Survey data and subjective beliefs in business cycle models*

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Abstract

Survey data on household forecasts for unemployment and inflation rates reveal large upward biases that are positively correlated and countercyclical. We develop a framework to analyze general equilibrium settings where agents' subjective beliefs are endogenous and shaped by time-varying concerns for model misspecification. Applying our framework to a New-Keynesian model with frictional labor markets, we find that, consistent with the survey evidence, an increase in concerns for model uncertainty generates large belief distortions, which reduce aggregate demand and propagate through frictional goods and labor market to cause a contraction. As part of our analysis we also develop solution techniques that preserve the effects of time-varying concerns for model misspecification in the class of linear solutions.

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1 Introduction

Survey data on households' expectations about future macroeconomic outcomes reveal significant systematic biases and comovement of these biases at business cycle frequencies. In this paper, we present a theory where subjective beliefs stem from households' concerns that the underlying model they use for decision-making is potentially misspecified. In doing so, we replace the rational expectations assumption with a tightly specified framework that links households' decisions to expectations adjusted for plausible misspecification fears. This departure is disciplined using data on macroeconomic variables along with survey data on households' expectations. We use this framework to quantify the magnitude and economic channels through which these subjective beliefs affect aggregate outcomes.

We begin by documenting several time-series and cross-sectional patterns in household forecasts for unemployment and inflation. Using the University of Michigan Survey of Consumers, we show that household forecasts for unemployment and inflation are significantly biased upwards on average and both biases fluctuate significantly over the business cycle, increasing during recessions. Furthermore, in the cross-section, households who forecast high inflation relative to the population also tend to forecast high unemployment. These results are corroborated by additional evidence from the Survey of Consumer Expectations conducted by the Federal Reserve Bank of New York.

To exploit these patterns and understand the channels through which subjective beliefs affect the macroeconomy, we develop a tractable framework that introduces agents with concerns for model misspecification into general equilibrium settings. Our theoretical foundation is an extension of the robust preference model of Hansen and Sargent (2001a,b). Agents endowed with robust preferences are concerned that the particular model they view as their 'benchmark' model is misspecified. Instead of using only the benchmark model, they consider a set of models that are statistically hard to distinguish from that benchmark. To ensure that their choices are robust against model misspecification, agents tilt their subjective beliefs in directions that lower their continuation values and make decisions using this alternative probability distribution. A key aspect of this framework is that subjective beliefs are an equilibrium object jointly pinned down with the rest of endogenous variables.[1]

[1] The construction of subjective beliefs is closely tied to the utility-minimizing prior in the multiple prior framework of Gilboa and Schmeidler (1989) and Epstein and Schneider (2003).

We extend the robust preference framework to allow for time variation in concerns for model uncertainty, which induces fluctuations in agents' subjective beliefs that we connect to our findings from the survey data.

Our work thus combines two sources of discipline for the role of subjective beliefs in the model. First, the robust preference framework yields testable cross-equation restrictions that constrain the dynamics of subjective beliefs. Second, the time-varying forecast biases that we construct from the survey data have directly measurable counterparts in the theoretical model and hence provide clear targets for calibration and estimation. In contrast, many popular alternative measures based on confidence indices or forecast disagreement provide qualitative information about model uncertainty faced by economic agents but do not have direct model counterparts. Instead, their mapping to equilibrium dynamics is inferred indirectly, from the evolution of macroeconomic quantities.

In order to study the quantitative role of households' subjective beliefs in driving aggregate outcomes, we introduce households endowed with model misspecification concerns into a New-Keynesian model with a frictional labor market. We take advantage of the restrictions imposed by our framework for endogenous subjective beliefs to calibrate the model, matching the moments of the forecast biases as well as moments from data on macroeconomic variables. The calibrated model explains most of the observed variation in the unemployment forecast bias and two thirds of the variation in inflation forecast bias. It also generates quantitatively plausible fluctuations in other key macroeconomic aggregates such as inflation, output and unemployment.

To quantify the feedback between subjective beliefs and macroeconomic outcomes, we consider the effects of a one-time one standard deviation increase in the concerns for model misspecification. The model replicates the joint increase in the unemployment and inflation belief biases. In addition, an increase in model uncertainty has contractionary effects, generating lower output, higher unemployment and lower inflation. Greater concerns for misspecification distort subjective beliefs more toward adverse outcomes, causing agents to perceive lower growth, a more persistent increase in uncertainty, and a simultaneous tightening of monetary policy.

In response to these perceptions, aggregate demand falls as consumers forecast a sequence of adverse shocks in the future, leading them to consume less today due to consumption smoothing. An increase in misspecification concerns also leads firms to expect lower productivity and higher marginal costs in the future, inducing lower labor demand. Moreover, in the presence of labor market frictions, agents' pessimistic evaluation of future surpluses leads to lower match creation, which increases unemployment and decreases output. Agents are concerned about states with lower productivity, higher marginal costs and tighter labor market conditions. This explains why our model generates countercyclical and positively correlated biases in inflation and unemployment forecasts.

We decompose the total effect of an increase in model uncertainty by analyzing the contribution of belief distortions of the separate economic actors in our setting: consumers, workers and firms. With rational firms, the increase in unemployment is about one half relative to the benchmark where all agents share the same subjective beliefs, emphasizing the interaction between aggregate demand effects from the consumers' distorted Euler equation and the equally important amplification through the frictional product and labor markets.

On the technical side, we develop a series expansion technique that incorporates the impact of time-varying model misspecification concerns in the first-order approximation of the model. The main challenge is that the belief distortions arise endogenously and need to be computed jointly with the equilibrium dynamics, as agents overweigh states with low utility realizations. The approximation method leads to a tractable linear solution for the equilibrium dynamics with a role for model misspecification concerns. Our series expansion technique extends to models in which agents have heterogeneous concerns for model misspecification and hence heterogeneous subjective beliefs. In our application, we use this flexibility to isolate the role of belief distortions of different economic agents.

The paper contributes to the growing literature that quantitatively assesses the role of model uncertainty in the macroeconomy, building on alternative decision-theoretic foundations by Gilboa and Schmeidler (1989), Epstein and Schneider (2003), Klibanoff et al. (2005, 2009), Ju and Miao (2012), Hansen and Sargent (2001a,b), Strzalecki (2011), Hansen and Sargent (2015) and others. Applications to macroeconomic models include Cagetti et al. (2002) and Bidder and Smith (2012). For a survey of applications in finance, see Epstein and Schneider (2010). Perhaps the closest to our paper is the work by Ilut and Schneider (2014) and Bianchi et al. (2017), who utilize the recursive multiple-prior preferences of Epstein and Schneider (2003). Both these papers use the cross-sectional variation in professional forecasts as a proxy for confidence and study real business cycle models. In contrast, we focus on a model with frictional goods and labor markets that speak to the systematic biases in unemployment and inflation that we extensively document using the survey data. We provide a more detailed discussion of how our modeling approach differs from Epstein and Schneider (2003) and other approaches in Section 6.

The paper is organized as follows. Section 2 describes key empirical findings from the survey data. Motivated by these findings, we introduce our extension of the robust preference framework in Section 3, link the implications of the theory to the belief biases in survey data, and develop a tractable solution technique for approximating the equilibrium dynamics. Section 4 is devoted to the construction and calibration of the structural business cycle model that embeds robust preferences. In Section 5, we discuss implications of the findings and the role of subjective beliefs in business cycle dynamics. Section 6 distinguishes our model of time-varying concerns for model misspecification from related notions of time-varying uncertainty, pessimism and discount rates, as well as alternative explanations for the belief biases. Section 7 concludes. The appendix contains detailed derivations of the approximation method, description of the data, and further results and robustness checks.

2 Survey expectations

We start by analyzing data on households' expectations from the University of Michigan Surveys of Consumers (Michigan Survey). This survey collects answers to questions about the households' own economic situation as well as their forecasts about the future state of the economy. We focus on the forecasts of future inflation and unemployment, studying their behavior over time, as well as the cross-sectional patterns across individual households. As a consistency check, we validate this evidence using data from the Survey of Consumer Expectations collected by the Federal Reserve Bank of New York (FRBNY Survey). A detailed description of the construction of the data is provided in Appendix C.

We are interested in deviations of these survey answers from rational expectations forecasts. We will refer to these deviations as *belief wedges*. The construction of belief wedges requires taking a stand on how to determine the probability measure that generates the data. To this end, we use a forecasting vector-autoregression (VAR), described in Appendix C.3. As a robustness check, we also document patterns for the belief wedges constructed using responses in Survey of Professional Forecasters (SPF) as the rational forecast.[2]

[2]While some studies report modest biases in SPF forecasts, these biases are an order of magni-

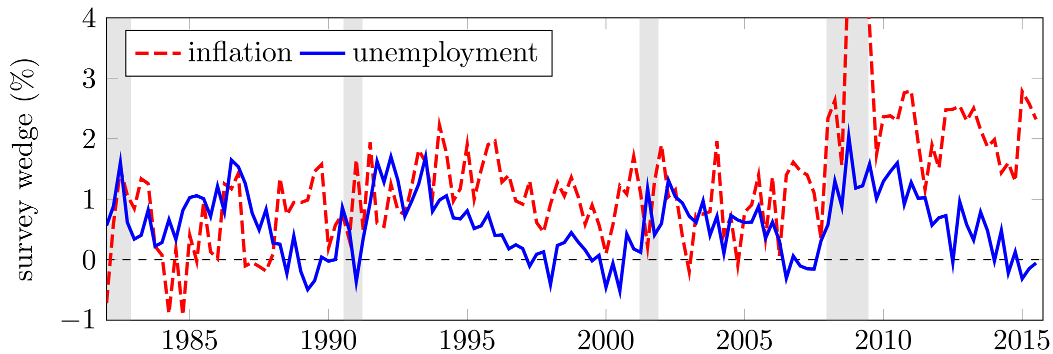

Figure 1: Difference between the mean one-year ahead forecasts from the Michigan Survey and corresponding statistical VAR forecasts. Details on the construction of the data series are in Appendix C. NBER recessions shaded.

Figure 1 shows the differences between the Michigan Survey average household expectations and the VAR forecasts for inflation and unemployment. The survey expectations are mean one-year ahead expectations in the survey samples, constructed using quarterly data for the period 1982Q1–2015Q4. The unemployment rate survey forecast is inferred from categorical answers by fitting a time series of parameteric distributions using the procedure from Carlson and Parkin (1975) and Mankiw et al. (2003). In Appendix C, we also report results for alternative definitions of the belief wedges based on average and median inflation forecasts, and an extended sample starting from 1960Q1.

The belief wedges in Figure 1 are large on average, vary over time and have a strong common component that is correlated with the business cycle. The average inflation and unemployment wedges over the sample period are 1.25% and 0.58% respectively. The wedges are also volatile, with standard deviations of 1.03% and 0.54% for inflation and unemployment respectively. Finally, the wedges consistently increase during the shaded NBER recessions. This means that households not only overestimate unemployment and inflation relative to the VAR forecast, but these biases are larger when measures of business activity are low. The correlations of the inflation and unemployment wedges with the output gap are −0.37 and −0.54 respectively.

These patterns are robust to several alternative ways of measuring the wedges.

tude smaller than those we find in household surveys, and not robust to the chosen time period. See, e.g., Elliott et al. (2008) and Capistrán and Timmermann (2009), who rationalize these biases by assuming forecasters have asymmetric loss functions.

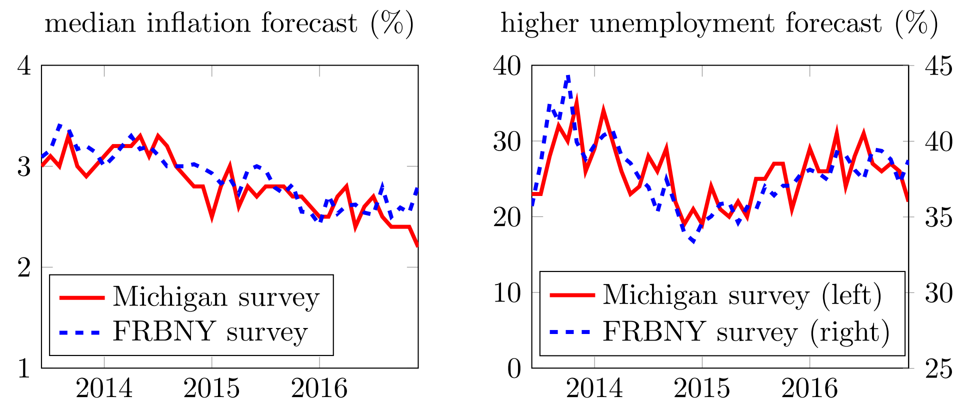

Figure 2: *Left panel:* Median inflation forecasts in the Michigan Survey and the Federal Reserve Bank of New York Survey of Consumer Expectations. *Right panel:* Share of respondents in the Michigan Survey stating that unemployment will be higher during the next 12 months, and the mean probability that unemployment will be higher one year from now in the FRBNY Survey. Details on the construction of the data series are in Appendix C.

A particularly insightful check is a comparison of our results from the Michigan Survey with the FRBNY Survey. The FRBNY survey contains a richer set of questions but only began in 2013. The left panel of Figure 2 shows that the median inflation forecasts from both surveys are very well aligned.[3] Since the two surveys do not ask the same questions about unemployment, the right panel shows two different unemployment forecast statistics. We report the mean probability that unemployment will be higher one year from now from the FRBNY Survey, and the share of respondents who predict that unemployment will be higher in the next 12 months from the Michigan Survey. While the levels of these statistics are not directly comparable, the statistics visibly comove over time. In Appendix C.4 we report the descriptive statistics using other ways of measuring the wedges, such as extending the data to a longer sample, using monthly survey data instead of quarterly data, taking forecasts from the Survey of Professional Forecasters (SPF) as the measure of unbiased estimates, and using the median response across households.

The large magnitude of the inflation wedge in household survey expectations is also consistent with the findings of Coibion and Gorodnichenko (2015) for the

[3]This comparison also provides another robustness check. The Michigan Survey forecast is constructed by aggregating point forecasts of individual households, which we interpret as the mean forecast under the subjective distribution in the quantitative model. The FRBNY survey aggregates mean forecasts from forecast.

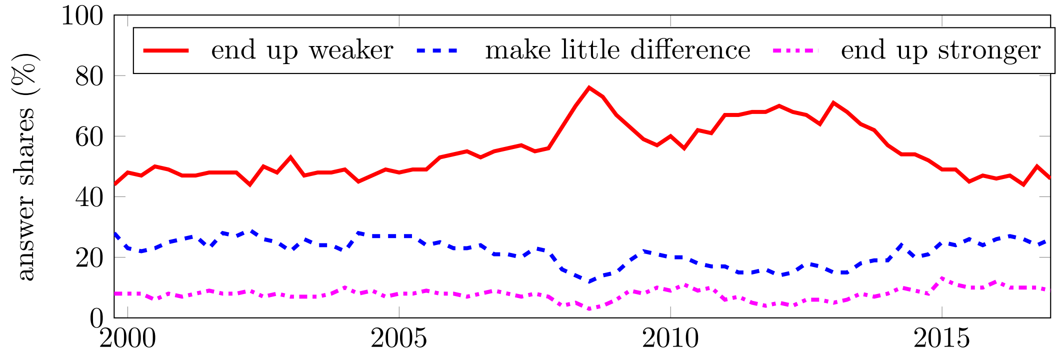

Figure 3: Bank of England Inflation Attitudes Survey, shares of answers to the question: "*If prices started to rise faster than they do now, do you think Britain's economy would ...*" Data sample 1999Q4–2017Q1.

U.S., as well as with international evidence. For example, Coibion et al. (2015) find large positive inflation biases in household and firm surveys in New Zealand, while Vellekoop and Wiederholt (2017) document large and persistent positive biases in a long panel survey of households in the Netherlands.

The Bank of England administers a quarterly Inflation Attitudes Survey in which households are asked, among other questions, what would the impact of an increase in inflation be on the U.K. economy. Figure 3 shows that over the sample, between 50 and 80 percent of households responded that an increase in inflation would weaken the economy. Moreover, this fear of an adverse impact of higher inflation is highest during the Great Recession, and the correlation of this share of households with U.K. GDP growth is -0.51. The household median inflation forecast averaged over the 1999Q4–2017Q1 sample is 2.71% while realized inflation rate over this period averaged to 2.05%. Therefore, U.K. households significantly overpredict inflation, associate high inflation with adverse economic outcomes, and tend have larger biases during recessions. These patterns are consistent with our findings for U.S. households.

2.1 Cross-sectional evidence

In addition to the time series, we also use household-level data to provide evidence that there is a positive cross-sectional correlation between the unemployment and inflation belief wedges and a strong comovement across time for several disaggregated demographic groups.

