014).)

We also identify important questions in the realm of capital structure, debt valuation, and default probability. Leland (1994), Leland (1998), and DeMarzo and Fishman (2007) all acknowledge the potential agency problem where a firm can alter the capital structure and affect the default probability. Broadie, Chernov, and Sundaresan (2007) study the complex relation between capital structure, debt valuation, bankruptcy, and the negotiation between shareholders and creditors, both of which have means to modify the real options modeling the firm. The last financial crisis led to stricter regulations concerning contingent capital to reduce the probability of default of US banks. However, in 2018, there is increasing pressure to relax the regulation. Does one solution fit all in the case of capital structure and default risk? What is the optimal regulation?

Our paper contributes to the literature by providing the first comprehensive study characterizing the exercise probability of real options. Our comprehensive results include both model-free propositions and general numerical methods. First, we study model-free cases for simple options. The general results we obtain, apply to a wide range of different real option payoffs. We generalize the non-monotone relation between risk and the exercise probability of real options. Our results show that more risk may lead to faster or slower investments, in general. We answer the question whether a government should promote investments by offering grants to reduce capital costs, or by increasing the present value of future cash flows by, e.g., offering tax credits. Our paper has policy implications. We provide an extension to the least-squares Monte Carlo method to estimate the exercise probability. We do not answer questions about the optimal capital structure, but we do provide the general theory upon which future research can build.

The rest of the paper is as follows. Section 2 reviews the literature on real options and positions our paper. Section 3 studies the effect of modifying the strike and the dynamics of the underlying in a distribution-free general case. Section 4 proposes a general simulation-based computational framework to study the exercise probability of options. Section 5 provides numerical examples with simple European and American options. Section 6 concludes.

## 2 Literature Review

One of the first finance/economics papers taking the real options approach would be Merton (1974) who uses option pricing theory to value corporate debt. Later, some authors studied the valuation of future cash flows under uncertainty, trying to improve the Net Present Value (NPV) method. Myers and Turnbull (1977) take a market equilibrium approach using the Capital Asset Pricing Model to determine the proper discount factor. Ross (1978) uses arbitrage theory to replicate future payoffs to value risky future cash flows. While Constantinides (1978) uses the Capital Asset Pricing Model to risk-adjust future cash flows and discount them using the risk-free rate.

Cash flow uncertainty is one aspect of real options, but management flexibility is where projects transform into real options. Multiple forms of managerial flexibility lead to different types of options and Trigeorgis (1993) categorizes them into five types: defer, abandon, contract, expand, and switch. Defer typically leads to call options; abandon to put options; contract, expand, and switch to more complex options. McDonald and Siegel (1985) recognize that a Net Present Value approach is insufficient to model management flexibility, so they model a project as an option. Brennan and Schwartz (1985) also specifically use an option-like model to value a project. Several others have used a real options approach in their research, among others: Bernanke (1983) uses a real options approach to explain cyclical fluctuations in investments; Titman (1985) proposes a valuation equation for vacant lots and provides an intuition to why it might be optimal to delay investment; and Pindyck (1988) studies the optimal capacity decision in a real options framework. For a general reference on the real options approach for capital budgeting, see Dixit and Pindyck (1994).

A field which benefited from the real options methodology is real estate in general and land valuation in particular. Titman (1985) uses a real options approach to analyze vacant lot development, and explain why some lots are left vacant in a densely populated area by the option value of the land. Several papers study land and real estate development. A good review of the literature can be found in Womack (2015). One of the many aspects of land development is gentrification. Cities, developers, current and prospective resident have different objectives concerning urban de-

velopment and gentrification. One may be interested in finding what would be a cost-efficient policy for a city to incite developers to renovate or tear-down and rebuild to facilitate gentrification.

Capital structure and debt valuation can be studied under a real options approach, as originally proposed by Merton (1974). Under Merton's (1974) model, equity is a call on the firm's assets, and creditors are short a put on the assets. The probability of default is an important aspect of this literature thoroughly reviewed by Sundaresan (2013). Black and Cox (1976) extend Merton's model to include safety covenants and explicitly expose the endogenous decision of capital structure and optimal default boundary. Leland (1998) finds that borrowing firms may increase the risk of assets after increasing debt to transfer wealth from creditors to borrowers, and increase the probability of default at the same time. The recent financial crisis of 2007 renewed this strand of literature which looks at contingent capital as a mean to prevent bankruptcy (exercising the put) or bank recapitalization at the expense of taxpayers (see, e.g., Glasserman and Nouri (2012)).

A strand of research already acknowledges that external stakeholders might influence investments in projects modeled by real options. For example, Blyth, Bradley, Bunn, Clarke, Wilson, and Yang (2007) investigate the effect of policy uncertainty on real option valuation. They find that in general, an increase in uncertainty leads to delays in project adoption. Zhu, Zhang, and Fan (2015) find that different tax systems affect real option values. Hud and Hussinger (2015) study government R&D subsidies and find that the efficiency of the program depends on the economic cycle. More recently, Kang and Létourneau (2016) consider the effect of policy uncertainty on the adoption of green energy. They find that the relation between uncertainty and exercise timing is not linear. In some cases, more uncertainty hastens investment.

The real options approach is being used in a multitude of fields now. Gordon, Loeb, Lucyshyn, and Zhou (2015) study the probability of investing in cybersecurity. Cyber threats can jeopardize the future of any firms. Take as an example the recent "wanna cry ransomware." It is important for a government to better understand what will promote the investment in cybersecurity to ensure the financial stability of its economy.

To the best of our knowledge, very few articles look at the deliberate actions of stakeholders to

modify a real option's exercise probability in a theoretical perspective. Sarkar (2000) is the closest to our paper. He studies the relationship between uncertainty and the probability of exercising a real option. However, he presents model specific analytic results. In fact, the results in Sarkar (2000) are a special case of our results.

### 3 Model Free Conditions

In this section, we derive the effects of general modifications of the properties of an option on its exercise probability. We start by obtaining analytical results for European style options. Next, we extend the results to options with early exercise possibilities. Finally, we discuss empirical applications of our results.

#### 3.1 Call and Put Options on Random Variables with General Distributions

We focus on the physical distribution of the underlying variable at the maturity of the European option because we study its exercise probability. The dynamics leading to the distribution at maturity is pointless. So we do not have to make assumptions about the discount rate or the dynamics of the underlying variable. We need, however, to assume the option payoff is not path-dependent. We provide results for calls and puts, but would like to note that calls and puts can model more general option-like payoffs. For example, an exchange option where the owner of the option can exchange asset A to obtain asset B can be modeled with a call option where the underlying asset is the spread process of A-B, and the strike of the option is zero.

To fix ideas, let us first establish a trivial result for a basic modification.

**Proposition 0** *Consider a call option where the exercise decision must be made one time in the future  $T$ . Let a random variable  $X(T)$  denote the value of the underlying process at time  $T$  and a real number  $K$  the strike of the option, and  $f_{X(T)}$  denote the distribution of  $X(T)$ . (When possible, we drop the time index for a lighter notation.) Define a modification of the real option as:*

$$g(X; \alpha) \equiv X + \alpha$$



where  $\alpha$  is a constant. We claim that:

1.  $\Pr [g(X; \alpha) > K] > \Pr [X > K] \iff \alpha > 0$ ,
2.  $\Pr [X + \alpha > K] = \Pr [X > K - \alpha]$ .

**Proof.** The proof of both claims is trivial. ■

Proposition 0 is about a simple additive transformation of the distribution of the process underlying a call option. Note from claim 2 that a pure translation of the distribution is equivalent to a translation in the strike. The effect on the exercise probability of the call is trivial. However, we find this intuitive result to be helpful in comprehending the next propositions. In the next propositions, we study the effect of a change in variance (along with a change in location) on the exercise probability of calls and puts.

**Proposition 1** *Consider a call option where the exercise decision must be made one time in the future  $T$ . Let a random variable  $X(T)$  denote the value of the underlying process at time  $T$  and a real number  $K$  the strike of the option, and  $f_{X(T)}$  denote the distribution of  $X(T)$ . (When possible, we drop the time index for a lighter notation.) Define a modification of the real option as:*

$$g(X; H(\cdot), e) \equiv (1 + e)(X - H(X)) + H(X)$$

where  $H(\cdot)$  is a real-valued function of a random variable satisfying  $H(aX + b) = aH(X) + b$  for any real numbers  $a$  and  $b$ , and  $e > -1$  is a real number. Examples of  $H(\cdot)$  include  $H(X) = \text{Median}[X]$ ,  $H(X) = E[X]$ ,  $H(X) = 5^{\text{th}} \text{percentile}[X]$  and  $H(X) = 0$ . We claim that:

1.  $H(g(X; H(\cdot), e)) = H(X)$ ;
2.  $\text{var}[g(X; H(\cdot), e)] = (1 + e)^2 \text{var}[X]$ ; and
3. (a) when  $H(X) > K$ ,  $\Pr [g(X; H(\cdot), e) > K] > \Pr [X > K] \iff e < 0$ ;  
 (b) when  $H(X) < K$ ,  $\Pr [g(X; H(\cdot), e) > K] > \Pr [X > K] \iff e > 0$ .

**Proof.** The first claim holds because of the property of  $H(aX + b) = aH(X) + b$ . The second claim trivially holds. To prove the third claim, first consider the case of  $H(X) > K$ :  $\Pr[g(X; H(\cdot), e) > K] = \Pr[(1 + e)(X - H(X)) + H(X) > K] = \Pr\left[X > \frac{K - H(X)}{1 + e} + H(X)\right] = \Pr\left[X > \frac{K + eH(X)}{1 + e}\right] > \Pr[X > K]$ , which is equivalent to  $\frac{K + eH(X)}{1 + e} < K$  and  $e(H(X) - K) < 0$ . The later is true only if  $e < 0$  because  $H(X) > K$ . Similar logic can be used to prove the case of  $H(X) < K$ . ■

Proposition 1 is all about a transformation which changes the variance of the underlying asset of a real option. A visual representation is proposed in Appendix B. The main take-away from Proposition 1 is that when the dynamics of the call's underlying asset is changed such that the variance is changed the probability of exercising the call might increase or decrease depending on the location measure,  $H(\cdot)$ , relative to the strike. Take for example  $H(X) = \text{Median}[X]$ . The transformation  $g(X; H(\cdot), e) = (1 + e)(X - H(X)) + H(X)$  where  $e > 0$  increases the variance of  $X$  without changing the median and increases the probability of exercising the call if  $K > \text{Median}[X]$ . In other words, if  $\Pr[X > K] < 50\%$  an increase in variance would increase the exercise probability of the call option. Proposition 1 tells us that a "pure" increase in variance does not always result in a decrease in exercise probability at maturity of a real call option. The result depends on how variance is changed and the current exercise probability. It is a powerful result that holds for any distribution of the underlying asset. Later, we discuss empirical evidence of our Proposition 1.

A modification of variance as in Proposition 1 applies well to a case where an increase in variance will have a small (read negligible) impact on the median of the distribution. However, in other cases, a modification of the variance will also modify important reference location measures such as the median. As such, we consider a transformation which changes both a location measure and variance. Furthermore, we decompose the effect on exercise probability into the effect of changing the location and that of changing the variance.

**Proposition 2** Consider a European call with maturity  $T$ . Define  $X$ ,  $H(\cdot)$ ,  $e$ , and  $K$  as defined

in Proposition 1. Furthermore, define

$$g(X; \varepsilon, H(\cdot), e) \equiv \varepsilon [(1 + e)(X - H(X)) + H(X)]$$

where a real number  $\varepsilon > 0$  represents a proportional increase ( $\varepsilon > 1$ ) or decrease ( $\varepsilon < 1$ ) given to  $X$ . Also, consider a real number  $p > 0$  denoting a proportional decrease ( $p < 1$ ) or increase ( $p > 1$ ) given to  $K$ . We claim that:

1.  $H(g(X; \varepsilon, H(\cdot), e)) = \varepsilon H(X)$ ;
2.  $\text{var}[g(X; \varepsilon, H(\cdot), e)] = \varepsilon^2 (1 + e)^2 \text{var}[X]$ ; and
3.  $\Pr[g(X; \varepsilon, H(\cdot), e) > pK] > \Pr[X > K]$  iff  $K(1 - \frac{p}{\varepsilon}) + e(K - H(X)) > 0$ .

**Proof.** The proofs of the first and second claims are similar to those in Proposition 1. The third claim is obtained from

$$\begin{aligned} \Pr[g(X; \varepsilon, H(\cdot), e) > K \cdot p] &= \Pr[\varepsilon [(1 + e)(X - H(X)) + H(X)] > K \cdot p] \\ &= \Pr\left[X > \frac{K \cdot \frac{p}{\varepsilon} - H(X)}{(1 + e)} + H(X)\right] > \Pr[X > K] \end{aligned}$$

iff  $\frac{K \cdot \frac{p}{\varepsilon} - H(X)}{(1 + e)} + H(X) < K$ . ■

In Proposition 2,  $g(X; \varepsilon, H(\cdot), e)$  is a modification changing a location measure  $H(X)$  to  $\varepsilon H(X)$  and  $\text{var}[X]$  to  $\varepsilon^2 (1 + e)^2 \text{var}[X]$ .  $e$  represents a variance modification separable to a location measure modification  $\varepsilon$ . The third claim tells us that increasing (decreasing) the location parameter by a factor  $\varepsilon$  has the same effect on the exercise probability as decreasing (increasing) the strike by a factor  $p = 1/\varepsilon$ . However, they are not economically equivalent. To illustrate this claim, consider three cases represented by at-the-money, in-the-money, and out-the-money call options. Let the location measure of reference be the median. Consider a first case where  $\text{Median}[X] = K = 100$ . In such a case,  $\Pr(X > K) = 50\%$ . Now, consider an incentive  $\varepsilon = 1.1$ . Proposition 2 tells us that increasing  $X$  by  $\varepsilon = 1.1$ , would be equivalent to decreasing the

strike by  $p = 1/1.1$ . However, increasing  $X$  would cost \$10, while decreasing the strike would cost \$9.09. In the case of a real call option where a stakeholder, perhaps the government, willing to increase the exercise probability, reducing the capital investment by \$9.09 would be less expensive than offering future tax cuts with a present value of \$10, but would have the same effect on the exercise probability. Second, consider a case where  $Median[X] = 120$ , and  $K = 100$ , such that the exercise probability  $\Pr(X > K) > 50\%$ . Again, reducing the capital investment by \$9.09 would have the same effect on the probability as increasing  $X$  by \$12. Third, consider a case where  $Median[X] = 80$ , and  $K = 100$ , such that the exercise probability  $\Pr(X > K) < 50\%$ . In this case, Proposition 2 tells us that increasing the present value of future cash flows by \$8 would be as efficient in increasing the exercise probability as reducing the capital investment by \$9.09. An optimal modification to the option to increase (or decrease) the exercise probability actually depends on the current exercise probability. This has important implications for policy design.

If  $p = \varepsilon$ , the third claim of Proposition 2 reduces to the third claim of Proposition 1. In addition, the effect of modifying the location parameter/strike ( $K(1 - \frac{p}{\varepsilon})$ ) is additively separate from that of the "pure" modification of variance (e.g.,  $e(K - H(X))$ ). Finally if the investment opportunity is out of the money, inflating  $\varepsilon$  (or deflating  $p$ ) and inflating "pure" variance modification  $e$  are complimentary; if the investment opportunity is in the money, inflating  $\varepsilon$  (or deflating  $p$ ) and deflating "pure" variance modification  $e$  are complimentary.

Policy implications of Propositions 2 are as follows. At least within a European call real option setting, to increase the exercise probability of an out-of-the-money real option, stakeholders should modify the underlying process to a more profitable one and increase the volatility of the present value of the project. The relationship between giving incentive and raising volatility is not a substitute but a compliment. This is in a sharp contrast with the conventional wisdom that a government should decrease volatility to stimulate investment.

Why is that important? Propositions 1 and 2 are very powerful because our results are distribution-free in the sense that they hold for any distribution of  $X$  at time  $T$ . Since our results are distribution free, they apply to any distribution of  $X$ . This means  $X$  could represent the

present value of a portfolio of options which would model a portfolio of projects and management flexibility. Our results would still hold. For instance, our results could be applied to an Exchange option where  $X$  would model the difference between the present value of the two assets being exchanged.

We complement our Propositions for call options with the put equivalent.

**Corollary 3** *Consider a put option with maturity  $T$  where  $X$  the underlying process, and  $K$  the strike. Define  $g(\cdot)$ ,  $H(\cdot)$ , and  $e$  as in Proposition 1. Then*

1. (a) *when  $H(X) > K$ ,  $\Pr[g(X; H(\cdot), e) < K] > \Pr[X < K] \iff e < 0$ ;*
- (b) *when  $H(X) < K$ ,  $\Pr[g(X; H(\cdot), e) < K] > \Pr[X < K] \iff e > 0$ .*

**Proof.** Proof of Corollary 3 is similar to proof of Proposition 1. ■

Corollary 3 is equivalent to Proposition 1, but for puts. What Corollary 3 tell us is that a "pure" increase in variance does not always result in a decrease in exercise probability at maturity of a real put option. The result depends on how the variance is changed and the current exercise probability. An example of where Corollary 3 can be applied is for an electrical power plant where the underlying process is the different between the stochastic revenues from producing electricity and the stochastic cost of producing it (the spark spread.) With an old production technology like Coal, a power plant cannot be shut down and fired up rapidly when the payoff becomes negative. However, management can predict the future spark spread to decide in advance to shutdown before the payoff becomes negative. Intuitively, when the spread is well favorable, the probability of executing the put option us low, and a reduction in variance is desired. However, when the exercise probability becomes high, a reduction in variance would only confirm the potential shutdown of the power plant. What Corollary 3 is saying is that in that case, an increase in variance would decrease the probability of shutting down the power plant.

**Corollary 4** *Consider a European put with maturity time  $T$ . Define  $X$ ,  $H(\cdot)$ ,  $e$ , and  $K$  as defined*

in Corollary 3. Furthermore, define

$$g(X; \varepsilon, H(\cdot), e) \equiv \varepsilon [(1 + e)(X - H(X)) + H(X)]$$

where a real number  $\varepsilon > 0$  represents a proportional increase ( $\varepsilon > 1$ ) or decrease ( $\varepsilon < 1$ ) given to  $X$ . Also, consider a real number  $p > 0$  denoting a proportional decrease ( $p < 1$ ) or increase ( $p > 1$ ) given to  $K$ . We claim that:

$$1. \Pr[g(X; \varepsilon, H(\cdot), e) < pK] > \Pr[X < K] \text{ iff } K\left(1 - \frac{p}{\varepsilon}\right) + e(K - H(X)) < 0.$$

**Proof.** Proof of Corollary 4 is similar to proof of Proposition 2.

$$\begin{aligned} \Pr[g(X; \varepsilon, H(\cdot), e) < K \cdot p] &= \Pr[\varepsilon [(1 + e)(X - H(X)) + H(X)] < K \cdot p] \\ &= \Pr\left[X < \frac{K \cdot \frac{p}{\varepsilon} - H(X)}{(1 + e)} + H(X)\right] > \Pr[X < K] \end{aligned}$$

$$\text{iff } \frac{K \cdot \frac{p}{\varepsilon} - H(X)}{(1 + e)} + H(X) > K. \blacksquare$$

### 3.2 Effect of Early Exercise Possibility

The results above easily apply to European options where one is concerned about the probability of exercising a real call or put option at maturity  $T$ . These results can be extended to options with multiple exercise possibilities such as American-style options.<sup>1</sup> For an option with early exercise possibilities, stakeholders will be interested in the cumulative probability, i.e., the probability that the option is exercised at or before the maturity date.