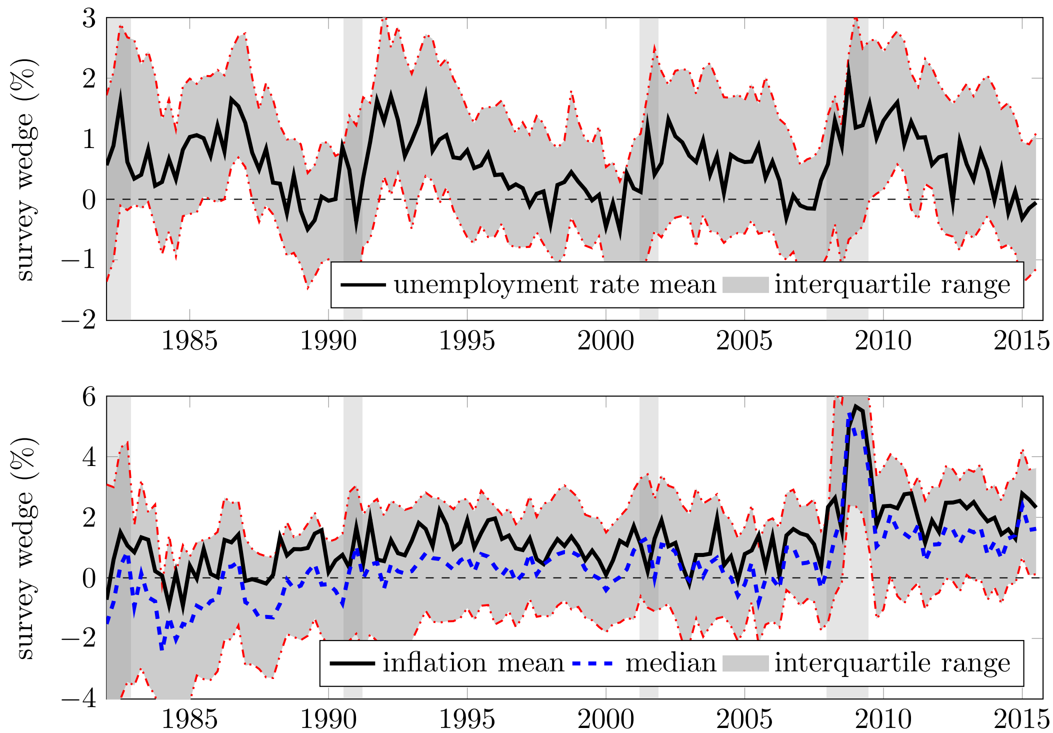

Figure 4: Dispersion in survey expectations in the Michigan Survey. The graphs show different quantiles of the distribution of responses in the Michigan survey, net of the mean VAR forecast. The top panel shows the unemployment responses, bottom panel the inflation responses. Details on the construction of the data series are in Appendix C. NBER recessions shaded.

We begin by plotting the dispersion of the data from the Michigan survey for the unemployment rate and inflation rate forecasts in Figure 4. For the inflation data, we have information on the quantiles of the cross-sectional distribution. For the unemployment rate forecast, we use the inferred distributions from categorical answers. There is indeed substantial cross-sectional dispersion in the survey answers across individual households, but the interquartile range appears to be stable over time (except for the inflation answers from early 1980s). The correlation between the mean and median inflation forecast is 0.94.

More importantly, the cross-sectional dispersion of the belief wedges exhibits systematic patterns across demographic groups and individual households—households with more upward biased inflation forecasts also exhibit larger positive biases in unemployment forecasts. Figure 5 displays evidence at the level of demographic

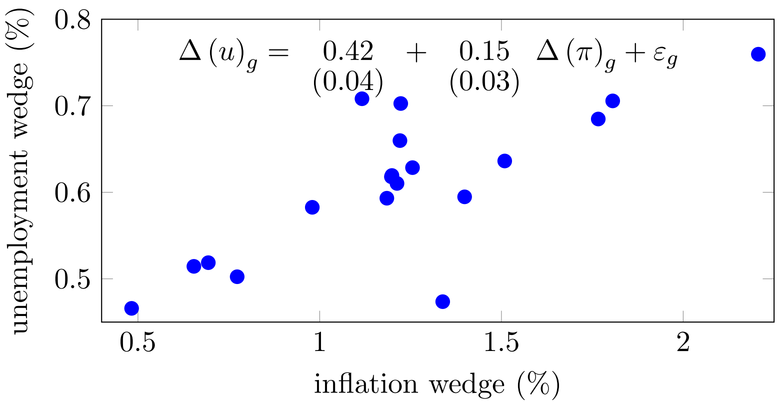

Figure 5: Relationship between the average inflation wedge $\Delta(\pi)_g$ of demographic group g and the corresponding average unemployment wedge $\Delta(u)_g$ in the Michigan Survey. Demographic groups are listed in Table 5 in Appendix C. Standard errors of regression coefficients in parentheses.

groups reported in the Michigan Survey for average wages over the examined period 1980–2015.[4] Demographic groups with larger average inflation wedges are also characterized by larger unemployment wedges. Consistent with existing evidence, households with lower reported education and lower reported income levels make more biased forecasts, but these biases remain nontrivial even for highly education and high-income households. It should be stressed that these relative cross-sectional patterns are independent of the construction of the underlying forecast under the data-generating process.

To show that this cross-sectional relationship is stable over time and holds at the level of individual households, we run, for each month t in the sample, the household-level regressions

$$\widetilde{u}_{i,t} = \alpha_t + \beta_t \widetilde{\pi}_{i,t} + \varepsilon_{i,t}, \tag{1}$$

where $\widetilde{u}_{i,t}$ and $\widetilde{\pi}_{i,t}$ are the unemployment and inflation forecasts, respectively, for household i in month t. Almost all the estimated slope coefficients $\widehat{\beta}_t$ are positive, 65% of them are larger than two standard errors, and the average t-statistic on $\widehat{\beta}_t$ is 2.48.[5] Hence the significant positive cross-sectional relationship between the

[4]This demographic classification includes alternative age groups, geographical regions, quartiles of the income distribution, men and women, and different levels of education. Table 5 in Appendix C provides additional details.

[5]Figure 13 in Appendix C.5 provides the smoothed time series of the t-statistics associated with

inflation and unemployment wedges is persistent over time and not driven by a particular subperiod in the data. We present additional cross-sectional evidence at various levels of disaggregation and controlling for demographic composition in Appendix C.5.

Finally, we also corroborate the cross-sectional patterns against those in the FRBNY survey. The richer set of questions in the FRBNY Survey to show that households who forecast higher inflation also assign a higher chance to a rise in the unemployment rate, a lower chance to an increase in the stock market, and have systematically more pessimistic beliefs about a range of questions concerning their own future situation. The details are provided in Table 9 of Appendix C.5.

Overall, the time series and cross-sectional evidence from the Michigan Survey, together with the alternative data sources, paints a clear picture—households on average expect higher unemployment and higher inflation relative to rational expectations and these biases are larger in recessions. In the next section, we propose a parsimonious theory of endogenous subjective beliefs that is sufficiently rich to generate cross-equation restrictions consistent with these empirical patterns.

3 Framework for subjective beliefs

Motivated by the empirical results from Section 2, we now introduce a decision-theoretic formulation that generates deviations of agents' subjective beliefs from the data-generating probability measure. Our formulation extends the robust preference framework of Hansen and Sargent (2001a,b) where concerns about the specification of the underlying model of the economy lead the agents to investigate a set of models that are close in terms of their statistical plausibility. Optimal decisions and subjective beliefs that rationalize them are pinned down by the desire of the household to bound utility losses from potential model misspecification.

Relative to Hansen and Sargent (2001a,b) we investigate a setting that allows for time variation in the degree of agents' misspecification concerns. We identify this time variation from survey data, while tightly restricting the structure of belief distortions across individual states, linking them to agents' continuation values and equilibrium dynamics. We develop an approximation technique that incorporates the effects of time-varying belief distortions in a tractable linear solution. Finally we explain how to exploit the restrictions imposed by our framework when we confront

estimates of β_t in the estimation in equation (1).

the model with survey data on agents' expectations.

3.1 Robust preferences

Agents' preferences are represented using a concave period utility $u(\cdot)$ and the continuation value recursion

$$V_t = \min_{\substack{m_{t+1}>0 \\ E_t[m_{t+1}]=1}} u(x_t) + \beta E_t[m_{t+1}V_{t+1}] + \frac{\beta}{\theta_t} E_t[m_{t+1}\log m_{t+1}] \tag{2}$$

$$\theta_t = \overline{\theta} x_t, \tag{3}$$

$$x_{t+1} = \psi(x_t, w_{t+1}). \tag{4}$$

Here, x_t is an $n \times 1$ vector of stationary economic variables that follows the Markovian law of motion (4), $\overline{\theta}$ is a $1 \times n$ vector of parameters and $w_{t+1} \sim N(0_k, I_{k\times k})$ is an iid vector of normally distributed shocks under the data-generating probability measure P. We take the function ψ as given for now, but later derive it as a solution to a set of equilibrium conditions. In Section 4, we also endow the agent with a set of controls, which gives rise to a min–max specification of the recursion.

The agent treats the measure P as an approximating or benchmark model and considers potential stochastic deviations from this model, represented by the strictly positive, mean-one random variable m_{t+1}. The minimization problem in (2) captures the search for a 'worst-case' model that serves as a basis for the agent's decisions. The agent considers models that are difficult to distinguish statistically from the benchmark model, and the degree of statistical similarity is controlled by the entropy penalty $E_t[m_{t+1}\log m_{t+1}]$, scaled by the penalty parameter θ_t. More pronounced statistical deviations that are easier to detect are represented by random variables m_{t+1} with a large dispersion that yields a large entropy. As $\bar{\theta} \to 0$, the penalty for deviating from the benchmark model becomes more severe, and the resulting preferences approach a utility-maximizing agent with rational expectations. The chained sequence of random variables m_{t+1} specifies a strictly positive martingale M recursively as $M_{t+1} = m_{t+1}M_t$ with $M_0 = 1$ that defines a probability measure $\widetilde{P}$ with conditional expectations

$$\widetilde{E}_t[x_{t+1}] \doteq E_t[m_{t+1}x_{t+1}].$$

A useful feature of this formulation is the *ex-post Bayesian* interpretation of the agents' behavior. Agents endowed with robust preferences act as subjective expected utility agents with endogenously determined beliefs given by the probability measure $\widetilde{P}$. Since $\widetilde{P}$ rationalizes their actions, we impose the hypothesis that these agents answer survey questions about economic forecasts according to the same $\widetilde{P}$ and relate the belief wedges of Section 2 to the difference between expectations under $\widetilde{P}$ and benchmark P.

This hypothesis is motivated by two observations. First, as we documented in Section 2 and consistent with the large literature on household survey expectations, household survey data on economic forecasts exhibit substantial and persistent biases. Second, subjective beliefs reported in surveys are found to be systematically related to real consumption behavior. Similar to us, Bachmann et al. (2015) and Malmendier and Nagel (2016) use the Michigan Survey to substantiate a significant relationship between survey responses on subjective expectations of economic outcomes and individual consumer spending, borrowing and lending decisions.[6] Other studies on business surveys (Bachmann et al. (2015) and Gennaioli et al. (2015)) show that subjective expectations of managers have predictive power for firm investment and production behavior. Lastly, Crump et al. (2015) exploit the FRBNY Survey of Consumer Expectations to estimate agents' inter-temporal elasticity of substitution using the relationship between subjective inflation expectations and expected spending behavior. All these findings support the rationale for associating the beliefs from survey answers to the subjective model households use in their decision making.

The solution to the minimization problem (2) satisfies

$$m_{t+1} = \frac{\exp\left(-\theta_t V_{t+1}\right)}{E_t\left[\exp\left(-\theta_t V_{t+1}\right)\right]} \tag{5}$$

and m_{t+1} completely characterizes the worst-case model distortions relative to the benchmark model. Substituting the worst-case belief distortion (5) into (2) yields the nonlinear recursion

$$V_t = u\left(x_t\right) - \frac{\beta}{\theta_t} \log E_t\left[\exp\left(-\theta_t V_{t+1}\right)\right]. \tag{6}$$

[6] In a closely related exercise, Ichiue and Nishiguchi (2015) use survey data on households in Japan to link inflation expectations and durable goods spending.

The variation in θ_t thus implies a time-varying model for the worst-case distortion, and $\theta_t = 0$ corresponds to $m_{t+1} \equiv 1$, in which case the one-period ahead conditional distribution of the worst-case model coincides with the benchmark model.

Notice that the distortion (5) implies a large value of m_{t+1} for low realizations of the continuation value V_{t+1}. The subjective model, represented by the probability measure $\widetilde{P}$, thus overweighs adverse states as ranked by the preferences of the agent. In this way, the preference model implies tightly restricted *endogenous pessimism* on the side of the agents, generated by concerns for model misspecification. The degree of pessimism is controlled by the evolution of θ_t but the relative biases across alternative economic variables depend on the properties of $\psi(x_t, w_{t+1})$ and continuation values $V_{t+1} = V(x_{t+1})$, both of which are endogenous objects.[7]

3.2 Solution method

We seek to incorporate the model of endogenous subjective beliefs in a large class of stochastic general equilibrium models. We therefore develop a novel approximation technique for the continuation values and belief distortions that incorporates time-varying effects of model misspecification concerns in a linear approximation of the equilibrium dynamics. The feedback between agents' subjective beliefs and the equilibrium law of motion requires jointly solving for the continuation value recursion (2), the endogenously determined probability measure $\widetilde{P}$, and the law of motion (4).

Our approximation is constructed using a perturbation technique in the spirit of Sims (2002) and Schmitt-Grohé and Uribe (2004). A wide range of dynamic stochastic general equilibrium models with robust agents can be cast as a solution to a system of expectational difference equations

$$0 = \widetilde{E}_t \left[g\left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right) \right], \tag{7}$$

where $g_{t+1} = g(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t)$ is an $n \times 1$ vector function.[8] This vector of

[7] Since θ_t is measurable with respect to the agent's information set at time t, the preferences are dynamically consistent. The linear specification of θ_t in general allows for negative values, in which case the conditional minimization problem in (2) turns into a maximization problem of an 'ambiguity-loving' agent, and the distortion (5) implies optimistic biases in survey responses.

[8] Our solution method, fully described in Appendix B, is able to handle heterogeneous belief distortions for different forward-looking equations of the equilibrium system. We abstract from this heterogeneity in the main text to simplify notation but utilize this flexibility in Section 5 to disentangle the effect of belief distortions on the side of households and firms in our structural model.

equations includes Euler equations of the robust household, which can be represented using subjective beliefs implied by m_{t+1}. Specifically, for i-th equation of the system,

$$0 = \widetilde{E}_t \left[g^i_{t+1} \right] = E_t \left[m_{t+1} g^i_{t+1} \right].$$

We are interested in deriving a tractable approximation of the equilibrium dynamics for x_t in the form of a Markovian law of motion (4) from the system of equations (7).

The challenge stems from the endogeneity of the subjective beliefs that agents use for their decisions. We address this using an approach that builds on the series expansion method used in Holmes (1995), Lombardo (2010) or Borovička and Hansen (2014). Consider a class of models indexed by a perturbation parameter q that approximate the dynamics (4) by scaling the volatility of the innovations w_{t+1}:

$$x_{t+1}(\mathsf{q}) = \psi\left(x_t(\mathsf{q}), \mathsf{q} w_{t+1}, \mathsf{q}\right). \tag{8}$$

Hence with each q, there is an associated state vector process $x_t(\mathsf{q})$ given by the law of motion (8), and $\mathsf{q} = 1$ recovers the original dynamics (4). The dynamics of $x_t(\mathsf{q})$ are approximated by constructing a first-order series expansion

$$x_t(\mathsf{q}) \approx \bar{x} + \mathsf{q} x_{1t}, \tag{9}$$

where the 'first-derivative' process x_{1t} represents the local dynamics in the neighborhood of the steady state $\bar{x}$ and does not depend on q. The steady state $\bar{x}$ is the solution to (8) evaluated at $\mathsf{q} = 0$, given implicitly by $\bar{x} = \psi(\bar{x}, 0, 0)$. Assuming that the function $\psi(x, w, \mathsf{q})$ is sufficiently smooth, we obtain the dynamics of x_{1t} by differentiating (8) with respect to q, utilizing (9) and evaluating at $\mathsf{q} = 0$:

$$x_{1t+1} = \psi_q + \psi_x x_{1t} + \psi_w w_{t+1}, \tag{10}$$

where ψ_q, ψ_x and ψ_w are conforming coefficient matrices representing the corresponding partial derivatives of $\psi(x, w, \mathsf{q})$ evaluated at the steady state, e.g., $\psi_x \doteq \frac{\partial}{\partial x} \psi(x, w, \mathsf{q})\big|_{(\bar{x},0,0)}$.

The key innovation in our approach is the approximation of the penalty parameter in the continuation value recursion (6) and the associated belief distortion (5).