Let  $c(t)$  be the value at time  $t$  of a European call option with maturity  $T$  and strike  $K$  and let  $C(t)$  be the value of an American call option with otherwise the same characteristics. Because the American option offers more than the European option,  $C(t) \geq c(t)$ . Let  $X(t)$  be the value of the asset underlying the option at time  $t$ . The probability of  $X(T) \geq K$  is independent of the number of possible early exercise and whether the option is exercised early. Before maturity,

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<sup>1</sup>An option with multiple exercise possibilities is a Bermudan style option, while an option with infinite number of exercise possibilities is an American style option. We refer only to American style option for simplicity.

the probability of exercising an American option is greater or equal zero. Thus, the cumulative probability of exercising the option at or before the maturity of an American option must be greater or equal than that of a European option.

Prior to maturity, an American call option will be exercised if  $X(t) > \kappa(t)$ , where  $\kappa(t)$  is the exercise boundary at time  $t$ . The exercise boundary is a function of the distribution of  $X(t), X(t+1), etc$ , the strike  $K$ , but not the current value of  $X(t)$ . The exercise boundary is generally not known analytically. In numerical methods, such as a binomial tree or Longstaff and Schwartz's (2001) Least Squares Monte Carlo method, it is determined by backward induction. The difficulty in finding general results for options with early exercise is that the exercise boundary will change when the option is modified.

**Proposition 5** *Consider an American call option. Define  $X, H(\cdot), e$ , and  $p$  as defined in Proposition 2. Define  $\kappa(X, K)$  as the exercise boundary, which is a function of the distribution of  $X$  and  $K$ . All variables and constants are at a time  $t < T$  where an early exercise is possible. Furthermore, define*

$$g(X; \varepsilon, H(\cdot), e) \equiv \varepsilon [(1 + e)(X - H(X)) + H(X)]$$

where a real number  $\varepsilon > 0$  represents a proportional incentive ( $\varepsilon > 1$ ) or penalty ( $\varepsilon < 1$ ) given to  $X$ . Now, define  $\omega$  as the effect on the exercise boundary caused by  $g(X; \varepsilon, H(\cdot), e)$ . A change in distribution of  $X$  leads to a change in the exercise boundary  $\kappa(X)$ , such that  $\kappa(g(X)) = \omega\kappa(X)$ . Then:

1.  $H(g(X; \varepsilon, H(\cdot), e)) = \varepsilon H(X)$ ;
2.  $var[g(X; \varepsilon, H(\cdot), e)] = \varepsilon^2 (1 + e)^2 var[X]$ ; and
3.  $\Pr[g(X; \varepsilon, H(\cdot), e) > p\omega\kappa(X)] > \Pr[X > \kappa(X)]$  iff  $\kappa(X) \left(1 - \frac{p\omega}{\varepsilon}\right) + e(K - H(X)) > 0$ .

**Proof.** The proof is similar to the proof of Proposition 2. ■

Essentially, Proposition 5 is equivalent to Proposition 2, except that  $\omega$  is determined by a modification on  $X$ . In general,  $\omega$  is not known analytically. However, Proposition 5 is still providing

some guidance for policy design. Proposition 5 is telling us that what applies for the European options in terms of increasing or decreasing the exercise probability also applies to American options, though, the effect might be mitigated or amplified by changes in the early exercise boundary. In a later section, we confirm this claim using a simple numerical application of Proposition 5.

### **3.3 Discussion of proposition and empirical evidences**

Our propositions and corollaries in this section have implications for owners, writers, and stakeholders of real options. In Merton's (1974) structural model framework, a shareholder of a public firm is the owner of a real call option on firms income, and a creditor is the real put option writer (Sundaresan, 2013). Both the shareholders and the creditors can modify the underlying process (operating/non-operating income) of the real option by influencing a firms strategic, production, and marketing decisions. Let us shift our attention to the exercise probability. Because of the costly nature of financial distresses, finance literature has studied the probability of default (PD). See, e.g., Elkamhi, Jacobs, and Pan (2014), among many others. Our propositions and corollaries 1 to 5 are relevant in this topic by providing model free conditions for increasing and decreasing physical measure exercise probabilities of European call and put-type real options. To reduce the PD times loss given default (LGD), both shareholders and creditors are enticed to modify the underlying process in the direction of increasing (decreasing) the physical probability of exercising a call (put) option. So is a government who is responsible for financial stability of national economy. So, owners, writers, and stakeholders want to decrease (increase) the variance if the real call option is in the money (out of the money).

The propositions and corollaries 1 to 5 also have implications to real option stakeholders such as a government considering incentive/penalty programs for private sectors investment in certain technologies. If the incentive/penalty program leads to a positive net present value (NPV), the government should reduce policy uncertainty to further encourage investment in the technology. Let us take an example of electric utilities investment in solar electricity generation facility. With production tax credit, the solar projects net present value (NPV) should be positive. However,



production tax credit is subject to renewal every year. Our model-free results propose that in order to increase the investment in green solar energy, the government should reduce policy uncertainty by making the production tax credit as a permanent policy. However, if the NPV is negative, our results tell us the government should increase uncertainty by sending a signal that the future may be different from the present; for example, there could be more tax cuts in the future. At this time, let us take an example of an early-stage information technology which gives a negative NPV to an investor. In order to give start-up entrepreneurs incentives to further invest in such an early-stage technology, government or venture capitalists should increase both the expectation and the variance by putting resources to research and development. In the context of capital budgeting, Proposition 5 is particularly important because most investment decisions are American-type. While propositions and corollaries in this section give various economic intuitions to many areas in finance and economics, these provide only qualitative and directional recommendations. To calculate a physical exercise probability, the next section relies on a numerical procedure called the Least Squares Monte Carlo method.

A realistic example of Proposition 1 is a portfolio manager reallocating his portfolio to change the level of risk. The real option of the portfolio manager can be modeled by a call option where the underlying process is the difference between the portfolio's and benchmark's performances, and the strike is set to zero.<sup>2</sup> Grinblatt and Titman (1989) highlights the potential agency problem related to performance-based contracts for portfolio managers. Carpenter (2000) solves the optimal dynamic investment policy for a risk-averse manager paid with a call option on the assets he controls. She finds that the manager's optimal decision might be to increase or decrease the risk of the assets he controls, depending on the current probability of exercising the option (receiving the incentive). Brown, Goetzmann, and Park (2001) find that portfolio managers behave similarly. Good performers in the first half of the year (who have a high probability of exercising their real option linked to their performance based contract) reduce the volatility of the portfolio under management, while bad performers increase the volatility. More recently, Huang, Sialm, and Zhang

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<sup>2</sup>The performance could be a simple return, a Sharpe ratio, or an asset pricing model's alpha.

(2011) study the impact of risk-shifting on portfolio performance. They find that funds increasing the volatility perform worse, suggesting that the risk-shifting is caused by ill-motivated trades. Whereas funds keeping their risk level stable perform better. In particular, for a portfolio manager, we can easily assume a change in variance of the portfolio has minimal impact on the median of the distribution. In such a case, Proposition 1 suggest that when the exercise probability is greater (lower) than 50%, the manager should decrease (increase) the variance to increase the exercise probability. Proposition 1 is confirmed in empirical studies such as Brown, Goetzmann, and Park (2001). Portfolio managers for which their real options embedded in their performance-based contract has a high (low) exercise probability tend to lower (increase) the risk of the portfolio to increase the exercise probability.

To the best of our knowledge, there are no empirical studies that fully test our Proposition 2. Current empirical researches only show partial applications of the proposition. For example, most papers studying project valuation using real options show an increase in variance causing a decrease in investments, and some papers show the inverse. This is manifestation of Proposition 2, claim 3. However, a complete test of the Proposition requires the observation of a real option before and after a modification is made. For example, Proposition 2 can be used to determine whether a grant or a tax credit is better to promote investment in a specific sector. An empirical test requires the estimation of the exercise probability prior to a policy change, observation of a policy change, and estimation of the exercise probability after the policy change. Such an empirical testing of our Proposition 2 deserve a research project on it's own. In this paper, we present numerical evidences for both European and American style real options.

## 4 Simulation Based Approach

The analytic solutions from the previous section provide guidelines, but in practice, specific results might be desired. However, oftentimes analytic solutions are not available. In this section, we provide a framework to study the exercise probability of more complex options using simulations. We study the optimal exercise of options for basic and more complex cases. First, we define the

option valuation framework. Next, we explain the methodology to evaluate option prices and evaluate the exercise probability.

#### 4.1 A Project as a Real Option

Here, we describe the general framework to value options and study the exercise probability. We first assume that time can be discretized. Thus, we assume that the option considered may be exercised at  $J$  early exercise points. We specify the potential exercise points as  $t_0 = 0 < t_1 \leq t_2 \leq \dots \leq t_J = T$ , with  $t_0$  and  $T$  corresponding to the current time and maturity of the option, respectively, where it is implicitly assumed that the option cannot be exercised at time  $t_0$ . Note that exercising at  $t_0$  makes the option irrelevant. An American-style real option can be approximated by increasing the number of early exercise points  $J$  and a European-style real option can be valued by setting  $J = 1$ . For special cases, we also allow  $T \rightarrow \infty$ . We assume a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  equipped with a discrete filtration  $(\mathcal{F}(t_j))_{j=0}^J$ . For simplicity, we assume a complete and arbitrage free market such that a risk-neutral measure  $\mathbb{Q}$  exists and is unique. See, e.g., Harrison and Pliska (1981) for an analysis of complete markets, martingales measures, and risk-neutral valuation. The later assumption is not required, but makes the option valuation and notation simpler. In an incomplete market, assumptions have to be made on investors preferences and can be generalized. The method proposed here does not critically depend on the completeness of the market.

The option's value depends on one or more underlying assets which are modeled by a Markovian process with state variables  $(X(t_j))_{j=0}^J$  adapted to the filtration. We denote by  $X(t_j)$  the state variable process under the risk-neutral measure  $\mathbb{Q}$  and by  $X'(t_j)$  the state variable process under the physical measure  $\mathbb{P}$ . We denote by  $(Z(t_j))_{j=0}^J$  an adapted payoff process satisfying  $Z(t_j) = \pi(X(t_j), t_j)$  for a suitable function  $\pi(\cdot, \cdot)$ , which is assumed to be square integrable.

The payoff function can accommodate all major real option payoffs and more. Consider as a first example a capital investment  $K$  that gives the right to a series of future cash flows with a present value of  $S(t_j)$ . This is similar to a vanilla call option and the payoff is defined as  $\pi^{call}(S, K, t_j) = \max(0, S(t_j) - K)$ . As a second example, consider a firm that owns a branch office and its building.

The firm is considering closing the branch office which currently has a present value of  $S(t_j)$  and selling the building for a value  $K$ . This is similar to a put option and the payoff is defined as  $\pi^{put}(S, K, t_j) = \max(0, K - S(t_j))$ . As a third example, consider a firm that has the option to invest in either of two technologies: A and B. Each have their own capital investment  $K^A(t_j)$  and  $K^B(t_j)$ , which can be functions of time and other state variables. Each technology will bring a series of futures cash flows which have a present value of  $S^A(t_j)$  and  $S^B(t_j)$ . The projects are mutually exclusives. The payoff is defined as  $\pi^{alt}(S, K, t_j) = \max(0, S^A(t_j) - K^A(t_j), S^B(t_j) - K^B(t_j))$ .

This notation is also sufficiently general to study capital structure problems through Merton's model and allow for non-constant interest rates (see, e.g., Glasserman (2004)). Furthermore, the present value of a project can include further optionality, such as the option to pause the project, or the option to suspend. Thus,  $S$  can be the value of an option that might need to be evaluated using this same methodology presented here.

Following, e.g., Karatzas (1988) and Duffie (1996), in the absence of arbitrage we can specify the American option price as

$$P(X(0)) = \max_{\tau(t_1) \in \mathcal{T}(t_1)} F(X(t_j)), \quad (1)$$

where  $\mathcal{T}(t_j)$  denotes the set of all stopping times with values in  $\{t_j, \dots, t_J\}$  and  $F$  is the expectation of the payoff  $Z$  conditional on the state variables  $X$  defined as

$$F(X(t_j)) \equiv E[Z(\tau(t_{j+1})) | X(t_j)]. \quad (2)$$

For an American-style option, at each potential exercise point, the holder of the option faces the decision to exercise its option, or hold until the next potential exercise point. The total value of the option at time  $t$  is thus

$$V(t_j) = \max(Z(t_j), F(t_j)). \quad (3)$$

The option will be held at time  $t_j$  if  $F(t_j) \geq Z(t_j)$  and will be exercised otherwise. It is possible

to derive the optimal stopping time iteratively using the following algorithm:

$$\begin{cases} \tau(t_J) = T \\ \tau(t_j) = t_j \mathbf{1}_{\{Z(t_j) \geq F(\tau(t_{j+1}))\}} + \tau(t_{j+1}) \mathbf{1}_{\{Z(t_j) < F(\tau(t_{j+1}))\}}, \quad 1 < j \leq J-1 \end{cases} \quad (4)$$

Based on (4), the value of the option in (1) can be calculated as

$$P(X(0)) = F(\tau(t_0)). \quad (5)$$

Chow, Robbins, and Siegmund (1971) (Theorem 3.2) provides the theoretical foundation for the algorithm in (4) and establishes the optimality of the derived stopping time and the resulting price estimate in (5).

## 4.2 Valuation of the Real Option Using Simulation

The theoretical solution of (4) is generally impossible to implement because the conditional expectations are unknown. This problem has been solved long ago using simulations. Because conditional expectations in (2) can be represented as a countable linear combination of basis functions we write  $F(X(t_j)) = \sum_{m=0}^{\infty} \phi_m(X(t_j)) c_m(t_j)$ , where  $\{\phi_m(\cdot)\}_{m=0}^{\infty}$  form a basis.<sup>3</sup> In order to make this operational we assume that it is possible to approximate well the conditional expectation function by using the first  $M+1$  terms such that  $F(X(t_j)) \approx \hat{F}_M(X(t_j)) = \sum_{m=0}^M \phi_m(X(t_j)) c_m(t_j)$  and that we can obtain an estimate of this function by

$$\hat{F}_M^N(X(t_j)) = \sum_{m=0}^M \phi_m(X(t_j)) \hat{c}_m^N(t_j), \quad (6)$$

where  $\hat{c}_m^N(t_j)$  are approximated or estimated using  $N \geq M$  independent simulated paths. If one was to use regular monomials for the basis, (6) would simply be an Ordinary Least Squares regression (OLS) with a polynomial of order  $M$  applied to the state variables.

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<sup>3</sup>This assumption is justified when approximating functions that are elements of the  $L^2$  space of square-integrable functions relative to some measure. Since  $L^2$  is a Hilbert space, it has a countable orthonormal basis (see, e.g., Royden (1988)).

Based on the estimates in (6) the optimal stopping time can be derived as:

$$\begin{cases} \hat{\tau}_M^N(t_J, n) = T \\ \hat{\tau}_M^N(t_j, n) = t_j 1_{\{Z(t_j) \geq \hat{F}_M^N(X(t_j))\}} + \hat{\tau}_M^N(t_{j+1}, n) 1_{\{Z(t_j) < \hat{F}_M^N(X(t_j))\}}, \quad 1 < j \leq J-1, \end{cases} \quad (7)$$

where  $\hat{\tau}_M^N(t_j, n)$  is the optimal stopping time for path  $n$  among  $N$  paths. This framework is equivalent to Longstaff and Schwartz's (2001) method. Stentoft (2014) shows that this framework, called stopping time iteration method, is the most efficient method.

Note that the approximations of  $\hat{F}_M^N(X(t_j))$  for each  $t_j$  are exclusively used to compare with  $Z(t_j)$ . Errors made in the approximation  $F$  are not carried over to subsequent steps of the algorithm. Nevertheless, the quality of the approximation of the optimal stopping time depends on the quality of the approximations of  $\hat{F}_M^N(X(t_j))$  for each  $t_j$ , especially at the intersection of  $Z$  and  $F$ .

Significant improvements can be achieved by using an Initial State Dispersion (ISD) proposed by Rasmussen (2005). ISD consists of dispersing the initial starting points of the simulated paths. Rasmussen (2005) uses a random dispersion based on the dynamics of  $X$ , though suggests a deterministic dispersion could be used. In fact, a deterministic dispersion is preferred because it reduces the variance of the estimates. The reason to disperse the paths is to make sure a portion of the simulated paths are in the optimal exercise region. This improves the estimation of  $F$  around the region where it intersects with  $Z$ , improving the estimation of this intersection, thus improving the approximation of the optimal stopping time.

From the algorithm in (7) a natural estimate of the option value in (5) is given by

$$\hat{P}_M^N(X(0)) = E[Z(\hat{\tau}_M^N(1)) | X(0)] = \hat{F}_M^N(X(0)), \quad (8)$$

where  $\hat{P}_M^N(X(0))$  is a function of the initial state variables and the price of the option is found by evaluating  $\hat{P}_M^N(X(0) = x)$ . In the case when all the paths are started at the same known values of

the state variable, i.e.  $X(0) = x$ , the estimate in (8) simplifies to a sample average as

$$\hat{P}_M^N(X(0) = x) = \frac{1}{N} \sum_{n=1}^N Z(n, \hat{\tau}_M^N(1, n)), \quad (9)$$

where  $Z(n, \hat{\tau}_M^N(1, n))$  is the payoff from exercising the option at the optimal stopping time  $\hat{\tau}_M^N(1, n)$  determined for path  $n$  according to (7).

### 4.3 Probability of Exercising the Option

The value of the option is essential, but the physical probability of exercising is of great interest for all stakeholders of a real option. The probability to exercise within a specific time horizon is an important piece of information for decision makers, stakeholders and the government. Here, we propose a general method to estimate that probability. The estimation is easiest when all simulated paths start at  $S(0) = x$ .<sup>4</sup> The probability of exercising under the risk-neutral measure  $\mathbb{Q}$  before a time horizon  $T^*$  is easily estimated by:

$$\hat{\mathcal{P}} = \frac{1}{N} \sum_{n=1}^N 1_{\{\hat{\tau}_M^N(0, n) \leq T^*\}} \times 1_{\{Z(\hat{\tau}_M^N(0, n)) > 0\}}, \quad (10)$$

where  $1_{\{\hat{\tau}_M^N(0, n) \leq T^*\}}$  indicates whether the stopping time was prior to or at the time horizon  $T^*$ , and  $1_{\{Z(\hat{\tau}_M^N(0, n)) > 0\}}$  indicates whether the option was exercised. The later is required for the cases when  $T^* = T$ , and the option is simply held until maturity, and never exercised. Note that the estimation of  $\hat{\tau}_M^N(0, n)$  is done under  $\mathbb{Q}$ .

In general, the risk-neutral probability of exercising is different from the physical probability of exercising. Note, however, that the value of the option  $P$ , the current payoff  $Z$ , and the conditional expectations  $F$  are the same, for all  $t_j$  under both  $\mathbb{P}$  and  $\mathbb{Q}$ . Estimating (10) under the physical measure requires a two-step procedure. First, estimate  $F(X(t_j))$  for all  $t_j$  using  $X(t_j)$  under  $\mathbb{Q}$ . An ISD is preferable for this step. Next, simulate  $X'(t_j)$ , without ISD, under  $\mathbb{P}$ . Then, estimate (7) using  $X'(t_j)$ . Finally, estimate (10) using the newly estimated  $\hat{\tau}_M^{\prime, N}$ . i.e., estimate  $F$  using

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<sup>4</sup>The ISD method could potentially be used in conjunction with importance sampling or stratification, but we leave this for future research.

$X'(t_j)$ , then evaluate  $F(X'(t_j))$  to determine the optimal stopping time  $\hat{\tau}'_M(0, n)$ . Note that in the case where  $T^*$  is a set of times,  $\mathcal{P}$  is a set of probabilities. For example, one can estimate the probability of exercising at each exercise possibilities  $t_k$ .