The perturbed continuation value recursion is given by

$$V_t(\mathsf{q}) = u(x_t(\mathsf{q}), \mathsf{q}) - \frac{\beta}{\theta_t(\mathsf{q})} \log E_t [\exp(-\theta_t(\mathsf{q}) V_{t+1}(\mathsf{q}))] . \tag{11}$$

The usual expansion of the perturbation parameter would lead to the following first-order approximation of the exponent in (11) and in the numerator of (5):

$$-\theta_t(\mathsf{q}) V_{t+1}(\mathsf{q}) \approx -\bar{\theta}(\bar{x} + \mathsf{q} x_{1t}) \left(\bar{V} + \mathsf{q} V_{1t+1}\right) \approx -\bar{\theta}\left(\bar{x} + \mathsf{q}\left(x_{1t}\bar{V} + \bar{x} V_{1t+1}\right)\right).$$

The scaling of the stochastic term by q indicates that as $\mathsf{q} \to 0$, the belief distortion in the perturbed model vanishes. In other words, in an economy that approaches its deterministic counterpart, agents do not need to fear that their perceived model of the world is misspecified. Consequently, the usual first-order approximation of (11) will not be affected by θ_t, a standard result due to the smoothness of the certainty-equivalent transformation $\log E_t [\exp(\cdot)]$.

Instead, we propose to use the perturbation

$$\theta_t(\mathsf{q}) = \overline{\theta} x_t(\mathsf{q}) \approx \frac{\overline{\theta}(\bar{x} + x_{1t})}{\mathsf{q}}. \tag{12}$$

Differentiating (11) with respect to q then yields a recursion for the first-derivative process V_{1t}

$$V_{1t} = u_x x_{1t} + u_q - \beta \frac{1}{\overline{\theta}(\bar{x} + x_{1t})} \log E_t \left[\exp\left(-\overline{\theta}(\bar{x} + x_{1t}) V_{1t+1}\right)\right]. \tag{13}$$

This recursion is the first-order approximation of (6) and the nonlinearity stems from the perturbation choice (12). Using the guess

$$V_{1t} = V_x x_{1t} + V_q,$$

recursion (13) yields a pair of equations for coefficients V_x and V_q. The equation for V_x is a Riccati equation whose solution can be found iteratively, see Appendix B.2. As a result, the zero-th order approximation of the belief distortion (5), i.e., the evaluation of the expansion of (5) at $\mathsf{q} = 0$, takes the form

$$m_{0t+1} = \frac{\exp\left(-\overline{\theta}(\bar{x} + x_{1t}) V_x \psi_w w_{t+1}\right)}{E_t\left[\exp\left(-\overline{\theta}(\bar{x} + x_{1t}) V_x \psi_w w_{t+1}\right)\right]}. \tag{14}$$

This expression reveals the effect of the perturbation choice (12). The volatility of the shocks $\mathsf{q}w_{t+1}$ in the perturbed economy (8) vanishes with $\mathsf{q} \to 0$ but, at the same time, agents' concerns about model misspecification (12) increase. These two effects offset each other such that in the economy that approaches its deterministic limit, agents' subjective model remains nontrivially distinct from the data-generating process.

When we approximate agents' subjective belief model $\widetilde{P}$ using the zero-th order term of the belief distortion (14), the vector of normally distributed innovations w_{t+1} in (4) under $\widetilde{P}$ has the distribution

$$w_{t+1} \sim N\left(-\overline{\theta}\left(\bar{x} + x_{1t}\right)\left(V_x \psi_w\right)', I_{k\times k}\right). \tag{15}$$

Instead of facing a vector of zero-mean shocks w_{t+1}, the agent perceives these shocks under her subjective belief as having a time-varying drift. The time variation is determined by a linear approximation to θ_t from equation (3), given by $\overline{\theta}\left(\bar{x} + x_{1t}\right)$. The relative magnitudes of the distortions of individual shocks are given by the sensitivity of the continuation value to the dynamics of the state vector, V_x, and the loadings of the state vector on individual shocks, ψ_w. An implication of (15) are the dynamics of the model (10) under the agents' subjective beliefs $\widetilde{P}$ that satisfy

$$\begin{aligned} x_{1t+1} &= \left[\psi_q - \psi_w \psi_w' V_x' \overline{\theta} \bar{x}\right] + \left[\psi_x - \psi_w \psi_w' V_x' \overline{\theta}\right] x_{1t} + \psi_w \widetilde{w}_{t+1} \\ &= \widetilde{\psi}_q + \widetilde{\psi}_x x_{1t} + \psi_w \widetilde{w}_{t+1}, \end{aligned} \tag{16}$$

where $\widetilde{w}_{t+1} \sim N\left(0_k, I_{k\times k}\right)$ is an iid vector of normally distributed shocks under the subjective probability measure $\widetilde{P}$. Detailed steps and formulas for implementing the linear approximation sketched out in this section are relegated to Appendix B.

3.3 Restrictions on subjective beliefs and connection to data

The subjective belief alters both the conditional mean and the persistence of economic shocks. Moreover, variables that tend to move θ_t and the continuation value in opposite directions tend to exhibit a higher persistence under the subjective beliefs.[9] Using the linearized model (10) and its dynamics under the subjective belief

[9] This statement is precisely correct in the scalar case, when $\psi_x^2 V_x \overline{\theta} < 0$.

of the agent given by (16), we obtain the model-implied belief wedges

$$\begin{aligned} \Delta_t^{(1)} &= \widetilde{E}_t\left[x_{t+1}\right] - E_t\left[x_{t+1}\right] = \psi_w \widetilde{E}_t\left[w_{t+1}\right] \\ &= -\overline{\theta}\left(\bar{x} + x_{1t}\right)\left(\psi_w \psi_w'\right) V_x'. \end{aligned} \tag{17}$$

Longer-horizon forecasts $\Delta_t^{(j)} = \widetilde{E}_t\left[x_{t+j}\right] - E_t\left[x_{t+j}\right]$ can be constructed correspondingly by iterating on the subjective dynamics (16). These formulas are provided in Appendix B.3.

The model thus predicts a one-factor structure in the dynamics of the belief wedges measured using the survey data. The relative distortions of survey answers on individual macroeconomic variables are given by the vector of constant loadings $-\left(\psi_w \psi_w'\right) V_x'$, while the factor that measures the overall magnitude of the belief distortions, $\theta_t \approx \overline{\theta}\left(\bar{x} + x_{1t}\right)$, varies over time. This one-factor structure is the key restriction that the robust preference model imposes on the joint dynamics of the survey answers and implies that the magnitudes of the belief wedges should comove over time, which is consistent with the evidence in Section 2.

The vector of loadings, $-\left(\psi_w \psi_w'\right) V_x'$, is the negative of the covariance of the innovations to the value function, $V_x \psi_w$, with innovations to the state variables, ψ_w. This represents the exact notion of pessimism in the model of robust preferences—households negatively distort forecasts about variables that comove with their value function, i.e. variables that reflect economic risks the household is facing. On the other hand, variables that do not comove with households' continuation values will not be distorted under the households' subjective beliefs.

Hence, the decision-theoretical model yields a set of testable economic restrictions, both for the cross-section and time-series properties of survey answers about various variables. We use these restrictions to compare the predictions from our structural macroeconomic model with data.

4 A structural business cycle model

In this section, we introduce the subjective beliefs framework from Section 3 into a calibrated version of a New-Keynesian model with a frictional labor market. In the absence of belief distortions, our environment is similar to Ravenna and Walsh (2008), Gertler et al. (2008) and Christiano et al. (2015). The search and matching environment and nominal rigidities provide well-defined notions of unemployment and inflation, which directly map to the survey questions. We use this model to interpret the patterns in survey data and investigate the economic channels through which subjective beliefs affect the macroeconomy.

4.1 Model

The model economy is populated by a representative household with robust preferences of the form (2), competitive producers of a homogeneous final good, and a two-tier structure of monopolistic producers of intermediate goods who hire workers in a frictional labor market. Here, we focus on key components of the model relevant for the analysis of the role of subjective beliefs—households' preferences, contracting in the labor market, and the specification of exogenous sources of variation in the model.

4.1.1 Representative household

The preferences of the representative household are given by the recursion

$$V_t = \min_{\substack{m_{t+1}>0 \\ E_t[m_{t+1}]=1}} \max_{C_t, I_t, B_{t+1}} \log\left(C_t\right) + \beta E_t\left[m_{t+1}V_{t+1}\right] + \frac{\beta}{\theta_t} E_t\left[m_{t+1}\log m_{t+1}\right], \tag{18}$$

with time preference coefficient β and an AR(1) process for θ_t

$$\theta_t = (1-\rho_\theta)\bar{\theta} + \rho_\theta \theta_{t-1} + \sigma_\theta w_t^\theta. \tag{19}$$

The magnitude of the belief distortion is determined by fluctuations in θ_t specified exogenously in (19). However, the equilibrium dynamics in the model endogenously determine the states that yield low continuation values V_{t+1}. These states are evaluated as adverse by the household, and are then perceived as more likely under the subjective model. Naturally, the dynamics of the subjective beliefs then endogenously depend on the structure of other shocks in the model, which we describe in Section 4.1.4.

The household consists of a unit mass of workers who perfectly share consumption risk. A fraction L_t is employed and earns a real wage ξ_t. A fraction $1 - L_t$ is unemployed and collects unemployment benefits with real value D financed through lump sum taxes. The household faces the nominal budget constraint

$$P_tC_t + B_{t+1} \leq (1 - L_t)\, P_tD + L_tP_t\xi_t + R_{t-1}B_t - T_t,$$

where P_t is the price of consumption goods, B_{t+1} denotes the one-period risk-free bonds purchased in period t with return R_t, and T_t are lump sum taxes net of profits.

4.1.2 Labor market

At the end of period t, employed workers separate with probability $1-\rho$ and join the pool of unemployed, who search for jobs at the beginning of period $t+1$. The total number of searchers at the beginning of period $t+1$ therefore is $1-\rho L_t$. Searchers then endogenously choose search effort e_{t+1} that affects their job finding probability f_{t+1}. The law of motion for the mass of employed workers thus is

$$L_{t+1} = \rho L_t + (1 - \rho L_t)\, f_{t+1} = (\rho + h_{t+1})\, L_t,$$

where

$$h_{t+1} = \frac{f_{t+1}\,(1 - \rho L_t)}{L_t}$$

is the hiring rate. Measured unemployment u_t is given by

$$u_t = 1 - L_t,$$

which includes people who do not re-join employment after searching at the beginning of the period.

We postulate as in Mukoyama et al. (2017) that the matching function takes the form

$$M\,(e_t, u_t, v_t) = \chi_m \left(\alpha_m e_t^{\psi_m} + (1 - \alpha_m) \left(\frac{v_t}{u_t} \right)^{\psi_m} \right)^{\frac{1}{\psi_m}} u_t,$$

where v_t is the number of vacancies posted by firms. The probability that a worker with search effort e_t finds a job is then given by

$$f_t = \frac{M\,(e_t, u_t, v_t)}{u_t} = \chi_m \left(\alpha_m e_t^{\psi_m} + (1 - \alpha_m)\, \zeta_t^{\psi_m} \right)^{\eta_m} \doteq f\,(e_t, \zeta_t)\,,$$

where $\zeta_t = v_t/u_t$ is the labor market tightness, and the vacancy-filling rate q_t is equal to

$$q_t = \frac{f_t}{\zeta_t}.$$

We now characterize workers' subjective valuations when they are employed, unemployed and searching. Let $s_{t+1} = \beta C_t / C_{t+1}$ denote the marginal rate of substitution between consumption today and consumption tomorrow. First define the present value of real wages conditional on the current job existing:

$$\xi_t^p = \xi_t + \rho \widetilde{E}_t \left[s_{t+1} \xi_{t+1}^p \right],$$

where $\widetilde{E}_t [\cdot]$ represents the expectation under the subjective belief of the household. The value of a job to the worker, J_t^w, consists of the present value of the wages ξ_t^p from the existing job defined above, plus the present value of benefits accrued when losing a job, A_t:

$$J_t^w = \xi_t^p + A_t.$$

The recursion for the unemployment value after search was attempted in period t if worker remains unemployed is

$$U_t = D + \widetilde{E}_t \left[s_{t+1} U_{t+1}^S \right],$$

where U_{t+1}^S is the value of the unemployed searcher at the beginning of period $t+1$. Search effort e_t incurs a cost $c(e_t)$ and is chosen as the solution to

$$U_t^S = \max_{e_t} -c(e_t) + \left[f(e_t, \zeta_t) J_t^w + (1 - f(e_t, \zeta_t)) U_t \right].$$

The searcher takes the stochastic discount factor s_{t+1} and the market tightness as given, which makes him a competitive agent in the labor market. The outside benefits A_t of being on a job follow the recursion

$$A_t = (1 - \rho) \widetilde{E}_t \left[s_{t+1} U_{t+1}^S \right] + \rho \widetilde{E}_t \left[s_{t+1} A_{t+1} \right].$$

The total value of the match between the firm and the worker is given by the present value of the worker's marginal product ϑ_t on the current job, obtained as a solution to

$$\vartheta_t^p = \vartheta_t + \rho \widetilde{E}_t \left[s_{t+1} \vartheta_{t+1}^p \right].$$

Consequently, the present value of the worker to the firm is

$$J_t = \vartheta_t^p - \xi_t^p.$$

To close the labor market we specify the free-entry condition and the wage setting protocol. Let κ_v be the flow cost of posting a vacancy and κ_h be the flow hiring cost incurred by the firm after the match is formed. The zero-profit condition for entering firms implies

$$J_t = \frac{\kappa_v}{q_t} + \kappa_h.$$

We follow Shimer (2008) and use standard Nash bargaining to split the match surplus using the rule

$$J_t = \frac{1-\eta}{\eta}\left(J_t^w - U_t\right),$$

where η is the bargaining power of the worker.

An important feature of the frictional labor market is the forward-looking nature of search and vacancy-posting decisions. When evaluating the distribution of future states, workers inherit the beliefs of the representative household. Similarly, firms maximize profits using equilibrium state prices obtained from households' preferences and beliefs. This implies that fluctuations in θ_t directly affect the incentives of firms to post vacancies and the search decision of the workers, through their effect on the valuation of the match surplus. This is a striking difference relative to the Walrasian spot market where workers are hired using only one-period employment contracts. In such an environment, model misspecification concerns are absent from the labor market decisions, since there is no uncertainty about economic conditions in the current period.

4.1.3 Production and market clearing

The frictional labor market is embedded in a New-Keynesian framework with Calvo (1983) price setting. A homogeneous final good Y_t with price P_t is produced in a competitive market using the production technology

$$Y_t = \left[\int_0^1 (Y_{i,t})^{\frac{1}{\lambda}}\, di\right]^{\lambda}, \qquad \lambda > 1$$

where $Y_{i,t}$ are specialized inputs with prices $P_{i,t}$. Final good producers solve the static competitive problem

$$\max_{Y_{i,t}} P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di,$$

leading to the first-order conditions

$$Y_{i,t} = \left(\frac{P_t}{P_{i,t}} \right)^{\frac{\lambda}{\lambda-1}} Y_t, \qquad i \in [0,1] \, .$$

Specialized inputs are produced by monopolist retailers indexed by i, using the production technology

$$Y_{i,t} = A_t l_{i,t} - \phi,$$

where $l_{i,t}$ is the quantity of labor services used in the production, A_t is the neutral technology level, and ϕ is a fixed cost of production. The retailer purchases labor services from a competitive sector that aggregates labor supply in the frictional labor market described in Section 4.1.2. The retailer is subject to the Calvo style price frictions and allowed to reset the price with probability $1 - \chi$. These infrequent adjustments imply that price setting is a dynamic problem affected by the belief distortions on the side of the firm.

The model is closed with an aggregate resource constraint

$$C_t + \left(\frac{\kappa_v}{q_t} + \kappa_h \right) h_t L_{t-1} = Y_t$$

and the market clearing condition for labor services

$$\int_0^1 l_{i,t} di = L_t.$$

4.1.4 Shock structure and monetary policy

We complete the model by specifying the sources of exogenous variation to the economy. The monetary authority follows the interest rate policy rule

$$\log \left(R_t / \overline{R} \right) = r_\pi \log \left(\pi_t / \overline{\pi} \right) + r_y \log \left(Y_t / Y^* \right) + \sigma_R w_t^R,$$

where w_t^R is an iid monetary policy shock and Y^* is the steady state value of Y_t. The neutral technology process A_t is specified as

$$\log A_{t+1} = \rho_a \log A_t + \sigma_a w_{t+1}^a$$

and the final source of exogenous variation is the shock process (19) that drives the agents' misspecification concerns. We assume that all innovations are independent under the data-generating measure P:

$$\left(w_t^R, w_t^A, w_t^\theta \right)' \overset{iid}{\sim} N\left(0, I\right).$$

As we have seen in Section 3, this property does not carry over to the subjective model where the joint distribution of future realizations of the innovations depends on the current level of θ_t.

4.2 Model solution and calibration

The equilibrium of the structural model sketched out in the previous section fits in the general framework that we developed in Section 3. We use the expansion methods from Section 3.2 to compute a linear approximation to the solution for the equilibrium dynamics.

We divide the parameters into 4 subsets: (a) preferences and goods market, (b) labor market, (c) monetary policy and (d) processes for exogenous shocks. We exploit the steady state relationships to calibrate several of the parameters in the goods and labor markets, follow the literature to pin down the monetary policy rule, and use the patterns in the time series of macro data and survey data that we emphasized in Section 2 to restrict the shock processes.

The subjective discount factor β is set to target a steady state real return of 2% per year. The parameters governing nominal frictions $\{\chi, \lambda\}$ are calibrated to match a markup of 20% and a frequency of price changes that corresponds to three quarters. For the monetary policy we use standard values for the inflation and output loadings of 2.0 and 0.125, respectively. We set the quarterly standard deviation of the monetary policy shock σ_R to be 0.15%.

For the labor market we have 9 parameters: the separation probability ρ, matching function parameters $\{\chi_m, \alpha_m, \psi_m\}$, cost of posting a vacancy κ_v, hiring cost κ_h, the Nash bargaining parameter η and flow value of unemployment D. Fol-

Parameters		Value
β	Discount factor	0.99
χ	Calvo price stickiness	0.75
λ	Price markup	1.20
r_π	Monetary policy rule: loading on inflation	2.00
r_y	Monetary policy rule: loading on output	0.13
g	Government consumption to output	0.20
	Labor market	
ψ_m	Curvature of matching function	1.17
α_m	Weight on search effort	0.16
χ_m	Matching efficiency	0.67
ρ	Job survival probability	0.90
$100c_0$	Search costs	3.71
$100\kappa_v$	Vacancy posting costs	0.80
$100\kappa_h$	Hiring costs	10.71
D	Flow benefits of unemployment	0.61
η	Worker's bargaining weight	0.75
	Shocks	
$\overline{\theta}$	Mean uncertainty	4.21
ρ_θ	Persistence of uncertainty	0.70
σ_θ	Volatility of uncertainty shock	3.29
ρ_a	Persistence of TFP shock	0.81
$100\sigma_a$	Volatility of TFP shock	0.83
$100\sigma_R$	Volatility of monetary policy shock	0.15

Table 1: Baseline parameter values. Model is calibrated at a quarterly frequency.

lowing Mukoyama et al. (2017), we impose a quadratic cost of search function $c(e) = -\frac{1}{2}c_0e^2$. Our framework provides two normalizations, the steady state market tightness $\zeta = 1$ and steady state search effort $e = 1$. We pick a separation rate of 10%, which pins down the job survival parameter $\rho = 0.9$. We impose a steady-state job-finding rate $f = 0.67$ which implies a steady state rate of unemployment of 5.1%.

We then target the flow value of unemployment to be 70% of wages for unemployed workers, which leads to $D = 0.61$. There is large uncertainty in the macro-labor literature about the flow value of unemployment and our target lies well within

Moment	**Data**	**Model**
Volatility of inflation wedge	1.03	0.62
Volatility of unemployment wedge	0.45	0.43
Mean of inflation wedge	1.25	0.77
Mean of unemployment wedge	0.54	0.54
Volatility of inflation	1.40	2.23
Volatility of output	2.33	2.38
Volatility of unemployment	1.65	0.90

Table 2: Moments targeted by the choice of the parameters for the shock processes $\left\{\bar{\theta}, \sigma_\theta, \rho_a, \sigma_a\right\}$. The sample period for the Data column is 1982Q1–2015Q4. Values in both columns are in percent and annualized.

the literature's broad range of 40%–95%.[10] We target a total cost of recruiting a worker at 15% of quarterly wages following Silva and Toledo (2009), who report estimates from 10% to 20%, and additionally require 90% of the total recruitment cost to be a fixed component representing post-match hiring costs.[11] For the remaining two parameters, we use the moments from Mukoyama et al. (2017), who document an elasticity of search effort with respect to labor market tightness of 15% and an in-sample standard deviation of job finding rates conditional on observed tightness and search effort of about 30%.

4.3 Pinning down the uncertainty process

An important feature of our setup is the endogeneity of subjective beliefs and the transmission of belief distortions to macroeconomic variables. The magnitudes of the belief wedges relative to each other and relative to the corresponding macro variables are all determined endogenously. This means that even though the process θ_t is exogenous, our model together with observable data on the belief wedges for in-

[10]Shimer (2008) pins the flow value of unemployment down at 40% by assuming that the only benefit for an unemployed worker is government unemployment insurance. Several authors including Rudanko (2011), or Mulligan (2012) reason that value of unemployment activities measures not only unemployment insurance, but also the total value of home production, self-employment, disutility of work, and leisure and use higher values up to 85%. Finally, Hagedorn and Manovskii (2008) argue that in a perfectly competitive labor market, this number should equal the value of employment and use an even higher value of 95%.

[11]The large share attributed to hiring costs is consistent with macro estimates in Christiano et al. (2016) and micro evidence in Yashiv (2000), Silva and Toledo (2009) and Cheremukhin and Restrepo-Echavarria (2014). This component includes training and other costs incurred by the firm after the match is formed.

flation and unemployment imposes tight restrictions on the parameters $\{\bar{\theta}, \rho_\theta, \sigma_\theta\}$. We exploit these cross-equation restrictions by simultaneously targeting means and standard deviations of the belief wedges and the standard deviations of macroeconomic variables.

First, we set the autocorrelation coefficient ρ_θ to 0.70, which matches the autocorrelation of the first principal component extracted from the time series of unemployment and inflation wedges. Our model predicts a one-factor structure of the belief wedges (17), and we treat the first principle component, which explains 76% of the common variation in the wedges, as being driven by fluctuations in the concern for model misspecification.

The remaining four parameters $\{\bar{\theta}, \sigma_\theta, \rho_a, \sigma_a\}$ are set to minimize an equally weighted sum of absolute percentage deviations of 7 moments listed in Table 2: the standard deviations of unemployment, inflation rate and output, the standard deviations of unemployment and inflation wedges, and the average unemployment and inflation wedges over the sample period 1982Q1–2015Q4. In addition, we impose that both the inflation and unemployment belief wedges rise in response to an increase in model uncertainty θ_t, thereby matching the positive comovement of the wedges documented in Section 2. The resulting parameters are reported in Table 1 and the model fit for the 7 moments is reported in Table 2.

In order to shed light on the type of restrictions the model imposes, consider the volatility σ_θ of the uncertainty process θ_t. As we show in Section 5, an increase in model uncertainty in our economy is contractionary, increasing the unemployment rate and simultaneously lowering inflation. Hence in an economy primarily driven by shocks to θ_t, adverse states would be associated with low inflation, and the worst-case model would be characterized by an inflation wedge that is negative on average and negatively correlated with the unemployment wedge. Increasing σ_θ therefore not only increases the volatility of the belief wedges, but also reverses sign of the inflation wedge by increasing the relative contribution of the uncertainty shock to total variability of consumption and welfare. The positive inflation wedge and robust positive relationship between inflation and unemployment forecasts in the data thus intuitively put an upper bound on σ_θ.

Our calibration matches the positive correlation between the unemployment and inflation wedge, and at the same time explains about 60% of the volatility of inflation wedge and a substantial 95% for the unemployment wedge. We somewhat overshoot the volatility of inflation and get a lower than targeted volatility of the inflation wedge. The relative volatility of the inflation wedge is limited by the discipline that the positive relationship between the wedges places on σ_θ in our model, as discussed above. We also inherit the well-known feature that simple frictional search and matching models predict small fluctuations in unemployment over the business cycle. These aspects of the model can be improved by adding features like habit preferences, capital formation, sticky wages or other mechanisms that generate wage inertia along the lines of Christiano et al. (2016). Nevertheless, unemployment volatility in our model is substantially closer to the data than in similar frictional models of the labor market (see Shimer (2010)), which is aided by the role belief fluctuations play in the labor market dynamics. We opt for the simple framework in this paper to keep the core transmission mechanism of beliefs transparent and relatively parsimonious, and leave estimating a richer DSGE model with belief distortions for future work.

5 Understanding the role of subjective beliefs

Using the model laid out above, we now analyze the mechanisms through which fluctuations in belief distortions propagate into the economy. In particular, we want to study the channels that generate the right comovement in belief wedges in response to an increase in model uncertainty, and investigate whether belief fluctuations consistent with those observed in survey data can have a quantitatively meaningful impact on the macroeconomy.

5.1 Belief wedges and responses under subjective model

Figure 6 depicts the impulse responses to the innovation w_t^θ of the model uncertainty process θ_t. A one standard deviation increase in concerns for misspecification leads to a fall of about 0.6% in output, an increase of 0.6 percentage points in the unemployment rate, and a decrease in inflation by 0.5 percentage points on impact. The contractionary effects of an increase in θ_t are about half of the response to a typical productivity shock: a one standard deviation fall in productivity leads output to fall by 1.2%. The bottom row of Figure 6 shows that households also increase their upward bias in inflation and unemployment forecasts, which is consistent with the survey data described in Section 2.

The structural model allows us to explain the economic mechanism underlying the role of the subjective beliefs. An increase in θ_t increases households' concerns about model misspecification and therefore alters their subjective model of the econ-

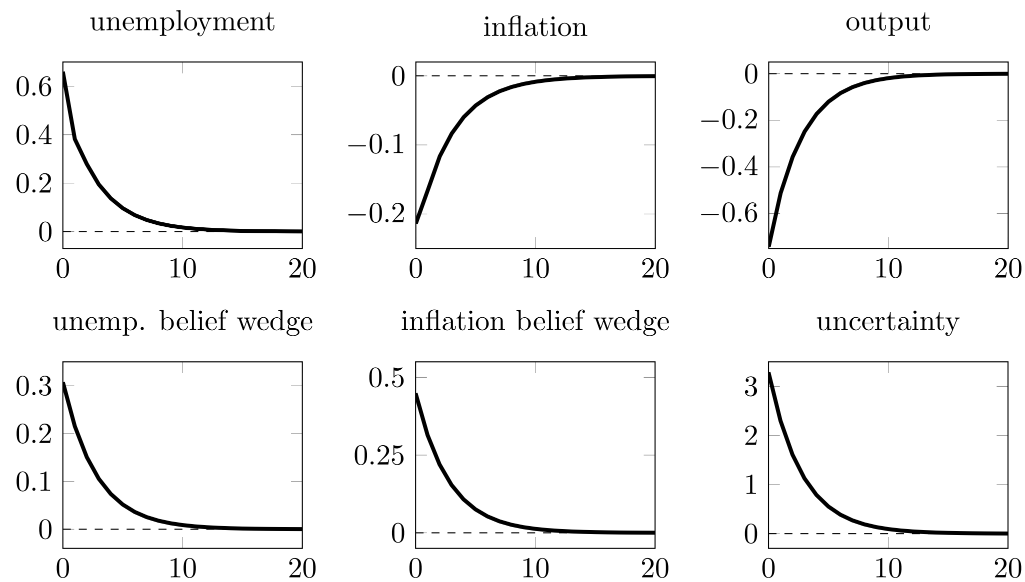

Figure 6: Impulse response functions to the model uncertainty shock w^θ in the structural model. The responses of output and inflation rate are reported in annualized percent, and unemployment rate is in percentage points. Horizontal axis in quarters.

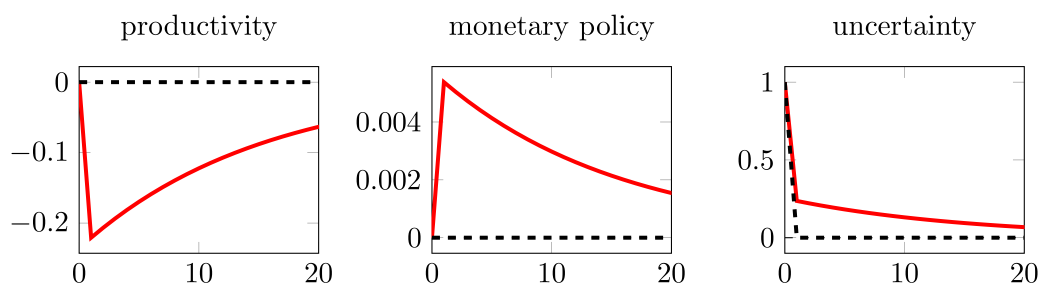

Figure 7: Impulse response functions to the model uncertainty shock w^θ in the structural model under the data-generating measure P (black dashed line) and the worst-case model $\widetilde{P}$ (red solid line). Horizontal axis in quarters.

omy $\widetilde{P}$. In order to understand the impact of the belief distortions, we therefore compare the impulse responses under the data-generating process P to those under the subjective beliefs $\widetilde{P}$. The former are observed by the rational econometrician, while the latter are perceived by the household in the model.

Figure 7 compares both types of responses to the innovation w_t^θ. Under the data-generating measure P, the individual exogenous shocks are uncorrelated, and the technology processes and the monetary policy shock do not respond to w_t^θ. In contrast, the household believes that the shocks are correlated in an adverse way. An increase in θ_t worsens households' expectations about the future path of the technology, and households expect this negative productivity shock to be accompanied by a monetary tightening. Moreover, the household forecasts a further sequence of positive innovations to model uncertainty, hence increasing the subjective persistence of the pessimistic shock.

The equilibrium mapping of shocks to endogenous variables then explains why households forecast higher unemployment, lower output growth, and higher inflation relative to the data-generating process. Households' inflation expectations increase relative to the rational forecast because expectations of lower productivity imply higher marginal costs, which pushes prices upwards through the optimal pricing behavior of firms.

This particular correlation structure arises because these three innovations to the exogenous processes all affect the continuation value V_t. Times of low productivity, exogenous monetary tightening through w_t^R, and high model uncertainty are all bad times, with a low continuation value V_t. The continuation value recursion (18) indicates that these bad times must be generated by low levels of current and future consumption under the households' subjective model. The first panel of Figure 8 confirms this intuition—the household that faces an increase in θ_t forecasts a large and very persistent drop in consumption relative to the data-generating process.

5.2 Economic channels

Figure 8 also shows that the increase in pessimism has a particularly pronounced contractionary effect on the labor market dynamics. The economic channels that underlie this transmission of subjective beliefs to macroeconomic outcomes can be uncovered by separating the role of beliefs of the key economic actors: consumers, workers and firms. Subjective beliefs affects all forward-looking decisions: the consumption-saving decision represented by the consumer's Euler equation, the dynamic pricing behavior of intermediate goods producers that determines the New-Keynesian Phillips curve, and the search effort of the worker and vacancy posting decisions of the firms in the labor market, driven by valuation of workers' and firms' surpluses from created matches.

To analyze how the different economic actors influence the role of subjective

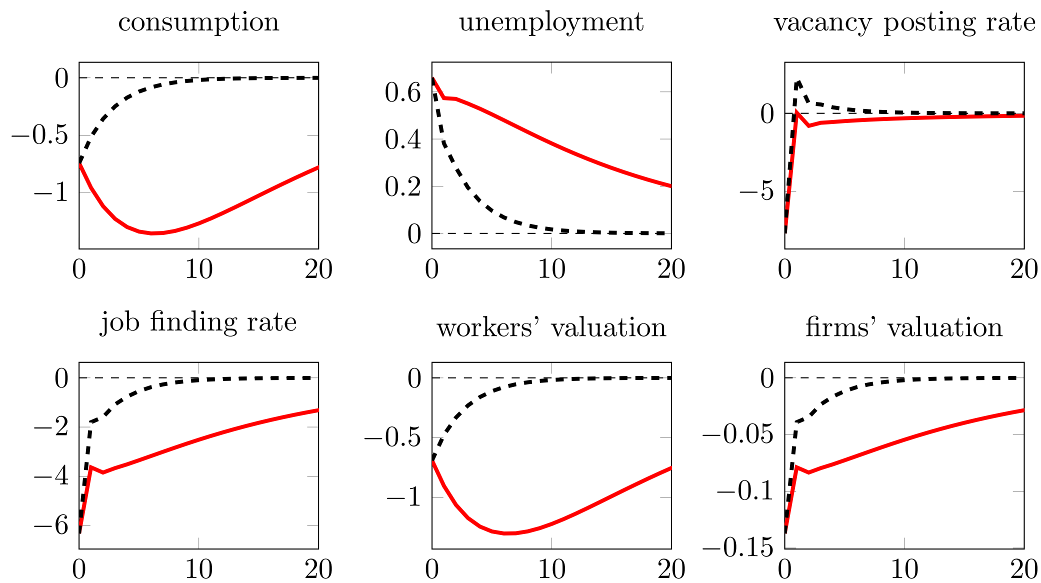

Figure 8: Impulse response functions to the model uncertainty shock w^{θ} in the structural model under the data-generating measure P (black dashed line) and the worst-case model $\widetilde{P}$ (red solid line). Horizontal axis in quarters.

beliefs in macroeconomy, we exploit the tractability of our framework to solve for an equilibrium in which "turn off" belief distortions on specific forward-looking equations. Formally, we look for the solution to the system of equations (7) modified as follows:

$$0 = E_t \left[\mathbb{M}_{t+1} g\left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right), \right]$$

where g is, as before, the $n \times 1$ vector of functions that includes Euler equations and market clearing conditions and $\mathbb{M}_{t+1} \equiv \operatorname{diag}\left\{m_{t+1}^{\sigma_1}, \ldots, m_{t+1}^{\sigma_n}\right\}$ are the separate belief distortions on each of the n equations. We consider two distinct belief distortions $\sigma_i \in \{0, 1\}$. $m_{t+1}^{0} \equiv 1$ denotes an undistorted equation under rational expectations, and

$$m_{t+1}^{1} \equiv \frac{\exp\left(-\theta_t V_{t+1}\right)}{E_t\left[\exp\left(-\theta_t V_{t+1}\right)\right]}$$

denotes, as in (5), an equation under the household's subjective beliefs. For details on implementation and a more general treatment of heterogeneous beliefs in this framework, see Appendix B.5.