In summary, one can simulate two sets of paths of state variables. One set under  $\mathbb{Q}$  using ISD ( $X(t_j)$ ) and one set under  $\mathbb{P}$  without ISD ( $X'(t_j)$ ). In the backward iteration algorithm, estimate  $\hat{F}$ , and evaluate  $Z$  using  $X(t_j)$  in order to determine  $\hat{\tau}$  under  $\mathbb{Q}$ . Next evaluate  $\hat{F}$  and  $Z$  using  $X'(t_j)$  to determine  $\hat{\tau}'$  under  $\mathbb{P}$ .

## 5 Numerical Examples

In this section, we discuss a numerical example of our general results. We take as a motivating example the energy industry and we replicate a simple option from Dixit and Pindyck (1994). This text book example is interesting because analytic solutions are available to compare to the simulation results. First, general conclusions are obtained using theory from Section 3. Second, model specific results are obtained using the analytic solution for Black Scholes Merton model (BSM hereafter). Finally, we study a similar option with multiple exercise possibilities using the simulation framework presented above. Results show the effect of multiple exercise opportunities on the cumulative exercise probability of real options and compare the incentives and penalties required to either promote or hinder the project.

### 5.1 Black-Scholes-Merton Option

As a first example, we wish to study a simple case with an analytic solution for easy comparison. Take as a motivating example the electricity industry which has alternative projects for a future electric power plant, one very polluting (brown power plant), one less polluting (green power plant). This is a timely example because according to U.S. Energy, Information Administration (2016), new coal plants are being built in the United States, and according to Hodge (2016) coal power plants might be on the rise again. Suppose the current market conditions are such that a polluting power plant is a better economical choice (because the firm does not pay for most of the externalities



of the pollution such as health issues, and waterways cleaning.) Now suppose the government, who is responsible for some of the externalities, wants to promote the green power plant and/or hinder the brown power plant. Because the firm has a real option, and because the government is a stakeholder which has control on some parameters of the real option, the government can modify the characteristics of the real option, change its value, and change the probabilities of exercise. For example, the government can offer a grant if the firm decides to build a green power plant. This would change the strike of the real option. The government could offer multiple future positive cash flows (for example tax credits) and this would change the dynamics of the underlying process of the real option. The government could hinder the brown power plant by either imposing a penalty at the building of the power plant and change the strike of the real option or impose future negative cash flows (for example with a carbon tax) and change the underlying asset dynamics.

Let us consider both the green and brown power plant as individual projects. Suppose the projected project's cash flows can be summarized as a stream of future cash flows. Let  $S(t_k)$  be a stochastic variable representing the present value of the future cash flows from the project. Let  $y$  be the continuous time income rate representing the cash flows from the project. Finally, let  $K$  be the capital investment for this project. In such a case, the real option can be modeled as a simple call option.

Furthermore, we assume that  $S$  follows a geometric Brownian motion dynamics under  $\mathbb{P}$  such that:

$$dS_t = (\mu - y) dt + \sigma dB_t, \quad (11)$$

where  $\mu$  is the growth rate of  $S$  under  $\mathbb{P}$ ,  $y$  is the dividend yield,  $\sigma$  is the constant volatility of log-variations in  $S$ , and  $B_t$  is a Brownian motion. In a market free of arbitrage and completed by a risk-free asset, the dynamics of  $S$  under the risk-neutral measure  $\mathbb{Q}$  becomes

$$dS_t = (r - y) dt + \sigma dB_t, \quad (12)$$

where  $r$  is the risk-free rate. We further assume that the firm has the option to wait until  $T$  to

Table 1: Option Value for a Reference Option with three different Capital Investment Requirements.

Case	Strike	Option value	Exer. Value	Prob. Exercise
1	80	\$22.56	\$20.00	63.79%
2	100	\$16.64	\$0.00	50.00%
3	120	\$12.32	\$0.00	38.66%

Note: This table shows the option value, immediate execution value, and exercise probability at maturity for a real option. Three cases are shown for a capital investment  $K \in \{80, 100, 120\}$ . The present value of the future cash flows is modeled by a Geometric Brownian Motion with  $S(0) = 100$ , a risk-free rate of  $r = 4\%$ , a dividend yield  $y = 4\%$ , and growing rate of  $\mu = 6\%$ , and a volatility of  $\sigma = 20\%$ . The option and exercise values are at  $t = 0$ , and the exercise probability is at  $t = T$ . The maturity of the option is 10 years.

exercise its option, but cannot exercise before that. The American-style option is introduced next.

In this Black-Scholes-Merton world, the value of the real option can easily be found using the analytic solution:

$$\begin{aligned}
 c(0) &= S(0) e^{-yT} N(d_1) - K e^{-rT} N(d_2), \\
 d_1 &= \frac{\ln(S(0)/C) + \left(r - y + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}, \\
 d_2 &= d_1 - \sigma \sqrt{T}.
 \end{aligned} \tag{13}$$

An external stakeholder, such as the government, will be interested in the probability of exercising under the physical measure  $\mathbb{P}$ . Under Black-Scholes-Merton, this is found by replacing  $r$  by  $\mu$  when computing  $N(d'_2)$ . We use  $d'_2$  to indicate that the value is under the physical measure as opposed to  $d_2$  under the risk-neutral measure.

Let  $S(0) = 100$ , the capital investment  $K \in \{80, 100, 120\}$ , the risk-free rate  $r = 4\%$ , the growth rate  $\mu = 6\%$ , the continuous time income yield  $y = 4\%$ , the volatility  $\sigma = 20\%$ , and the maturity of the option  $T = 10$  years. We use these numbers for both the green and brown power plants to cover many different cases.

Table 1 shows the values of these reference real options for three different capital investments. The in the money (ITM) option has an option value which is greater than the execution value. Thus, it is optimal to hold on the option. Both the at the money (ATM) and out the money (OTM) options should also be held until maturity. The physical exercise probability is the probability of

exercising the option at maturity. Obviously, the ITM option has a higher exercise probability. The three cases will be used to show what happens when the initial exercise probability is above, at, or below 50%.

We wish to study the effect of modifying the option specification on the exercise probability. Suppose that the government wants to promote the green power plant project, and wants to impede the brown power plant project. First, we use the general results from Section 3 to determine which action would be optimal to promote and impede these projects. Next, we present numerical results specific to the BSM model.

First, we present the general results for the promotion of the green power plant project. Suppose the government has a limited budget of \$10 to promote the execution of the real option. The government has the choice between offering a grant at the exercise of the option, essentially reducing the capital investment, or to offer future tax credits (in some form) that will essentially increase the present value of the future cash flows, i.e.  $S(0)$ . We suppose that the present value of the future cash flows is \$10. Proposition 2 tells us that for the ITM and ATM option, the most efficient method to increase the exercise probability will be to apply the \$10 to reduce the capital investment. The alternative would be to apply the \$10 to increase the present value of the future cash flows, but that would increase the total variance and decrease the exercise probability. For the OTM option, it will be the inverse. It will be more efficient to offer multiple future positive cash flows with a present value of \$10, i.e., increase  $S(0)$ .

Second, we present general results when the government wishes to hinder the brown power plant project. Suppose the government wants to impede the project by either imposing a penalty on the execution of the real option, or impose an additional tax that reduces the present value of the future cash flows. Corollary 4 tells us that for the ITM and ATM option, applying a penalty at the exercise to increase the capital investment should reduce the exercise probability more than imposing multiple future smaller penalties. For the OTM option, applying a penalty in the form of multiple future cash flows, such as a carbon tax should be more efficient in decreasing the exercise probability of the option.

Table 2: Promoting the real option with a \$10.00 incentive.

Case	Init. Prob.	Apply \$10 to	New Prob.	Preferred?
1	63.79%	dec. Capital cost	71.36%	***
		inc. Future C.F.	69.27%	
2	50.00%	dec. Capital cost	56.62%	***
		inc. Future C.F.	55.99%	
3	38.66%	dec. Capital cost	44.01%	***
		inc. Future C.F.	44.53%	

Note: This table shows the effect of offering a \$10 incentive to promote a real option. Three cases are shown for a capital investment  $K \in \{80, 100, 120\}$ . The present value of the future cash flows is modeled by a Geometric Brownian Motion with  $S(0) = 100$ , a risk-free rate of  $r = 4\%$ , a dividend yield  $y = 4\%$ , and growing rate of  $\mu = 6\%$ , and a volatility of  $\sigma = 20\%$ . The option and exercise values are at  $t = 0$ , and the exercise probability is at  $t = T$ . The maturity of the option is 10 years. For the three cases, the incentive is either applied to the capital investment or to the present value of the future cash flows. "\*\*\*" represent the method that has the largest impact on the exercise probability.

Table 3: Impeding the real option with a \$10.00 incentive.

Case	Init. Prob.	Apply \$10 to	New Prob.	Preferred?
1	63.7889%	inc. Capital cost	56.62%	***
		dec. Future C.F.	57.39%	
2	50.0000%	inc. Capital cost	44.01%	***
		dec. Future C.F.	43.38%	
3	38.6568%	inc. Capital cost	33.91%	***
		dec. Future C.F.	32.46%	

Note: This table shows the effect of offering a \$10 incentive to promote a real option. Three cases are shown for a capital investment  $K \in \{80, 100, 120\}$ . The present value of the future cash flows is modeled by a Geometric Brownian Motion with  $S(0) = 100$ , a risk-free rate of  $r = 4\%$ , a dividend yield  $y = 4\%$ , and growing rate of  $\mu = 6\%$ , and a volatility of  $\sigma = 20\%$ . The option and exercise values are at  $t = 0$ , and the exercise probability is at  $t = T$ . The maturity of the option is 10 years. For the three cases, the incentive is either applied to the capital investment or to the present value of the future cash flows. "\*\*\*" represent the method that has the largest impact on the exercise probability.