We decompose the contribution of separate beliefs distortions in explaining the dynamic response of the macroeconomy to an increase in model uncertainty by

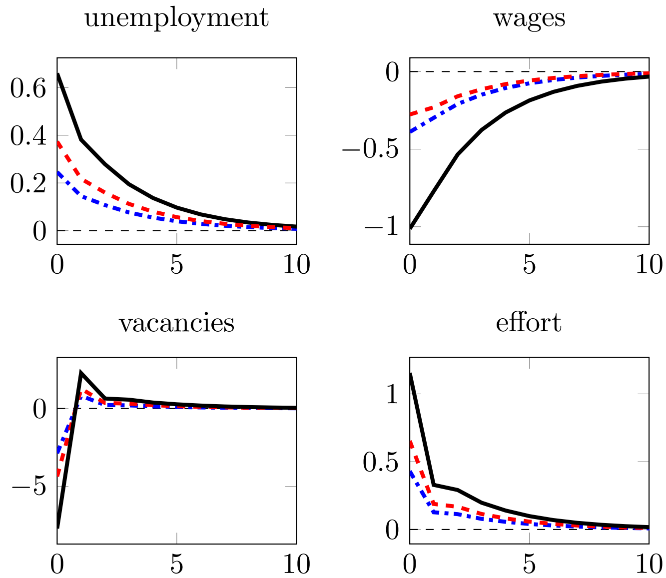

Figure 9: Impulse response functions to the model uncertainty shock w^{θ} in the structural model under benchmark (black line) and the economy with rational firms (blue dash-dotted line) and rational firms and workers (red dashed line). Horizontal axis in quarters.

comparing them to two alternative economies—one where consumers have distorted beliefs but workers and firms are rational, and a case where consumers and workers have distorted beliefs while firms have rational beliefs.

Distorted beliefs in the consumption Euler equation of the representative consumer have a direct impact on aggregate demand. As we saw in Figure 8, an increase in θ_t leads the consumer to forecast periods of low consumption in the future. Given these beliefs, optimal consumption smoothing leads her to lower current consumption. Since output is partly demand-determined due to nominal rigidities, this leads to lower labor demand, lower prices and higher unemployment. The forces described here are also present in Basu and Bundick (2017), who study responses to an increase in the volatility of subjective discount rates in a model of demand-determined output. The red dashed lines in Figure 9 show the response to an increase in θ_t when consumers use subjective beliefs and workers and firms use rational beliefs, thus isolating the effect of this channel. We interpret these impulse responses as indicating that the decrease in aggregate demand alone accounts for about a half (57% on impact) of the increase in unemployment relative to the case with all agents using subjective beliefs. The remaining part is attributed to the role fluctuations in subjective beliefs play in the frictional goods and labor markets.

First, with price setting rigidities, monopolistic producers of intermediate goods have to forecast future marginal costs and price changes. Higher model uncertainty means that they expect lower productivity under their subjective belief and hence higher marginal costs and higher prices in the future. This lowers their incentives to produce and thereby lowers labor demand. Additionally, with search and matching rigidities, hiring and bargaining decisions are based on the value of the discounted future surplus generated by a match. Lower expected productivity and higher expected interest rates lower the value of the match from the perspective of the worst-case beliefs shared by the worker-firm pair. This lowers equilibrium hiring rates, and lower employment also implies lower output.

We are also interested in separating out the role of the subjective belief for the worker and the firm in the frictional labor market. This leads us to study a specification where only firms are endowed with rational beliefs while consumers and workers have subjective beliefs determined by the worst-case model. Workers and firms engage in Nash bargaining when they meet, and they agree to disagree about their subjective valuation of the match, in the sense of Harrison and Kreps (1978) and Morris (1995).[12]

The blue dashed lines in Figure 9 show the response to an increase in θ_t in this environment. The response of unemployment is muted relative to the case where firms and workers have homogeneous rational beliefs. This difference comes from the wage setting process in the frictional labor markets. In the top right panel of Figure 9 we see that wages fall more when workers and firms have heterogeneous beliefs. Keeping the wage process unchanged, an increase in θ_t decreases workers' subjective valuation of the wage income but, at the same time, also reduces the value of the outside option by lowering the subjective job finding probability rates. Since job finding rates are volatile and comove strongly with the continuation values, pessimism about the outside option dominates the wage valuation effect. With Nash bargaining, a lower outside option for the workers results in lower equilibrium wages. Firms therefore have higher profits and post more vacancies. As a result, labor market tightness falls by less and worker search effort is lower. The responses of vacancies and search effort are shown in the bottom panel of Figure 9.

[12] With direct empirical evidence on the beliefs of firms, we could also endow firms with their own subjective belief that is distinct from that of the worker or the rational belief.

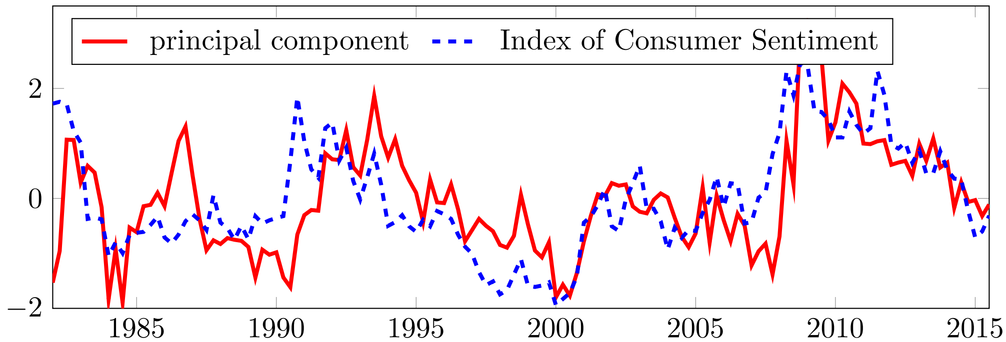

Figure 10: First principal component of the belief wedges (red solid line) and the negative of the Michigan Survey Index of Consumer Sentiment (blue dashed line), normalized to mean zero and standard deviation one.

6 Subjective beliefs and measures of model uncertainty

The belief wedges we extract from the survey data and their structural interpretation through the lens of the robust preference household provide a direct quantitative measure of model misspecification concerns. The model endogenously generates subjective beliefs that contain time-varying pessimistic biases driven by the model uncertainty process θ_t. We discipline the dynamics of θ_t by aligning its persistence with the persistence of the first principal component of the unemployment of inflation wedges, and choosing its volatility to line up the volatility of the wedges in the model and in the data.

Figure 10 depicts the common (first principal) component of the belief wedges against the negative of the Michigan Survey Index of Consumer Sentiment (ICS). The ICS and similar empirical measures have been frequently used in the macroeconomics and finance literature as qualitative reduced-form proxies for notions of time-varying uncertainty, pessimism, confidence or discount rates. Contrary to our belief wedges, these proxies have no direct quantitative counterpart in theoretical models, which complicates calibration of these features of agents' decision problems.

The figure shows that both measures are highly correlated—household confidence is low in recessions, when our belief wedges indicate largest pessimistic deviations in households' beliefs. The correlation coefficient between the two time series is -0.62. In contrast to Angeletos et al. (2016), who also obtain a measure of pessimism and show that it correlates strongly with the ICS, we directly map our model of subjective beliefs to household forecasts in the Michigan Survey. This yields additional restrictions on the magnitude of the belief fluctuations in the model, and also fully characterizes the shape of the belief distortions informed by the relative magnitudes of the belief wedges.

This feature also provides an interpretation and empirical discipline for the discount rate shocks in Hall (2017) and their role in labor market dynamics. When misspecification concerns increase, agents evaluate more pessimistically cash flows that positively correlate with households' continuation values. Since the match surplus is procyclical, it is effectively discounted at a higher rate. While the discount rate shocks in Hall (2017) affect all cash flows roughly equally with little heterogeneity based on cash flow risk, our notion of time variation in pessimism is tightly tied to exposures to states that are endogenously determined to be adverse by the robust household.

The uncertainty shocks in Leduc and Liu (2016), modeled as time variation in the standard deviation of innovations to the technology shock, also increase unemployment through fluctuations in the discount rate. Leduc and Liu (2016) calibrate the stochastic volatility process to the magnitude of fluctuations in the share of households in the Michigan Survey who report 'uncertain future' as a reason for why will it be a bad time to purchase a vehicle in the next 12 months. While this measure meaningfully proxies for model misspecification concerns and consumer sentiment (the authors show that innovations to their uncertainty measure also move the ICS, and, in fact, the time series of their measure is similar to that for the negative of ICS reported in Figure 10), its direct mapping to objects in the model is absent.

Piazzesi et al. (2015) use survey expectations of financial forecasters about future interest rates to estimate the role of subjective beliefs for the dynamics of the yield curve and associated bond risk premia in an affine term structure model. Like us, Szöke (2017) interprets these survey forecasts as representing a subjective worst-case probability distribution of investors facing model misspecification concerns. He embeds these survey data in an estimated endowment economy model and finds that the cross-equation restrictions implied by the robust preference framework are consistent with the joint dynamics of survey forecasts and the yield curve dynamics.

Ilut and Schneider (2014) and Bianchi et al. (2017) use the dispersion in the SPF forecasts as a proxy of household confidence in the forecasting model. Their building block is the recursive multiple-prior preferences of Epstein and Schneider (2003) and the main difference between their approach and ours to model uncertainty is how the set of potential misspecifications is parameterized in each case. The multiple-prior framework does not restrict the relative magnitudes of individual shock distortions under the worst-case model, and thus introduces a heavier burden on identification through observable data. In our setting, exposures of household continuation values to the underlying shocks endogenously determine the distortions. These exposures are pinned down by cross-equation restrictions that arise from optimizing behavior of forward-looking agents and impose consistency between the worst-case model and implied continuation values. We explicitly characterize these restrictions and utilize them in our calibration.

Another difference is that we use data on cross-sectional average distortions in household survey answers, for which our theory has direct quantitative predictions, to calibrate the model uncertainty shock process. Ilut and Schneider (2014) instead use the forecast *dispersion* as a proxy for confidence and show an empirically plausible relation of this measure to the notion of model uncertainty. In related work, Mankiw et al. (2003), Bachmann et al. (2012) and others use measures of cross-sectional forecast dispersion as a proxy for economic uncertainty, based on the presumption that a higher dispersion is indicative of more difficulty in estimating the forecast distribution, and therefore implies more uncertainty. While it may be appealing to use cross-sectional dispersion in forecasts as a proxy for the model misspecification concerns of each individual household, our theory does not provide such a direct link. We seek to keep model misspecification concerns separate from the notion *disagreement* in forecasts across households. The model we develop in this paper is based on a representative agent framework that does not feature heterogeneity in individual forecasts, and therefore yields no predictions about forecast dispersion measures. Despite these differences, we view both approaches as complementary.

At the same time, it is possible to extend the framework by introducing heterogeneity in agents' concerns for uncertainty, as illustrated in Section 5 and Appendix B.5. More generally, agents with differing model misspecification concerns or differing in exposures to shocks will deduce alternative worst-case models, which then generates dispersion in forecasts in the model. The evidence in Section 2 corroborates this narrative for the cross-section of households. Rozsypal and Schlafmann (2017) and Das et al. (2017) study these cross-sectional forecast patterns in the Michigan Survey and explore implications for heterogeneity in households' consumption-saving behavior. Similarly, survey information such as in Coibion et al. (2015) can be used to inform heterogeneity in beliefs between firms and households.

While conceptually interesting, a full-fledged quantitative analysis of heterogeneous subjective beliefs is beyond the scope of this paper.

Fluctuations in macroeconomic survey forecasts have also been investigated through the lens of other theories of subjective beliefs. While we view these theories as complementary to ours, we argue that the mechanism investigated in this paper yields distinct testable predictions consistent with quantitatively large belief biases that cannot be explained by those alternatives.

Barsky and Sims (2012) study the impact of innovations to the measure of consumer confidence from the Michigan Survey and decompose these innovations into the contribution of news shocks, representing arrival of information about future productivity (Pigou (1927), Beaudry and Portier (2004), Jurado (2016)), and 'animal spirits' that capture fluctuations in agents' subjective beliefs. We address the decomposition problem by constructing the belief wedge as the difference between households' and professionals' forecasts in the robustness check in Appendix C, thus differencing out the impact of news shocks while preserving the role of fluctuations in subjective beliefs in the form of the households' worst-case model.

Carroll (2003), Reis (2009), Coibion and Gorodnichenko (2012) and many others contribute to the large literature on learning and information acquisition in macroeconomics, imposing alternative learning mechanisms on the side of economic agents. Learning is a plausible way of introducing a wedge between agents' beliefs and the data-generating measure, but does not explain the large and systematic pessimistic biases observed in household survey responses. Further, it is generally inconsistent with the cross-equation restrictions imposed by our framework, which imply larger biases for shocks with a more adverse utility impact. Finally, learning models imply slow adjustment of agents' beliefs to economic shocks, and would therefore predict a relatively optimistic bias in recessions, as agents do not fully incorporate the adverse realization of the current state. We observe the opposite correlation between belief wedges and the business cycle in the data. We consider a combination of model misspecification concerns and learning to be an appealing extension, see Epstein and Schneider (2007), Hansen and Sargent (2007, 2010) or Bhandari (2015), but as in the case of belief heterogeneity leave it for further research.

7 Conclusion

We develop a framework in which agents' subjective beliefs are driven by time variation in concerns for model misspecification. Our framework deviates from rational expectations but provides tight restrictions on agents' subjective forecasts and equilibrium dynamics. We use survey data to document time-series and cross-sectional patterns in household forecasts of macroeconomic aggregates, and then exploit the cross-equation restrictions that our theory provides to embed subjective beliefs into quantitative general equilibrium models. We demonstrate the workings of our framework using an application that studies frictional labor and goods markets. Our findings suggest that subjective beliefs have an economically significant role in driving macroeconomic outcomes, especially labor market quantities.

The solution technique we develop is applicable to an extensive class of models that feature agents with robust preferences. The linear approximation captures time variation in model misspecification concerns and is easy to implement and estimate, which facilitates quantitative analysis of dynamic stochastic equilibrium models with robust agents. Our method also extends to settings with heterogeneous agents.

We conclude by outlining two extensions of the methods developed in this paper. A natural question given our findings is how to conduct optimal policy in settings with subjective beliefs as in the theoretical studies of Hansen and Sargent (2012), Adam and Woodford (2012), Karantounias (2013), Orlik and Presno (2013), or Kwon and Miao (2013). In parallel work, we study the implications of this framework for optimal monetary policy. The second extension is to analyze subjective beliefs in economies with uninsurable idiosyncratic risk. With incomplete markets, heterogeneous exposure of continuation values to shocks generates endogenous heterogeneity in beliefs and has implications for savings, portfolio choices, and labor market behavior. Such an extension can naturally exploit the cross-sectional and panel dimension of survey data. We leave this for future research.

Appendix

A Subjective beliefs and belief wedges

In this section, we derive formulas for the belief distortions in the linearized version of the dynamic model described in Section 3, extended to include nonstationary shocks as in Appendix B.7. Let $(\Omega, \{\mathcal{F}_t\}_{t=0}^{\infty}, P)$ be the filtered probability space generated by the innovations $\{w_{t+1}\}_{t=0}^{\infty}$, with $w_{t+1} \sim N(0_k, I_{k\times k})$ iid. The subjective probability measure $\widetilde{P}$ is formally defined by specifying a strictly positive martingale M_{t+1} with one-period increment

$$m_{t+1} = \frac{M_{t+1}}{M_t} = \exp\left(-\frac{1}{2}|\nu_t|^2 + \nu_t' w_{t+1}\right). \tag{20}$$

The conditional mean of the innovation vector under $\widetilde{P}$ then satisfies $\widetilde{E}_t[w_{t+1}] = \nu_t$. We consider linear model dynamics given by

$$\begin{aligned} x_t &= \widehat{x}_t + z_t \\ \widehat{x}_{t+1} &= \psi_q + \psi_x \widehat{x}_t + \psi_w w_{t+1} \\ z_{t+1} - z_t &= \phi_q + \phi_x \widehat{x}_t + \phi_q w_{t+1}. \end{aligned} \tag{21}$$

The vector x_t of economic variables therefore has a stationary component $\widehat{x}_t$ and a nonstationary component z_t that has a stationary growth rate.[13] We impose a restriction on the belief distortion (20):

$$\nu_t = \overline{H} + HF\widehat{x}_t$$

where F is a $1 \times n$ vector and H, $\overline{H}$ are $k \times 1$ vectors. The belief distortion derived in the structural model is a special case of this restriction. In particular, in the case of the linear approximation of the stationary model developed in Section 3, we have $z_t \equiv 0$ and $x_{1t} = \widehat{x}_t$. Equation (15) implies that $\nu_t = -\overline{\theta}(\bar{x} + x_{1t})(V_x \psi_w)'$ and hence

$$\overline{H} = -\overline{\theta}\bar{x}(V_x\psi_w)' \qquad H = -(V_x\psi_w)' \qquad F = \overline{\theta}.$$

In the case of the nonstationary model from Appendix B.7, the expressions for $\overline{H}$, H and F are given in equation (55).

Let $\zeta_t = Zx_t = Z(\widehat{x}_t + z_t)$ be an $m \times 1$ vector of variables for which we have observable data on households' expectations where Z is an $m \times n$ selection matrix. We are interested in τ-period ahead belief wedges

$$\Delta_t^{(\tau)} = \widetilde{E}_t[\zeta_{t+\tau}] - E_t[\zeta_{t+\tau}].$$

Guess that

$$\begin{aligned} E_t[\zeta_{t+\tau} - \zeta_t] &= G_x^{(\tau)}\widehat{x}_t + G_0^{(\tau)} \\ \widetilde{E}_t[\zeta_{t+\tau} - \zeta_t] &= \widetilde{G}_x^{(\tau)}\widehat{x}_t + \widetilde{G}_0^{(\tau)} \end{aligned}$$

where $G_x^{(\tau)}$, $G_0^{(\tau)}$, $\widetilde{G}_x^{(\tau)}$ and $\widetilde{G}_0^{(\tau)}$ are conformable matrix coefficients with initial conditions

$$G_0^{(\tau)} = \widetilde{G}_0^{(\tau)} = 0_{m\times 1} \qquad G_x^{(\tau)} = \widetilde{G}_x^{(\tau)} = 0_{m\times n}.$$

[13] The linear approximation of the model specified in Section 3 maps directly into this framework. We drop the subindices denoting first-order derivative processes for convenience.

We can then establish a recursive formula for the expectations under the data-generating measure

$$\begin{aligned}
G_x^{(\tau)}\widehat{x}_t + G_0^{(\tau)} &= E_t\left[\zeta_{t+\tau} - \zeta_t\right] = \\
&= E_t\left[Z\left(x_{t+1} - x_t\right) + G_x^{(\tau-1)}\widehat{x}_{t+1} + G_0^{(\tau-1)}\right] \\
&= G_0^{(\tau-1)} + Z\phi_q + \left(Z + G_x^{(\tau-1)}\right)\psi_q + \left[\left(Z + G_x^{(\tau-1)}\right)\psi_x + \left(Z\phi_x - Z\right)\right]\widehat{x}_t \\
&\quad + \left(\left(Z + G_x^{(\tau-1)}\right)\psi_w + Z\phi_w\right)E_t\left[w_{t+1}\right].
\end{aligned} \tag{22}$$

Since $E_t\left[w_{t+1}\right] = 0$, we obtain

$$\begin{aligned}
G_x^{(\tau)} &= \left(Z + G_x^{(\tau-1)}\right)\psi_x + \left(Z\phi_x - Z\right) \\
G_0^{(\tau)} &= G_0^{(\tau-1)} + Z\phi_q + \left(Z + G_x^{(\tau-1)}\right)\psi_q.
\end{aligned}$$

Under the subjective measure, the derivation is unchanged, except the last line in (22) that now involves the subjective expectation $\widetilde{E}_t\left[w_{t+1}\right] = \overline{H} + HF\widehat{x}_t$. Then

$$\begin{aligned}
\widetilde{G}_x^{(\tau)} &= \left(Z + \widetilde{G}_x^{(\tau-1)}\right)\psi_x + \left(Z\phi_x - Z\right) + \left(\left(Z + \widetilde{G}_x^{(\tau-1)}\right)\psi_w + Z\phi_w\right)HF \\
\widetilde{G}_0^{(\tau)} &= \widetilde{G}_0^{(\tau-1)} + Z\phi_q + \left(Z + \widetilde{G}_x^{(\tau-1)}\right)\psi_q + \left(\left(Z + \widetilde{G}_x^{(\tau-1)}\right)\psi_w + Z\phi_w\right)\overline{H}
\end{aligned}$$

Consequently

$$\Delta_t^{(\tau)} = \left(\widetilde{G}_x^{(\tau)} - G_x^{(\tau)}\right)\widehat{x}_t + \widetilde{G}_0^{(\tau)} - G_0^{(\tau)}.$$

When the dynamics (21) are stationary and demeaned, $\overline{H}$, ϕ_q, ϕ_x, ϕ_w and ϕ_q are all zero, and we get explicit expressions

$$\begin{aligned}
G_x^{(\tau)} &= Z\left(\psi_x\right)^{\tau} \\
G_0^{(\tau)} &= Z\sum_{i=0}^{\tau-1}\left(\psi_x\right)^i\psi_q = Z\left(I - \psi_x\right)^{-1}\left(I - \left(\psi_x\right)^{\tau}\right)\psi_q \\
\widetilde{G}_x^{(\tau)} &= Z\left(\psi_x + \psi_w HF\right)^{\tau} \\
\widetilde{G}_0^{(\tau)} &= Z\sum_{i=0}^{\tau-1}\left(\psi_x + \psi_w HF\right)^i\psi_q = Z\left(I - \left(\psi_x + \psi_w HF\right)\right)^{-1}\left(I - \left(\psi_x + \psi_w HF\right)^{\tau}\right)\psi_q.
\end{aligned}$$

B Linear approximation of models with robust preferences

The linear approximation in this paper builds on the series expansion method used in Holmes (1995), Lombardo (2010) and Borovička and Hansen (2014). The innovation in this paper consists of adapting the series expansion method to an approximation of models with robust preferences to derive a linear approximation that allows for endogenously determined time-varying belief distortions. The critical step in the expansion lies in the joint perturbation of the shock vector w_t and the penalty process θ_t.

B.1 Law of motion

We start with the approximation of the model for the law of motion (4) with a sufficiently smooth policy rule ψ. We consider a class of models indexed by the scalar perturbation parameter q that scales the volatility of the shock vector w_t

$$x_{t+1}(\mathsf{q}) = \psi(x_t(\mathsf{q}), \mathsf{q}w_{t+1}, \mathsf{q}) \tag{23}$$

and assume that there exists a series expansion of x_t around $\mathsf{q} = 0$:

$$x_t(\mathsf{q}) \approx \bar{x} + \mathsf{q}x_{1t} + \frac{\mathsf{q}^2}{2}x_{2t} + \ldots.$$

The processes x_{jt}, $j = 0, 1, \ldots$ can be viewed as derivatives of x_t with respect to the perturbation parameter, and their laws of motion can be inferred by differentiating (23) j times and evaluating the derivatives at $\mathsf{q} = 0$, assuming that ψ is sufficiently smooth. Here, we focus only on the approximation up to the first order:

$$\begin{aligned} \bar{x} &= \psi(\bar{x}, 0, 0) \qquad (24) \\ x_{1t+1} &= \psi_q + \psi_x x_{1t} + \psi_w w_{t+1}. \end{aligned}$$

We begin with a case where the equilibrium dynamics of x_t is stationary. Extensions to non-stationary environments are considered in Appendix B.7.

B.2 Continuation values

We now focus on the expansion of the continuation value recursion. Substituting the worst-case belief distortion (5) into the recursion (2) yields

$$V_t = u(x_t) - \frac{\beta}{\theta_t}\log E_t\left[\exp(-\theta_t V_{t+1})\right]. \tag{25}$$

We are looking for an expansion of the continuation value

$$V_t(\mathsf{q}) \approx \bar{V} + \mathsf{q}V_{1t}. \tag{26}$$

In order to derive the solution of the continuation value, we are interested in expanding the following perturbation of the recursion:

$$V_t(\mathsf{q}) = u(x_t(\mathsf{q}), \mathsf{q}) - \beta\frac{\mathsf{q}}{\overline{\theta}(\bar{x} + x_{1t})}\log E_t\left[\exp\left(-\frac{\overline{\theta}(\bar{x} + x_{1t})}{\mathsf{q}}V_{t+1}(\mathsf{q})\right)\right]. \tag{27}$$

Here, we utilized the fact that $\theta_t = \overline{\theta}x_t \approx \overline{\theta}(\bar{x} + x_{1t})$. More importantly, the perturbation scales jointly the volatility of the stochastic processes for V_t and $u(x_t)$ with the magnitude of the penalty parameter θ_t. In particular, the penalty parameter in the perturbation of equation (2) becomes $\mathsf{q}/\left[\overline{\theta}(\bar{x} + x_{1t})\right]$ and decreases jointly with the volatility of the shock process. This assumption will imply that the benchmark and worst-case models do not converge as $\mathsf{q} \to 0$, and the linear approximation around a deterministic steady state yields a nontrivial contribution of the worst-case dynamics.

Using the expansion of the period utility function

$$u(x_t(\mathsf{q}), \mathsf{q}) \approx \bar{u} + \mathsf{q}u_{1t} = \bar{u} + \mathsf{q}(u_x x_{1t} + u_q)$$

we get the deterministic steady state (zero-th order) term by setting $\mathsf{q} = 0$:

$$\bar{V} = (1 - \beta)^{-1} \bar{u}.$$

The first-order term in the expansion is derived by differentiating (27) with respect to q and is given by the recursion

$$V_{1t} = u_{1t} - \beta \frac{1}{\overline{\theta} (\bar{x} + x_{1t})} \log E_t \left[\exp \left(-\overline{\theta} (\bar{x} + x_{1t}) V_{1t+1} \right) \right] \tag{28}$$

Since $\bar{x}$ is constant and x_{1t} has linear dynamics (24), we hope to find linear dynamics for V_{1t} as well. Since $u_t = u(x_t)$, we can make the guess that $V_t^i(\mathsf{q}) = V^i(x_t(\mathsf{q}), \mathsf{q})$ which leads to the following expressions for the derivative of V_t:

$$V_{1t} = V_x x_{1t} + V_q.$$

Using this guess and comparing coefficients, equation (28) leads to a pair of algebraic equations for the unknown coefficients V_x and V_q:

$$\begin{aligned} V_x &= u_x - \frac{\beta}{2} V_x \psi_w \psi_w' V_x' \overline{\theta} + \beta V_x \psi_x \\ V_q &= u_q - \frac{\beta}{2} \overline{\theta} \bar{x} V_x \psi_w \psi_w' V_x' + \beta V_x \psi_q + \beta V_q \end{aligned}$$

The first from this pair of equations is a Riccati equation for V_x, which can be solved for given coefficients ψ_x and ψ_w.

B.3 Distortions and belief wedges

With the approximation of the continuation value at hand, we can derive the expansion of the one-period belief distortion m_{t+1} that defines the worst-case model relative to the benchmark model. As in (27), we scale the penalty parameter θ_t jointly with the volatility of the underlying shocks:

$$m_{t+1}(\mathsf{q}) = \frac{\exp\left(-\frac{1}{\mathsf{q}} \theta_t V_{t+1}(\mathsf{q})\right)}{E_t\left[\exp\left(-\frac{1}{\mathsf{q}} \theta_t V_{t+1}(\mathsf{q})\right)\right]} \approx m_{0,t+1} + \mathsf{q} m_{1,t+1}.$$

It turns out that in order to derive the correct first-order expansion, we are required to consider a second-order expansion of the continuation value

$$V_t(\mathsf{q}) \approx \bar{V} + \mathsf{q} V_{1t} + \frac{\mathsf{q}}{2} V_{2t},$$

although the term V_{2t} will be inconsequential for subsequent analysis. Substituting in expression (26) and noting that $\bar{V}$ is a deterministic term, we can approximate m_{t+1} with

$$m_{t+1}(\mathsf{q}) \approx \frac{\exp\left(-\overline{\theta}(\bar{x} + x_{1t})\left(V_{1t+1} + \frac{\mathsf{q}}{2} V_{2t+1}\right)\right)}{E_t\left[\exp\left(-\overline{\theta}(\bar{x} + x_{1t})\left(V_{1t+1} + \frac{\mathsf{q}}{2} V_{2t+1}\right)\right)\right]}$$

Differentiating with respect to q and evaluating at $\mathsf{q} = 0$, we obtain the expansion

$$
\begin{aligned}
m_{0t+1} &= \frac{\exp\left(-\overline{\theta}\left(\bar{x}+x_{1t}\right)V_{1t+1}\right)}{E_t\left[\exp\left(-\overline{\theta}\left(\bar{x}+x_{1t}\right)V_{1t+1}\right)\right]} \\
m_{1t+1} &= -\frac{1}{2\overline{\theta}\left(\bar{x}+x_{1t}\right)}m_{0t+1}\left[V_{2t+1}-E_t\left[m_{0t+1}V_{2t+1}\right]\right]
\end{aligned} \tag{29}
$$

This expansion is distinctly different from the standard polynomial expansion familiar from the perturbation literature. First, observe that m_{0t+1} is not constant, as one would expect for a zeroth-order term, but nonlinear in V_{1t+1}. However, since $E_t\left[m_{0t+1}\right]=1$, we can treat m_{0t+1} as a change of measure that will adjust the distribution of shocks that are correlated with m_{0t+1}. We will show that with Gaussian shocks, we can still preserve tractability. Further notice that $E_t\left[m_{1t+1}\right]=0$.

The linear structure of V_{1t} also has an important implication for the worst-case distortion constructed from m_{0t+1}. Substituting into (29) yields

$$
m_{0t+1}=\frac{\exp\left(-\overline{\theta}\left(\bar{x}+x_{1t}\right)V_x\psi_w w_{t+1}\right)}{E_t\left[\exp\left(-\overline{\theta}\left(\bar{x}+x_{1t}\right)V_x\psi_w w_{t+1}\right)\right]}.
$$

This implies that for a function $f\left(w_{t+1}\right)$ with a shock vector $w_{t+1}\sim N\left(0,I\right)$, the first-order approximation is given by

$$
\begin{aligned}
\widetilde{E}_t\left[f\left(w_{t+1}\right)\right] &= E_t\left[m_{t+1}f\left(w_{t+1}\right)\right] \\
&\approx f_0\left(w_{t+1}\right)+E_t\left[m_{0t+1}f_1\left(w_{t+1}\right)\right].
\end{aligned} \tag{30}
$$

The distortion generating the $\widetilde{P}$ (worst-case) measure is therefore approximated by the 'zero-th' order term m_{0t+1}, and the vector w_{t+1} has the following distribution:

$$
w_{t+1}\sim N\left(-\overline{\theta}\left(\bar{x}+x_{1t}\right)\left(V_x\psi_w\right)',I_k\right). \tag{31}
$$

The mean of the shock is therefore time-varying and depends on the linear process x_{1t}.

It follows that the belief wedges for the one-period-ahead forecast of the vector of variables x_t are given by

$$
\Delta_t^{(1)}=\widetilde{E}_t\left[x_{t+1}\right]-E_t\left[x_{t+1}\right]=\psi_w\widetilde{E}_t\left[w_{t+1}\right]=-\overline{\theta}\left(\bar{x}+x_{1t}\right)\left(\psi_w\psi_w'\right)V_x'.
$$

Belief wedges for longer-horizon forecasts are then computed using formulas from Appendix A, observing that we can set

$$
F=\overline{\theta},\qquad H=-\left(V_x\psi_w\right)',\qquad \overline{H}=-\overline{\theta}\bar{x}\left(V_x\psi_w\right)'.
$$

The terms ψ_w and V_x are functions of structural parameters in the model solved in the following section.