We can verify the previous results and quantify the change for the model specified. Suppose the government has a budget of \$10 to promote the green power plant, and wants to apply a penalty with a present value of \$10 to impede the brown power plant. For each, ITM, ATM, and OTM options, we apply the grant and penalty to either the strike or underlying asset and then compute the exercise probability under the physical measure using the analytic solution. Table 2 presents the results when promoting the real option. For the OTM option which has an initial exercise probability of 63.79%, decreasing the capital investment increases the exercise probability

to 71.36%, whereas increasing the underlying increases the exercise probability only to 69.27%. This confirms the theoretical results presented above. For the ATM option, we get to the same conclusions, however, the difference in probabilities of exercise is less. For the OTM option, apply the \$10 to the underlying asset is more efficient and increases the exercise probability to 44.53%. Table 3 presents the results when impeding the project with a penalty of \$10. For the ITM option, it is more efficient to apply the penalty on the strike of the option to impede the project. The same goes for the ATM option. This is inline with the theoretical results presented above. However, for the OTM option, imposing the penalty to the underlying asset reduces the exercise probability to 32.46% as opposed to 33.41% when imposing the penalty to the strike.

Next, we study a different situation. Suppose the government wants to promote a project and has a specific target for the exercise probability at maturity of the real option. We determine what is the grant/penalty required to promote/hinder the project to a specific target probability. At the same time, we determine if it is more efficient to apply the grant/penalty towards the capital investment, or the present value of the future cash flow. For the BSM model analytic solutions exist. The analytic solutions are derived in Appendix A. In this section, we present the numerical results. Let us first consider the promotion of the green power plant. As an example, suppose that the government wants to promote the green power plant and wishes to increase the exercise probability to at least 90%.

Table 4 presents the results for promoting the project such that the exercise probability becomes 90%. For Cases 1 and 2, theory states that applying an amount of money to the strike will be more efficient. This is also what we observe with the numerical results. For Case 1, to increase the exercise probability from 63.79% to 90.00%, it will cost \$35.54 to modify the strike, whereas the underlying asset would need to be increased by \$79.93. Confirming that modifying the strike would be preferable. For Case 2, to increase the exercise probability from 50.00% to 90.00%, it will cost \$55.54 to modify the strike, whereas the underlying asset would need to be increased by \$124.91. For Case 3, the theory cannot answer our question specifically because the current probability is very low and the target is very high. It is possible that both cases of Proposition 2 be encountered

Table 4: Promoting the real option to a target exercise probability of 90%.

Case	Init.Prob.	Target Prob.	Cost for C	Cost or S
1	63.79%	90.00%	\$ 35.54	\$ 79.93
2	50.00%	90.00%	\$ 55.54	\$124.91
3	38.66%	90.00%	\$ 75.54	\$169.89
3'	38.66%	90.00%	\$ 66.65	\$ 20.00

Note: This table shows the cost of promoting the real option such that the exercise probability becomes 90%. Three cases are shown for a capital investment  $K \in \{80, 100, 120\}$ . The present value of the future cash flows is modeled by a Geometric Brownian Motion with  $S(0) = 100$ , a risk-free rate of  $r = 4\%$ , a dividend yield  $y = 4\%$ , and growing rate of  $\mu = 6\%$ , and a volatility of  $\sigma = 20\%$ . The option and exercise values are at  $t = 0$ , and the exercise probability is at  $t = T$ . The maturity of the option is 10 years. For the three cases, the incentive is either applied to the capital investment or to the present value of the future cash flows. For the third case, we also consider a program for which money is applied toward the present value of future cash flow until the exercise probability is 50%, then more money is applied towards the capital investment until the probability of exercise is 90%.

Table 5: Impeding the real option to a target exercise probability of 10%.

Case	Init.Prob.	Target Prob.	Cost for C	Cost or S
1	63.79%	10.00%	\$144.91	\$ 64.43
2	50.00%	10.00%	\$124.91	\$ 55.54
3	38.66%	10.00%	\$104.91	\$ 46.65
1'	63.79%	10.00%	\$ 20.00	\$ 55.54

Note: This table shows the penalty required of impede the real option such that the exercise probability becomes 10%. Three cases are shown for a capital investment  $K \in \{80, 100, 120\}$ . The present value of the future cash flows is modeled by a Geometric Brownian Motion with  $S(0) = 100$ , a risk-free rate of  $r = 4\%$ , a dividend yield  $y = 4\%$ , and growing rate of  $\mu = 6\%$ , and a volatility of  $\sigma = 20\%$ . The option and exercise values are at  $t = 0$ , and the exercise probability is at  $t = T$ . The maturity of the option is 10 years. For the three cases, the incentive is either applied to the capital investment or to the present value of the future cash flows. For the first case, we also consider a program which a penalty is applied toward the capital investment until the exercise probability is 50%, then further penalty is applied to the present value of future cash flows until the probability of exercise is 10%.

while the exercise probability is modified. That is, the optimal modification might be a combination of both modifying the strike and the underlying asset dynamics. We thus present two scenarios from Case 3. Case 3 shows the results when modifying either the strike or the underlying. Even if the current exercise probability is below 50%, it is more efficient to change the strike than change the underlying asset. For Case 3' we modify the underlying asset until the exercise probability equal 50%, then modify the strike. Thus, for case 3', the cost to change the exercise probability to 90% is the sum of both amounts, that is \$86.65.

Next, we want to study a case where the government wants to impede the brown power plant

project. As an example, we consider a case where the government wants to decrease the exercise probability of the real option to 10%. Table 5 shows the required penalty to impede the project to a exercise probability of 10%. Results show the exact opposite when one wishes to promote the project. The penalty required to impede the project is less if imposed to the underlying asset than if imposed to the strike. However, note that here, the special occurs when the initial exercise probability is above 50%. This is why we created Case 1' where the penalty is imposed to both the underlying asset and the strike.

## 5.2 Vanilla American Call Option

Suppose an option similar to the one described in the previous section, but with early exercise possibilities. Let  $S(0) = 100$ , the capital investment  $K \in \{80, 100, 120\}$ , the risk-free rate  $r = 4\%$ , the growth rate  $\mu = 6\%$ , the continuous time income yield  $y = 4\%$ , the volatility  $\sigma = 20\%$ , and the maturity of the option  $T = 10$  years. We consider multiple exercise possibilities and present results for  $J \in \{1, 2, 10, 50\}$ . Analytic solutions cannot be used to solve the problem with multiple exercise possibilities and numerical methods have to be used. The European option is replicated with  $J = 1$  to compare numerical method results with analytic solution results.

We evaluate the option using Least Squares Monte Carlo and the framework described above. First  $X(t_j)$  are simulated with an ISD. The ISD is created using a simple uniform distribution. Next, all  $F(t_j)$  are estimated. The price is estimated by valuing  $F(t_0)$ . A second set of paths is simulated under  $\mathbb{P}$  without ISD to estimate the physical probabilities of exercising. We repeat 100 times and estimate the probability of exercising under  $\mathbb{P}$  has the average of the 100 repetition. Since all repetitions are independent, we estimate the standard deviation of the estimates using those 100 repetitions, which we report below.

It is well known that for two options that differ only on the number of possible early exercise, the option with more opportunities to exercise will have at least as much value as the other option, and possibly more. Intuitively, if the option to hold has more value, we would expect less exercise. However, it is not the case.

Table 6: American Option Estimates as a Function of the number of Exercise Dates.

Panel A: Price Estimates

Case	K	Number of exercise dates			
		J=1	J=2	J=10	J=50
1	80	22.54 (0.13)	25.54 (0.11)	27.23 (0.08)	27.46 (0.08)
2	100	16.62 (0.12)	18.25 (0.10)	19.15 (0.08)	19.32 (0.09)
3	120	12.30 (0.11)	13.18 (0.09)	13.77 (0.08)	13.90 (0.09)

Panel B: Cumulative Probability Estimates

Case	K	Number of exercise dates			
		J=1	J=2	J=10	J=50
1	80	63.80% (0.16)	68.32% (0.24)	70.73% (0.23)	70.66% (0.24)
2	100	50.01% (0.16)	53.12% (0.23)	53.96% (0.20)	54.01% (0.24)
3	120	38.66% (0.15)	40.60% (0.19)	41.05% (0.19)	41.13% (0.21)

Note: This table shows the option value, and the cumulative exercise probability at maturity for a real option. Three cases are shown for a capital investment  $K \in \{80, 100, 120\}$ . The present value of the future cash flows is modeled by a Geometric Brownian Motion with  $S(0) = 100$ , a risk-free rate of  $r = 4\%$ , a dividend yield  $y = 4\%$ , and growing rate of  $\mu = 6\%$ , and a volatility of  $\sigma = 20\%$ . The option and exercise values are at  $t = 0$ , and the exercise probability is at  $t = T$ . The maturity of the option is 10 years, and six cases are shown for the number of possible early exercise  $J \in \{1, 2, 10, 50\}$ . The optimal exercise strategy is approximated using 100,000 simulated paths under  $\mathbb{Q}$  and an ISD. The option value is approximated using a second set of 100,000 simulated paths under  $\mathbb{Q}$  without an ISD. The physical exercise probability at each possible early exercise date is estimated with a third set of 100,000 simulated paths under  $\mathbb{P}$  without an ISD. We report the cumulative exercise probability at or prior to maturity. The simulations are repeated 100 times and we report the mean and standard deviation over the 100 repetitions.

Table 6 presents the value and cumulative exercise probability for the American-style option with increasing number of exercise possibilities. Panel A presents the value of the real option for three different strikes. When  $J = 1$ , note that the estimates are inline with the European values using analytic solutions. As the number of possible exercise possibilities increases, the value of the option increases and that was expected. More exercise possibilities increases the value of the option (and decreases the probability of exercising very early). However, more exercise possibilities increases the cumulative exercise probability. Panel B presents the cumulative exercise probability. For  $J = 1$ , the cumulative exercise probability is the probability of exercising at maturity of the options. For  $J = 2+$ , the cumulative exercise probability is the probability of exercising at or before maturity. For  $J = 1$ , the exercise probability is equal the exercise probability of the European option. For  $J = 2$ , the probability of exercising at maturity is the same as for the European option, however, there are paths which are optimally exercised prior to maturity. This



Table 7: Promoting the American Style Option with a fixed budget of \$10.  
 Panel A: Option value estimates when \$10 is applied to reduce the Capital Investment

Case	K	Number of exercise dates			
		J=1	J=2	J=10	J=50
1	80	26.27 (0.14)	30.27 (0.11)	32.82 (0.08)	33.12 (0.07)
2	100	19.35 (0.12)	21.56 (0.10)	22.76 (0.08)	22.96 (0.09)
3	120	14.29 (0.11)	15.48 (0.10)	16.21 (0.08)	16.34 (0.09)

Panel B: Cumulative exercise probability when \$10 is applied to reduce the Capital Investment

Case	K	Number of exercise dates			
		J=1	J=2	J=10	J=50
1	80	71.37% (0.13)	76.44% (0.23)	80.53% (0.24)	80.66% (0.29)
2	100	56.62% (0.16)	60.44% (0.24)	61.81% (0.23)	61.79% (0.26)
3	120	44.02% (0.16)	46.49% (0.22)	47.07% (0.20)	47.15% (0.22)

Panel C: Option value estimates when \$10 is applied to increase the present value of future cash flows

Case	K	Number of exercise dates			
		J=1	J=2	J=10	J=50
1	80	27.72 (0.15)	31.79 (0.12)	34.29 (0.08)	34.58 (0.08)
2	100	20.99 (0.14)	23.36 (0.11)	24.64 (0.09)	24.86 (0.10)
3	120	15.93 (0.12)	17.28 (0.11)	18.09 (0.09)	18.24 (0.10)

Panel D: Cumulative exercise probability when \$10 is applied to increase the present value of future cash flows

Case	K	Number of exercise dates			
		J=1	J=5	J=10	J=50
1	80	69.28% (0.14)	74.23% (0.24)	77.81% (0.25)	77.83% (0.26)
2	100	56.00% (0.16)	59.75% (0.25)	61.05% (0.22)	61.05% (0.25)
3	120	44.54% (0.16)	47.06% (0.22)	47.66% (0.20)	47.74% (0.23)

Note: This table shows the option value, and the cumulative exercise probability at maturity for a real option when promoting the project with a budget of \$10 applied to either the strike or the underlying asset. Three cases are shown for a capital investment  $K \in \{80, 100, 120\}$ . The present value of the future cash flows is modeled by a Geometric Brownian Motion with  $S(0) = 100$ , a risk-free rate of  $r = 4\%$ , a dividend yield  $y = 4\%$ , and growing rate of  $\mu = 6\%$ , and a volatility of  $\sigma = 20\%$ . The option and exercise values are at  $t = 0$ , and the exercise probability is at  $t = T$ . The maturity of the option is 10 years, and six cases are shown for the number of possible early exercise  $J \in \{1, 2, 10, 50\}$ . The optimal exercise strategy is approximated using 100,000 simulated paths under  $\mathbb{Q}$  and an ISD. The option value is approximated using a second set of 100,000 simulated paths under  $\mathbb{Q}$  without an ISD. The physical exercise probability at each possible early exercise date is estimated with a third set of 100,000 simulated paths under  $\mathbb{P}$  without an ISD. We report the cumulative exercise probability at or prior to maturity. The simulations are repeated 100 times and we report the mean and standard deviation over the 100 repetitions.

is why the cumulative exercise probability increases.

Table 7 presents the results for promoting a real option with early exercise possibilities. Panel A shows the value of the option when a grant is applied at the exercise of the option, which decreases

the strike; while Panel B shows the cumulative exercise probability. We can observe the same patterns as in Table 6. It is interesting to note that as the number of early exercise possibilities increases, the cumulative exercise probability increases too. This suggests that if the government wants to evaluate the required budget to promote a project, it is important to consider the early exercise possibility. Suppose the government wants to promote the project and has a target of 80% for the probability of exercising at or prior to maturity. If the government uses an European option to model the option, it will be under the impression that a budget of \$10 will not be sufficient, but if in reality the firm has multiple exercise possibilities, then a budget of \$10 is sufficient. Panel C and Panel D show the price and cumulative exercise probability when the promotion is offered through the modification of the underlying asset. Interestingly, the value of the option (to wait) is greater when the underlying asset is modified instead of the strike, but the exercise probability is lower. Results in Table 7 compare to results in Table 2. Observe first that the early exercise possibilities do not change the preferred method for promoting the project. Second, the early exercise possibilities actually improve the effectiveness of the promotion policy as the cumulative exercise probability is increased.

Table 8 shows the results when impeding the project with a fixed amount of \$10. Panel A and Panel B show the results when the penalty is applied to the strike, while Panel C and Panel D show the results when the penalty is applied to the underlying asset. If one is strictly interested about knowing whether it is better to impose a penalty to the strike or the underlying asset to impede the project, the conclusions drawn from the European options appears to be sufficient for this case. Again, we observe that when the current exercise probability is greater than 50%, imposing the penalty on the strike reduces the exercise probability more, and this holds when early exercise is possible. When the current exercise probability is lower than 50%, then it is better to impose the penalty onto the underlying asset. The difference when early exercises are possible is that the exercise probability is greater. Thus, if one wants to reduce the exercise probability, the possibility to exercise early will mean that the required penalty will be greater than if early exercise was not possible. Table 8 compare to Table 3. Observe first that the optimal method to hinder the project

Table 8: Impeding the American Style Option with a fixed value of \$10.

Panel A: Option value estimates when \$10 is applied to increase the Capital Investment

Case	K	Number of exercise dates			
		J=1	J=2	J=10	J=50
1	80	19.35 (0.12)	21.56 (0.10)	22.76 (0.08)	22.96 (0.09)
2	100	14.29 (0.11)	15.48 (0.10)	16.21 (0.08)	16.34 (0.09)
3	120	10.62 (0.10)	11.25 (0.09)	11.75 (0.08)	11.86 (0.08)

Panel B: Cumulative exercise probability when \$10 is applied to increase the Capital Investment

Case	K	Number of exercise dates			
		J=1	J=2	J=10	J=50
1	80	56.62% (0.16)	60.44% (0.24)	61.81% (0.23)	61.79% (0.26)
2	100	44.02% (0.16)	46.49% (0.22)	47.07% (0.20)	47.15% (0.22)
3	120	33.91% (0.14)	35.42% (0.19)	35.79% (0.18)	35.91% (0.17)

Panel C: Option value estimates when \$10 is applied to decrease the present value of future cash flows

Case	K	Number of exercise dates			
		J=1	J=2	J=10	J=50
1	80	17.71 (0.11)	19.77 (0.09)	20.89 (0.07)	21.07 (0.07)
2	100	12.65 (0.10)	13.68 (0.09)	14.32 (0.07)	14.44 (0.08)
3	120	9.10 (0.09)	9.62 (0.08)	10.04 (0.07)	10.13 (0.07)

Panel D: Cumulative exercise probability when \$10 is applied to decrease the present value of future cash flows

Case	K	Number of exercise dates			
		J=1	J=2	J=10	J=50
1	80	57.39% (0.16)	61.29% (0.25)	62.75% (0.24)	62.71% (0.25)
2	100	43.39% (0.16)	45.80% (0.22)	46.36% (0.20)	46.44% (0.22)
3	120	32.46% (0.13)	33.84% (0.19)	34.20% (0.17)	34.32% (0.17)

Note: This table shows the option value, and the cumulative exercise probability at maturity for a real option when impeding the project with a budget of \$10 applied to either the strike or the underlying asset. Three cases are shown for a capital investment  $K \in \{80, 100, 120\}$ . The present value of the future cash flows is modeled by a Geometric Brownian Motion with  $S(0) = 100$ , a risk-free rate of  $r = 4\%$ , a dividend yield  $y = 4\%$ , and growing rate of  $\mu = 6\%$ , and a volatility of  $\sigma = 20\%$ . The option and exercise values are at  $t = 0$ , and the exercise probability is at  $t = T$ . The maturity of the option is 10 years, and six cases are shown for the number of possible early exercise  $J \in \{1, 2, 10, 50\}$ . The optimal exercise strategy is approximated using 100,000 simulated paths under  $\mathbb{Q}$  and an ISD. The option value is approximated using a second set of 100,000 simulated paths under  $\mathbb{Q}$  without an ISD. The physical exercise probability at each possible early exercise date is estimated with a third set of 100,000 simulated paths under  $\mathbb{P}$  without an ISD. We report the cumulative exercise probability at or prior to maturity. The simulations are repeated 100 times and we report the mean and standard deviation over the 100 repetitions.

is not affected by the early exercise possibilities. However, the hindering policy is mitigated by the early exercise possibilities. This means that when designing a policy to prevent the execution of a real option, care needs to be taken when early exercise is possible as this will mitigate the effect of

the policy.

## 6 Conclusions

The terms and conditions of financial options are contractually binding and do not change after a derivative transaction is made. In contrast, real options can be modified, because real option holders, writers, and stakeholders may have legitimate influence on the strike price and/or the underlying process of the real option. This unique feature of real options has implications to government policy, debt evaluation, and many other areas in the literature and practice of finance and economics. However, literature to date has not paid enough attention to the fundamental theory of this important topic.

In this paper, we provide model-free results that can provide guidance on determining whether it is better to change the strike of an option or the dynamics of the underlying to increase or decrease an exercise probability. We also propose a general computational framework to study the exercise probabilities of realistic American-style real options. We find that early exercise possibilities helps increase the exercise probability. This will play in favor of one who wants to increase an exercise probability of a real option, but will play against one who wishes to decrease a exercise probability. Finally we document that the optimal way of modifying the real option exercise probability depends on the current exercise probability.

The results and methods documented in this paper will open an avenue for new research. First, our results have the advantage of being very general. However, they only provide general guidance for specific cases. Future research should aim at finding analytic results for more specific cases. Second, energy and environmental economists focusing on practical applications may use our approach to investigate policy implications of a green incentive/penalty policy. To do so, a researcher may want the payoff of American-style real options to be more complex than the examples in this paper. Third, corporate finance researchers or practitioners may use our framework to assess the impact of debt forgiveness and many other topics relevant in a corporates financial decision-making. Forth, motivated by our paper, an empiricist may investigate whether a corporation tries

to improve the underlying process, its own earnings, after writing a real option on its firm value.