B.4 Equilibrium conditions

We assume that equilibrium conditions in our framework can be written as

$$
0=E_t\left[\widetilde{g}\left(x_{t+1},x_t,x_{t-1},w_{t+1},w_t\right)\right] \tag{32}
$$

where $\widetilde{g}$ is an $n\times 1$ vector function and the dynamics for x_t is implied by (4). This vector of equations includes expectational equations like Euler equations of the robust household, which can be represented using worst-case belief distortions m_{t+1}. We therefore assume that we can write the j-th component of $\widetilde{g}$ as

$$\widetilde{g}^j \left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right) = m_{t+1}^{\sigma_j} g^j \left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right).$$

where $\sigma_j \in \{0, 1\}$ captures whether the expectation in the j-th equation is under the household's worst-case model. In particular, all nonexpectational equations and all equations not involving agents' preferences have $\sigma_j = 0$. System (32) can then be written as

$$0 = E_t \left[\mathbb{M}_{t+1} g \left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right)\right] \tag{33}$$

where $\mathbb{M}_{t+1} = \operatorname{diag}\left\{m_{t+1}^{\sigma_1}, \ldots, m_{t+1}^{\sigma_n}\right\}$ is a diagonal matrix of the belief distortions, and g is independent of θ_t. The zero-th and first-order expansions are

$$\begin{aligned} 0 &= E_t \left[\mathbb{M}_{0t+1} g_{0t+1}\right] = g_{0t+1} \\ 0 &= E_t \left[\mathbb{M}_{0t+1} g_{1t+1}\right] + E_t \left[\mathbb{M}_{1t+1} g_{0t+1}\right] = E_t \left[\mathbb{M}_{0t+1} g_{1t+1}\right] \end{aligned}$$

where the last equality follows from $E_t \left[m_{1t+1}\right] = 0$.

For the first-order derivative of the equilibrium conditions, we have

$$0 = E_t \left[\mathbb{M}_{0t+1} g_{1t+1}\right] \tag{34}$$

The first-order term in the expansion of g_{t+1} is given by

$$\begin{aligned} g_{1t+1} &= g_{x+} x_{1t+1} + g_x x_{1t} + g_{x-} x_{1t-1} + g_{w+} w_{t+1} + g_w w_t + g_q = \\ &= \left[\left(g_{x+}\psi_x + g_x\right)\psi_x + g_{x-}\right] x_{1t-1} + \left[\left(g_{x+}\psi_x + g_x\right)\psi_w + g_w\right] w_t + \\ &\quad + \left(g_{x+}\psi_x + g_{x+} + g_x\right)\psi_q + g_q + \left(g_{x+}\psi_w + g_{w+}\right) w_{t+1} \end{aligned} \tag{35}$$

where symbols x_+, x, x_-, w_+, w, q represent partial derivatives with respect to $x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t$ and q, respectively. Given the worst-case distribution of the shock vector (31), we can write

$$\widetilde{E}_t \left[w_{t+1}\right] = -\left(V_x \psi_w\right)' \overline{\theta} \left[\left(\bar{x} + \psi_q\right) + \psi_x x_{1t-1} + \psi_w w_t\right]$$

Let $[A]^i$ denote the i-th row of matrix A. Notice that

$$\left[g_{x+}\psi_w + g_{w+}\right]^i \left(V_x \psi_w\right)' \overline{\theta}$$

is a $1 \times n$ vector. Construct the $n \times n$ matrix $\mathbb{E}$ by stacking these row vectors for all equations $i = 1, \ldots, n$:

$$\mathbb{E} = \operatorname{stack}\left\{\sigma_i \left[g_{x+}\psi_w + g_{w+}\right]^i \left(V_x \psi_w\right)' \overline{\theta}\right\}$$

which contains non-zero rows for expectational equations under the worst-case model. Using matrix $\mathbb{E}$, we construct the conditional expectation of the last term in g_{1t+1} in (35). In particular

$$\begin{aligned} 0 &= E_t \left[\mathbb{M}_{0t+1} g_{1t+1}\right] = \\ &= \left[\left(g_{x+}\psi_x + g_x\right)\psi_x + g_{x-}\right] x_{1t-1} + \left[\left(g_{x+}\psi_x + g_x\right)\psi_w + g_w\right] w_t + \\ &\quad + \left(g_{x+}\psi_x + g_{x+} + g_x\right)\psi_q + g_q - \mathbb{E}\left[\left(\bar{x} + \psi_q\right) + \psi_x x_{1t-1} + \psi_w w_t\right] \end{aligned}$$

Equation (34) is thus a system of linear second-order stochastic difference equations. There are well-known results that discuss the conditions under which there exists a unique stable equilibrium path to this system (Blanchard and Kahn (1980), Sims (2002)). We assume that such conditions are satisfied. Comparing coefficients on x_{1t-1}, w_t and the constant term implies that

$$
\begin{aligned}
0 &= \left(g_{x+}\psi_x + g_x - \mathbb{E}\right)\psi_x + g_{x-} && (36)\\
0 &= \left(g_{x+}\psi_x + g_x - \mathbb{E}\right)\psi_w + g_w && (37)\\
0 &= \left(g_{x+}\psi_x + g_{x+} + g_x\right)\psi_q + g_q - \mathbb{E}\left(\bar{x} + \psi_q\right) && (38)
\end{aligned}
$$

These equations need to be solved for ψ_x, ψ_w, ψ_q and V_x where

$$V_x = u_x - \frac{\beta}{2} V_x \psi_w \psi_w' V_x' \overline{\theta} + \beta V_x \psi_x \tag{39}$$

and

$$\mathbb{E} = \text{stack}\left\{\sigma_i \left[g_{x+}\psi_w + g_{w+}\right]^i \left(V_x \psi_w\right)' \overline{\theta}\right\}. \tag{40}$$

B.5 Multiple belief distortions

We proceeded with the derivation of the approximation under the assumption that there is only a single belief distortion affecting the equilibrium equations. This has been done for notational simplicity, and the extension to a framework with multiple agents endowed with heterogeneous belief distortions stemming from robust preferences is straightforward. Let us assume that there are J agents with alternative belief distortions characterized by $\left(V_t^j, m_{t+1}^j, \overline{\theta}^j\right)$, $j = 1, \ldots J$. The system of equilibrium conditions (33) given by

$$0 = E_t \left[\mathbb{M}_{t+1} g\left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right)\right]$$

with $\mathbb{M}_{t+1} = \text{diag}\left\{m_{t+1}^{\sigma_1}, \ldots, m_{t+1}^{\sigma_n}\right\}$ can then be extended to include alternative belief distortions indexed by $\sigma_i \in \{0, 1, \ldots, J\}$ where $m_{t+1}^0 \equiv 1$ denotes an undistorted equation. Subsequently, there are J distorted means of the innovations

$$\widetilde{E}_t^j \left[w_{t+1}\right] = -\left(V_x^j \psi_w\right)' \overline{\theta}^j \left[\left(\bar{x} + \psi_q\right) + \psi_x x_{1t-1} + \psi_w w_t\right]$$

that distort individual equations. Matrix $\mathbb{E}$ in (40) that collects the distortions of the equilibrium conditions then becomes

$$\mathbb{E} = \text{stack}\left\{\left[g_{x+}\psi_w + g_{w+}\right]^i \left(V_x^{\sigma_i} \psi_w\right)' \overline{\theta}^{\sigma_i}\right\}.$$

where $\sigma_i = 0$ corresponds to no distortion and hence i-th row is then a row of zeros. The structure of the system (36)–(40) remains the same except that we now have J recursions for V_x^j in (39) and a modified matrix $\mathbb{E}$.

B.6 Special case: θ_t is an exogenous AR(1) process

In the application, we consider a special case that restricts θ_t to be an exogenous AR(1) process. With a slight abuse in notation, this restriction can be implemented by replacing the vector of variables x_t with $\left(x_t', f_t\right)'$ where f_t is a scalar AR(1) process representing the time variation in the concerns for model misspecification as an exogenously specified 'belief' shock:

$$f_{t+1} = (1-\rho_f)\,\bar{f} + \rho_f f_t + \sigma_f w^f_{t+1}. \tag{41}$$

The dynamics of the model then satisfies

$$x_t = \psi\left(x_{t-1}, w_t, f_t\right) \tag{42}$$

with steady state $\left(\bar{x}', \bar{f}\right)'$. The vector $\overline{\theta}$ in (3) is then partitioned as $\overline{\theta}' = \left(\overline{\theta}'_x, \overline{\theta}_f\right) = (0_{1\times n-1}, 1)$ and thus $\theta_t = f_t$. Constructing the first-order series expansion of (42), we obtain

$$\begin{pmatrix} x_{1t+1} \\ f_{1t+1} \end{pmatrix} = \begin{pmatrix} \psi_q \\ 0 \end{pmatrix} + \begin{pmatrix} \psi_x & \rho_f \psi_{xf} \\ 0 & \rho_f \end{pmatrix} \begin{pmatrix} x_{1t} \\ f_{1t} \end{pmatrix} + \begin{pmatrix} \psi_w & \sigma_f \psi_{xf} \\ 0 & \sigma_f \end{pmatrix} \begin{pmatrix} w_{t+1} \\ w^f_{t+1} \end{pmatrix}$$

where w_{t+1} and w^f_{t+1} are uncorrelated innovations. The matrices ψ_x and ψ_w thus do not involve any direct impact of the dynamics of the belief shock f_{1t} and the matrix ψ_{xf} captures how the dynamics of f_{1t} influences the dynamics of endogenous state variables.

Let us further assume that the system (32) represents the equilibrium restrictions of the model *except* equation (41). In this case, the function g does not directly depend on f. Repeating the expansion of the equilibrium conditions from Section B.4 and comparing coefficients on x_{t-1}, f_{t-1}, w_t and the constant term yields the set of conditions for matrices ψ_x, ψ_w, ψ_{xf} and ψ_q:

$$\begin{aligned}
0 &= \left(g_{x+}\psi_x + g_x\right)\psi_x + g_{x-} && (43)\\
0 &= \left(g_{x+}\rho_f\psi_{xf} - \mathbb{E}\right) + \left(g_{x+}\psi_x + g_x\right)\psi_{xf} && (44)\\
0 &= \left(g_{x+}\psi_x + g_x\right)\psi_w + g_w && (45)\\
0 &= \left(g_{x+}\psi_x + g_{x+} + g_x\right)\psi_q + g_q - \mathbb{E}\bar{f} && (46)
\end{aligned}$$

with

$$\begin{aligned}
V_x &= u_x + \beta V_x \psi_x && (47)\\
V_f &= u_f - \frac{\beta\overline{\theta}}{2}\left(V_f^2\sigma_f^2 + 2V_x\psi_{xf}\sigma_f^2 V_f + V_x\left(\sigma_f^2\psi_{xf}\psi'_{xf} + \psi_w\psi'_w\right)V'_x\right) && (48)\\
&\quad + \beta\left(V_f\rho_f + V_x\psi_{xf}\rho_f\right)\\
\mathbb{E} &= \text{stack}\left\{\sigma^i\left[g_{x+}\psi_{xf}\sigma_f^2\left(V_f + V_x\psi_{xf}\right) + \left(g_{x+}\psi_w + g_{w+}\right)\psi'_w V'_x\right]^i\right\}\overline{\theta}. && (49)
\end{aligned}$$

This set of equations is the counterpart of equations (36)–(40) and can be solved sequentially. First, notice that equations (43) and (45) can be solved for ψ_x and ψ_w, and these coefficients are not impacted by the dynamics of f_t. But the equilibrium dynamics of x_t is affected by movements in f_t through the coefficient ψ_{xf}. The coefficient $\rho_f\psi_{xf}$ introduces an additional component in the time-varying drift of x_t, while $\sigma_f\psi_{xf}$ is an additional source of volatility arising from the shocks to household's concerns for model misspecification.

We solve this set of equations by backward induction. First, we use (36), (40) and (47) to find the rational expectations solution for ψ_x, ψ_w, V_x. Then we postulate that (42) is in fact a time-dependent law of motion

$$x_t = \psi^t\left(x_{t-1}, w_t, f_t\right)$$

with terminal condition at a distant date T

$$x_T = \psi^T\left(x_{T-1}, w_T, 0\right).$$

This corresponds to assuming that starting from date T, misspecification concerns are absent in the model. Plugging this guess to the set of equilibrium conditions, we obtain the set of algebraic equations

$$\begin{aligned}
0 &= \left(g_{x+}\psi_{xf}^{t+1}\rho_f - \mathbb{E}^{t+1}\right) + \left(g_{x+}\psi_x + g_x\right)\psi_{xf}^t && (50)\\
V_f^t &= u_f - \frac{\beta\overline{\theta}}{2}\left(\left(V_f^{t+1}\sigma_f\right)^2 + 2V_x\psi_{xf}^{t+1}\sigma_f^2 V_f^{t+1} + V_x\left(\sigma_f^2\psi_{xf}^{t+1}\left(\psi_{xf}^{t+1}\right)' + \psi_w\psi_w'\right)V_x'\right) && (51)\\
&\quad +\beta\rho_f\left(V_f^{t+1} + V_x\psi_{xf}^{t+1}\right)\\
\mathbb{E}^{t+1} &= \left[g_{x+}\psi_{xf}^{t+1}\left(V_f^{t+1} + V_x\psi_{xf}^{t+1}\right)\sigma_f^2 + \left(g_{x+}\psi_w + g_{w+}\right)\psi_w' V_x'\right]\overline{\theta}. && (52)
\end{aligned}$$

Equation (50) can then be solved for

$$\psi_{xf}^t = \left(g_{x+}\psi_x + g_x\right)^{-1}\left(\mathbb{E}^{t+1} - g_{x+}\psi_{xf}^{t+1}\rho_f\right) \tag{53}$$

Iterating backwards on equations (51)–(53) backward until convergence yields the stationary solution of the economy with model misspecification concerns as a long-horizon limit of an economy where these concerns vanish at a distant T. The system converges as long as its dynamics are stationary under the worst-case model. Once we find the limit $\lim_{t\to-\infty}\mathbb{E}^t = \mathbb{E}$, we can also determine

$$\psi_q = \left(g_{x+}\psi_x + g_{x+} + g_x\right)^{-1}\left(\mathbb{E}\bar{f} - g_q\right).$$

B.7 Nonstationary models

For the purpose of applying the expansion method, we assumed that the state vector x_t is stationary. Our framework can, however, deal with deterministic or stochastic trends featured in macroeconomic models. Specifically, let us assume that there exists a vector-valued stochastic process z_t such that the dynamics of x_t can be written as

$$\begin{aligned}
x_t &= \widehat{x}_t + z_t && (54)\\
z_{t+1} - z_t &= \phi\left(\widehat{x}_t, w_{t+1}\right)
\end{aligned}$$

where $\widehat{x}_t$ is a stationary vector Markov process that replaces dynamics (4):

$$\widehat{x}_{t+1} = \psi\left(\widehat{x}_t, w_{t+1}\right).$$

The process z_t thus has stationary increments and x_t and z_t are cointegrated, element by element. A typical example of an element in z_t is a productivity process with a permanent component. Once we solve for the stationary dynamics of $\widehat{x}_t$, we can obtain the dynamics of x_t in a straightforward way using (54).