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## A Black-Merton-Scholes Modification with Target Probability

As a first example, we wish to study a simple case with an analytic solution for easy comparison. Suppose the projected project’s cash flows can be summarized as a stream of future cash flows. Let  $S(t_k)$  be a stochastic variable representing the present value of the future cash flows from the project. Let  $y$  be the continuous time income rate representing the cash flows from the project. Finally, let  $C$  be the capital investment for this project. In such a case, the real option can be modeled as a simple call option.

Furthermore, we assume that  $S$  follows a geometric Brownian motion dynamics under  $\mathbb{P}$  such that:

$$dS_t = (\mu - y) dt + \sigma dB_t, \quad (14)$$

where  $\mu$  is the growth rate of  $S$  under  $\mathbb{P}$ ,  $y$  is the dividend yield,  $\sigma$  is the constant volatility of  $S$ , and  $B_t$  is a Brownian motion. In a market free of arbitrage and completed by a risk-free asset, the dynamics of  $S$  under the risk-neutral measure  $\mathbb{Q}$  becomes

$$dS_t = (r - y) dt + \sigma dB_t, \quad (15)$$

where  $r$  is the risk-free rate. We further assume that the firm has the option to wait until  $T$  to exercise its option, but cannot exercise before that. We will consider the American option next.

In this Black-Scholes-Merton world, the value of the real option can easily be found using the

analytic solution:

$$\begin{aligned}
p(0) &= S(0) e^{-yT} N(d_1) - K e^{-rT} N(d_2), \\
d_1 &= \frac{\ln(S(0)/C) + \left(r - y + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}, \\
d_2 &= d_1 - \sigma\sqrt{T}.
\end{aligned}$$

Under the risk-neutral measure, the probability of exercising is  $N(d_2) = \text{Prob}^{\mathbb{Q}}(S(T) \geq K)$ . Under the physical measure, the probability of exercising is  $N(d'_2) = \text{Prob}^{\mathbb{P}}(S(T) \geq K)$ .

It is possible to solve the change in strike necessary to attain a target exercise probability

$$D^* = K - \frac{S(0)}{\exp\{N^{-1}(\mathcal{P}^*)\sigma\sqrt{T} - \left(\mu - y - \frac{\sigma^2}{2}\right)T\}},$$

where  $\mathcal{P}^*$  is the target probability.

It is also possible to solve for the required change in  $S(0)$  to attain a target probability

$$D^{**} = S(0) - K \times \exp\{N^{-1}(\mathcal{P}^*)\sigma\sqrt{T} - \left(\mu - y - \frac{\sigma^2}{2}\right)T\} \quad (16)$$

For this option, consider a program that pays (or collects) the amount  $D$  at  $T$  if the option is exercised. For an incentive program  $D > 0$ , and  $D < 0$  for a penalty program. The value of the option becomes

$$P(0) = E^{\mathbb{Q}} \left[ e^{-rT} \left( S(T) \times 1_{\{S(T) \geq K\}} - K \times 1_{\{S(T) \geq K\}} + D \times 1_{\{S(T) \geq K\}} \right) \right]. \quad (17)$$

One can easily solve (17) and find

$$P(0) = S(0) e^{-yT} N(d_1) - (K - D) N(d_2) \quad (18)$$

Suppose we are interested in the  $\mathbb{P}$  probability of exercising the option when an incentive program

is present. The probability of exercising is  $N(d'_2)$ , where

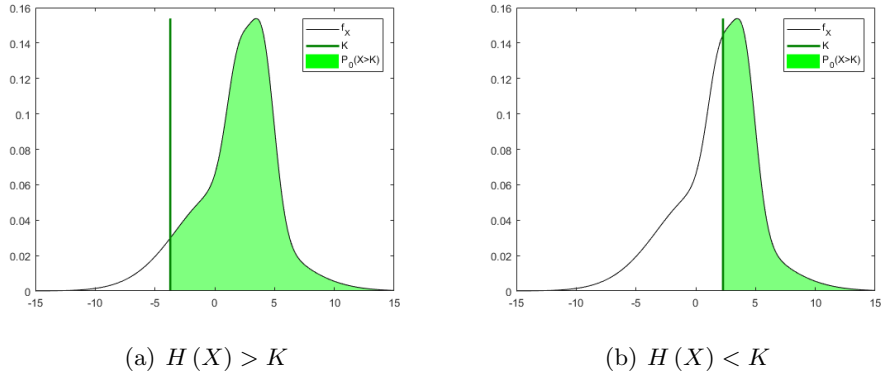
$$d'_2 = \frac{\ln(S(0)/(C-K)) + \left(\mu - y - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}. \quad (19)$$

Once can solve for the required incentive level such that the probability of exercising is  $\mathcal{P}^*$ . Since  $\mathcal{P}^* = N(d'_2)$ , then

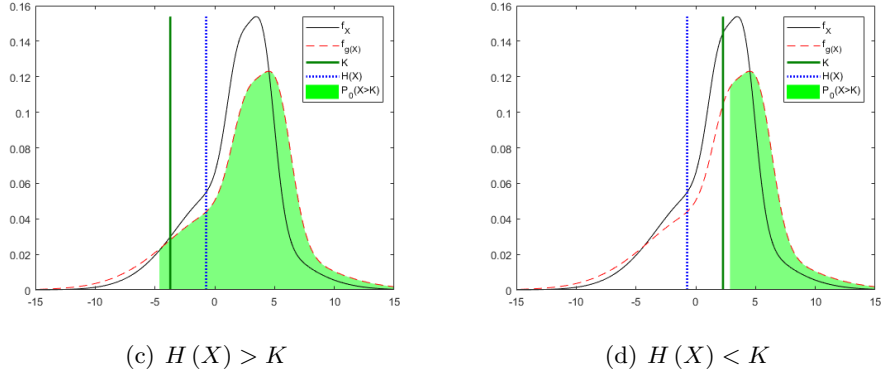
$$D^* = C - \frac{S(0)}{\exp\{N^{-1}(\mathcal{P})\sigma\sqrt{T} - \left(\mu - y - \frac{\sigma^2}{2}\right)T\}} \quad (20)$$

## B Visual Representation of Theoretical Results

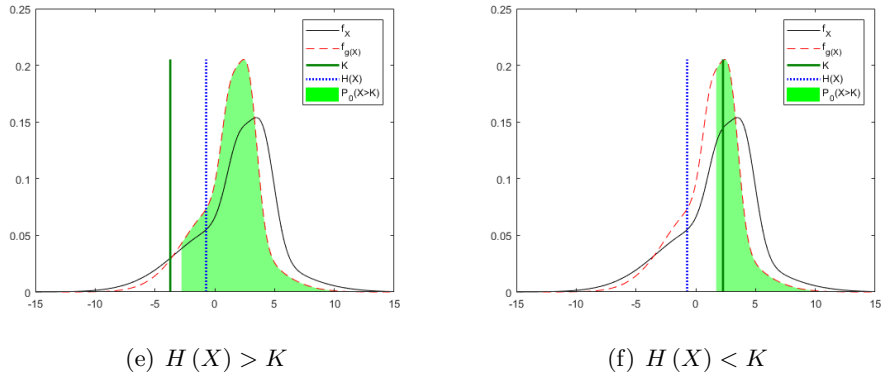
To get better insights into Proposition 1, we provide a visual representation. Figure 1 shows the effect of increasing(decreasing) the variance of a random variable  $X$  on the probability of  $\Pr(X > K)$ . A general distribution is generated for  $X$  using a convolution of multiple Gaussian distributions with different means and variance. The resulting distribution has a  $\text{Mean}(X) = 1.8805$ , a  $\text{Median}(X) = 2.3738$ , a  $\text{Variance}(X) = 11.7798$ , a  $\text{Skewness}(X) = -0.4835$ , and a  $\text{Kurtosis}(X) = 4.1012$ . At the top of the figure, the initial distribution is represented, the green shaded area highlights  $\Pr(X > K)$ . We apply to modification  $g(X; H(\cdot), e) = (1 + e)(X - H(X)) + H(X)$ , where  $e = 0.25$  ( $e = -0.25$ ). We select  $H(X)$  as the second quintile. We find the proportion of the distribution of  $X$  which is above the strike  $K$ , and we represent this region using shading on the transformed distribution. This shading helps visualize whether the probability has increased or decreased. On the left side, we have the case where  $H(X) > K$ , as in Proposition 1, (3a). Observe that when the variance is increased, a portion of the distribution that was above the strike  $K$  is now below (see the shaded region below  $K$ ), thus the probability is decreased by the increase in variance. Whereas the probability is increased when the variance is decreased. On the right side, we have the case where  $H(X) < K$ . The effects of increasing and decreasing the variance are reversed.



Initial situation



Increasing Variance



Decreasing Variance

**Figure 1: Illustration of Proposition 1.**

This Figure shows the effect of modifying the variance of a general distribution on the exercise probability of a call option. We set  $H(X)$  as the second quantile. The black solid line represents the original general distribution of  $X$  (a convolution of multiple normal distributions).  $\text{Mean}(X) = 1.8805$ ,  $\text{Median}(X) = 2.3738$ ,  $\text{Variance}(X) = 11.7798$ ,  $\text{Skewness}(X) = -0.4835$ , and  $\text{Kurtosis}(X) = 4.1012$ . The red dashed lines represents  $g(X)$ . The modification increases (or decreases) the variance using  $g(X; H(\cdot), e) = (1 + e)(X - H(X)) + H(X)$ , where  $e = 0.25$  ( $e = -0.25$ ). We are interested in how  $\Pr[g(X; H(\cdot), e) > K]$  compares to  $\Pr[X > K]$ . The shaded area represent the proportion of  $X > K$  projected onto  $g(X)$  In (c), and (f) the probability decreases. In (d), and (e) the probability increases.