Assume that the period utility function can be written in the form

$$u\left(x_t\right) = \widehat{u}\left(\widehat{x}_t\right) + Z^u z_t$$

where Z^u is a selection vector that selects the appropriate scaling from the vector z_t. For example,

$$u(x_t) = \log C_t = \log\left[\widehat{C}_t \exp(Z^u z_t)\right] = \log \widehat{C}_t + Z^u z_t$$

where $Z^u z_t$ is the nonstationary component of the logarithm of consumption $\log C_t$, and $\widehat{C}_t = \widehat{C}(\widehat{x}_t)$ is the stationary part. It follows from equation (25) that we can write

$$V_t = \widehat{V}(\widehat{x}_t) + (1-\beta)^{-1} Z^u z_t$$

and the stationary component of the continuation value $\widehat{V}(\widehat{x}_t)$ satisfies the recursion

$$\widehat{V}(\widehat{x}_t) = \widehat{u}(\widehat{x}_t) - \frac{\beta}{\theta_t} \log E_t\left[\exp\left(-\theta_t\left(\widehat{V}(\widehat{x}_{t+1}) + (1-\beta)^{-1} Z^u \phi(\widehat{x}_t, w_{t+1})\right)\right)\right].$$

The first-order expansion of ϕ yields

$$\begin{aligned} \bar{z}_{t+1} - \bar{z}_t &= \phi(\bar{x}, 0) \\ z_{1t+1} - z_{1t} &= \phi_q + \phi_x \widehat{x}_{1t} + \phi_w w_{t+1} \end{aligned}$$

where $\bar{x}$ is the steady state of $\widehat{x}_t$. We can now proceed as in the stationary case except using the expansion of functions $\widehat{u}$ and $\widehat{V}$. We have

$$\bar{V} = (1-\beta)^{-1}\left[\bar{u} + \beta(1-\beta)^{-1} Z^u \phi(\bar{x}, 0)\right]$$

and

$$\widehat{V}_{1t} = V_x \widehat{x}_{1t} + V_q$$

with

$$\begin{aligned} V_x &= u_x + \beta\left[V_x \psi_x + (1-\beta)^{-1} Z^u \phi_x\right] - \frac{\beta}{2}\left|V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w\right|^2 \overline{\theta} \\ V_q &= u_q + \beta\left[V_q + V_x \psi_q + (1-\beta)^{-1} Z^u \phi_q\right] - \frac{\beta}{2}\overline{\theta}\bar{x}\left|V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w\right|^2. \end{aligned}$$

The zero-th order distortion is consequently given by

$$m_{0t+1} = \frac{\exp\left(-\overline{\theta}(\bar{x} + \widehat{x}_{1t})\left(V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w\right) w_{t+1}\right)}{E_t\left[\exp\left(-\overline{\theta}(\bar{x} + \widehat{x}_{1t})\left(V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w\right) w_{t+1}\right)\right]}$$

so that under the worst-case model,

$$w_{t+1} \sim N\left(-\overline{\theta}(\bar{x} + \widehat{x}_{1t})\left(V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w\right)', I_k\right).$$

Equation (16) then becomes

$$\begin{aligned}
\widehat{x}_{1t+1} &= \psi_q - \overline{\theta}\bar{x}\psi_w \left(V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w \right)' \\
&\quad + \left[\psi_x - \psi_w \left(V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w \right)' \overline{\theta} \right] \widehat{x}_{1t} + \psi_w \widetilde{w}_{t+1} \\
&= \widetilde{\psi}_q + \widetilde{\psi}_x x_{1t} + \psi_w \widetilde{w}_{t+1}.
\end{aligned}$$

Comparing these dynamics under worst-case beliefs with those under the data-generating process, we can again construct belief wedges for longer-horizon forecasts as in Section B.3. Under the nonstationary dynamics, these wedges $\Delta_t^{(j)} = \widetilde{E}_t \left[x_{t+j} \right] - E_t \left[x_{t+j} \right]$ are computed using the recursive calculations outlined in Appendix A, imposing

$$\begin{aligned}
F &= \overline{\theta} \\
H &= -\left(V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w \right)' \\
\overline{H} &= -\left(\overline{\theta} \bar{x} \right) \left(V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w \right)'.
\end{aligned} \tag{55}$$

In order to solve for the equilibrium dynamics, notice that we are still solving the set of equations (36)–(38) but now with V_x and $\mathbb{E}$ given by

$$\begin{aligned}
V_x &= u_x + \beta \left[V_x \psi_x + (1-\beta)^{-1} Z^u \phi_x \right] - \frac{\beta}{2} \left| V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w \right|^2 \overline{\theta} \\
\mathbb{E} &= \text{stack} \left\{ \sigma_i \left[g_{x+} \psi_w + g_{w+} \right]^i \left(V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w \right)' \overline{\theta} \right\}.
\end{aligned}$$

In the special case described in Section B.6, the belief shock f_t is modeled as an exogenous AR(1) process. The first-order dynamics of the stochastic growth rate can be expressed as

$$z_{1t+1} - z_{1t} = \phi_q + \phi_x \widehat{x}_{1t} + \phi_{xf} f_{1t} + \phi_w w_{t+1} + \phi_{wf} w_{t+1}^f.$$

The only modifications appearing in the model solution are those related to the continuation value recursion and the shock distortion in $\mathbb{E}$. Specifically,

$$\begin{aligned}
V_x &= u_x + \beta \left[V_x \psi_x + (1-\beta)^{-1} Z^u \phi_x \right] \\
V_f &= u_f + \beta \left(\rho_f V_f + \rho_f V_x \psi_{xf} + (1-\beta)^{-1} Z^u \phi_{xf} \right) \\
&\quad - \frac{\beta\overline{\theta}}{2} \left| V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w \right|^2 - \frac{\beta\overline{\theta}}{2} \left| V_x \psi_{xf} \sigma_f + V_f \sigma_f + (1-\beta)^{-1} Z^u \phi_{wf} \right|^2 \\
\mathbb{E} &= \text{stack} \left\{ \sigma^i \left[\left(g_{x+} \psi_w + g_{w+} \right) \left(V_x \psi_w + (1-\beta)^{-1} Z^u \phi_w \right)' \right]^i \right\} \overline{\theta} \\
&\quad + \text{stack} \left\{ \sigma^i \left[g_{x+} \psi_{xf} \sigma_f \left(V_f \sigma_f + V_x \psi_{xf} \sigma_f + (1-\beta)^{-1} Z^u \phi_{wf} \right) \right]^i \right\} \overline{\theta}
\end{aligned}$$

In the recursive form, V_f and $\mathbb{E}$ can be solved by iterating on the pair of equations

$$\begin{aligned}
V_f^t &= u_f + \beta\left(\rho_f V_f^{t+1} + \rho_f V_x \psi_{xf}^{t+1} + (1-\beta)^{-1} Z^u \phi_{xf}\right) \\
&\quad -\frac{\beta\overline{\theta}}{2}\left|V_x\psi_w + (1-\beta)^{-1} Z^u \phi_w\right|^2 - \frac{\beta\overline{\theta}}{2}\left|V_x\psi_{xf}^{t+1}\sigma_f + V_f^{t+1}\sigma_f + (1-\beta)^{-1} Z^u\phi_{wf}\right|^2 \\
\mathbb{E}^{t+1} &= \text{stack}\left\{\sigma^i\left[\left(g_{x+}\psi_w + g_{w+}\right)\left(V_x\psi_w + (1-\beta)^{-1} Z^u\phi_w\right)'\right]^i\right\}\overline{\theta} \\
&\quad +\text{stack}\left\{\sigma^i\left[g_{x+}\psi_{xf}^{t+1}\sigma_f\left(V_f^{t+1}\sigma_f + V_x\psi_{xf}^{t+1}\sigma_f + (1-\beta)^{-1} Z^u\phi_{wf}\right)\right]^i\right\}\overline{\theta}.
\end{aligned}$$

together with equation (53) which remained unchanged.

C Data and further empirical evidence

Macroeconomic data are collected from the Federal Reserve Bank of St. Louis database (FRED).[14] The data on households' expectations are obtained from the University of Michigan Survey of Consumers.[15] We also use data from the Survey of Consumer Expectations administered by the Federal Reserve Bank of New York,[16] and data from the Survey of Professional Forecasters collected from the Federal Reserve Bank of Philadelphia website.[17] See Table 3 for details.

We use the Consumer Price Index for All Urban Consumers: All Items (CPIAUSCL in FRED) to compute the rate of inflation in the data. Computing the belief wedges using the Personal Consumption Index (PCE) from the Bureau of Economic Analysis as an alternative would leave the cyclical component of the inflation wedge almost unchanged because the two series are highly correlated. However, the PCE series has a substantially lower mean (by 0.4% annually between 1982 and 2015) so using PCE as observations from the data-generating process would make households appear to overestimate inflation significantly more than in the case of CPI. We prefer CPI because its weighting is based on surveys of the composition of households' purchases, and is based on out-of-pocket expenditures, which are arguably more salient for the formation of households' beliefs.

For the rate of unemployment, we use the Civilian Unemployment Rate (UNRATE in FRED) as the data counterpart. Since households in the Michigan Survey are asked about the *change* in the rate of unemployment, the potential issue with different average levels of alternative measures of unemployment that households could envision becomes irrelevant. We construct the level forecast as the realized UNRATE measure in the month when the forecast is made, plus the forecasted change in the unemployment rate from the Michigan Survey.

C.1 Survey data

For the inflation rate in the Michigan Survey, we record the cross-sectional mean, median and quartile answers. The survey question on unemployment rate only records up/same/down responses. We use the method from Carlson and Parkin (1975) and Mankiw et al. (2003) to fit a time series of normal distributions to these qualitative responses. Let q_t^u, q_t^s and q_t^d be the fractions of survey answers up, same, down,

[14] https://research.stlouisfed.org/fred2/

[15] http://www.sca.isr.umich.edu/ See also Thomas (1999) for details on the survey methodology.

[16] https://www.newyorkfed.org/microeconomics/sce

[17] https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/

Households' expectations (Michigan Survey)

$\widetilde{E}_t\left[\sum_{j=1}^{4}\pi_{t+j}\right]$	Expected change in prices during the next year (Table 32, variable PX1), mean and median responses and quartiles of the cross-sectional distribution of individual answers. Questions: "*During the next 12 months, do you think that prices in general will go up, or go down, or stay where they are now?*" and "*By about what percent do you expect prices to go up, on the average, during the next 12 months?*"
$\widetilde{E}_t\left[\frac{1}{n}\sum_{j=1}^{4}u_{t+j}\right]$	Expected unemployment rate during the next year (Table 30, variable UMEX), construction of mean response and the dispersion detailed in the text. Question: "*How about people out of work during the coming 12 months – do you think there will be more unemployment than now, about the same, or less?*" We also report results interpreting answers to this question as expected unemployment in one year, $\widetilde{E}_t\left[u_{t+4}\right]$.

Households' expectations (FRBNY Survey)

$\widetilde{E}_t\left[\sum_{j=1}^{4}\pi_{t+j}\right]$	Median one-year ahead expected inflation rate (used in Figure 2). The time series is constructed by aggregating probabilistic responses to question: "*In your view, what would you say is the percent chance that, over the next 12 months... the rate of inflation will be between x_i% and x_{i+1}%*" for a range of brackets across individual households. See Armantier et al. (2016) for details.
$\widetilde{P}_t[u_{t+4}]$	Probability of unemployment being higher in one year than today (used in Figure 2). Mean response to question: "*What do you think is the percent chance that 12 months from now the unemployment rate in the U.S. will be higher than it is now?*"

Survey of Professional Forecasters

$E_t\left[\sum_{j=1}^{4}\pi_{t+j}\right]$	Forecasted CPI inflation rate, seasonally adjusted (CPI). Forecast at time t is constructed as the mean survey forecast made in second month of quarter $t+1$, for CPI inflation rate between quarters t and $t+4$.
$E_t[\frac{1}{n}\sum_{j=1}^{4}u_{t+j}]$	Forecasted unemployment rate, seasonally adjusted (UNEMP). Forecast at time t is constructed as the mean survey forecast made in second month of quarter $t+1$, for the average unemployment rate in quarters $t+1$ to $t+4$.

Macroeconomic variables (FRED)

π_t	Consumer price index for all urban consumers: All items, seasonally adjusted (CPIAUCSL). Quarterly logarithmic growth rate, last month to last month of quarter.
u_t	Civilian unemployment rate, quarterly, seasonally adjusted (UNRATE).
$\log\left(\mathcal{Y}_t/\mathcal{Y}_{t-1}\right)$	Real gross domestic product, quarterly, seasonally adjusted annual rate (GDPC96). Quarterly logarithmic growth rate.
$\log\left(\mathcal{Y}_t/\bar{\mathcal{Y}}_t\right)$	Output gap. Difference between real gross domestic product, quarterly, seasonally adjusted annual rate (GDPC96) and real potential output (GDPPOT).

Table 3: Data definitions for key macroeconomic and survey variables.

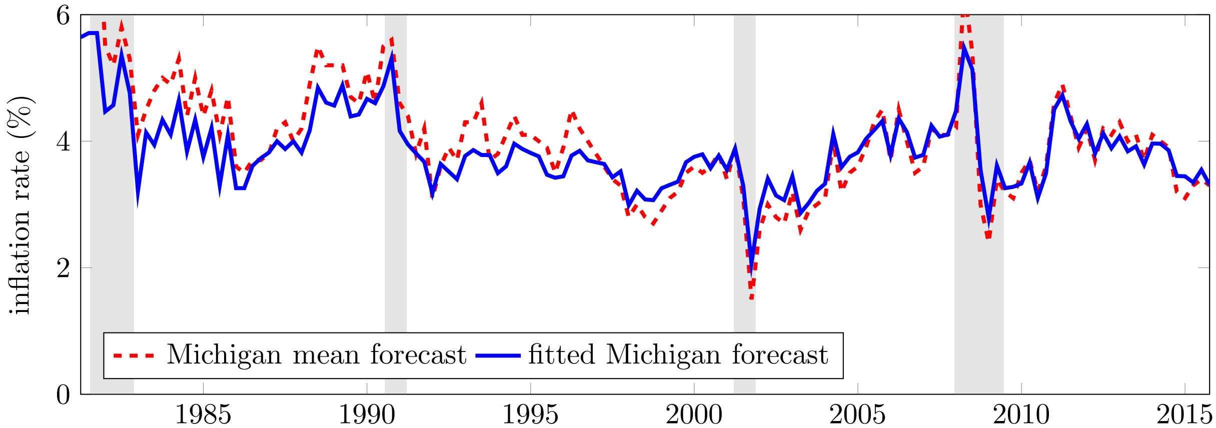

Figure 11: Mean one-year ahead inflation rate forecast in the Michigan Survey (red dashed) and the fitted mean forecast constructed using the Carlson and Parkin (1975) and Mankiw et al. (2003) method from categorical data. NBER recessions shaded.

respectively, recorded at time t. We assume that these categories are constructed from a continuous cross-sectional distribution of responses with normal density $N\left(\mu_t, \sigma_t^2\right)$. In particular, there exists a response threshold a such that an answer on the interval $[-a, a]$ is recorded as 'same'. This implies

$$q_t^d = \Phi\left(\frac{-a-\mu_t}{\sigma_t}\right) \qquad q_t^u = 1 - \Phi\left(\frac{a-\mu_t}{\sigma_t}\right)$$

and thus

$$-a-\mu_t = \sigma_t \Phi^{-1}\left(q_t^d\right) \qquad a-\mu_t = \sigma_t \Phi^{-1}\left(1-q_t^u\right)$$

and therefore

$$\begin{aligned} \sigma_t &= \frac{2a}{\Phi^{-1}\left(1-q_t^u\right) - \Phi^{-1}\left(q_t^d\right)} \\ \mu_t &= a - \sigma_t \Phi^{-1}\left(1-q_t^u\right) \end{aligned}$$

The constant a is then determined so that the time-series average of the cross-sectional dispersions σ_t divided by the observed average cross-sectional dispersion for the SPF forecast corresponds to the analogous ratio for the inflation responses, for which we have dispersion data readily available. We use the resulting means μ_t as the time series of mean unemployment rate forecasts.

In order to verify that the obtained time series μ_t provides a meaningful fit to the actual mean forecast, we verify the methodology using the inflation forecast data. We categorize individual numerical inflation forecast responses in each period into three bins, $< 3\%$, $3-5\%$ and $> 5\%$, and then fit a time series of normal distributions as described above, using the three time series of answer shares in each of the bins as input. Figure 11 compares the time series of actual mean forecasts with the time series of fitted means constructed using categorical data. The correlation between the two series is 92.8% and the time series averages differ only by 0.12%, providing strong support for the methodology as a plausible approximation of the actual mean forecast.

C.2 Information sets

The construction of belief wedges requires taking a stance on how to align information sets available to surveyed households and the econometrician. We use a quarterly VAR for our baseline forecast under the data-generating (rational) measure. The Michigan Survey contains aggregated data at the quarterly frequency starting from 1960. We use these quarterly time series for the time period 1960Q3–2015Q4 in Figure 1 and Panel A of Table 4. We use the responses reported during quarter $t+1$ as those made with information available to the households at the end of quarter t. The forecasting horizon is assumed to span quarters $t+1$ to $t+4$.

We use monthly data from the Michigan Survey and available micro data from the monthly cross-sections of the survey for the period 1982Q1–2015Q4. When computing the belief wedges relative to the VAR forecast, we use responses from the first month of quarter $t+1$ as those made by households with information available at the end of quarter t. Time series moments for the wedges in this sample are summarized in Panel B of Table 4.

The Survey of Professional Forecasters is administered during the second month of each quarter. To compute the belief wedge relative to the SPF forecast, we therefore use Michigan Survey responses from the second month of each quarter as well to align information sets for the two forecasts. We again use the time period 1982Q1–2015Q4. Forecasts made in the second month of quarter $t+1$ are assumed to span quarters $t+1$ to $t+4$ in the quarterly analysis. Panel C of Table 4 summarizes the data.

C.3 Forecasting VAR

We use a standard quarterly forecasting VAR to compute the forecasts of inflation and unemployment under the data-generating measure. All time series are downloaded from FRED for the period 1960Q1–2015Q4: CPI inflation (CPIAUCSL, percent change to a year ago), real GDP (nominal series GDP divided by GDP deflator GDPCTPI, annualized percentage quarterly change), unemployment rate (UNRATE), log change in the relative price of investment goods (PIRIC), capital utilization rate (CUMFNS), hours worked (HOANBS), consumption rate ((PCDG+PCEND+PCESV)/GDP), investment rate (GPDI/GDP) and the Federal Funds rate (FEDFUNDS). The VAR is estimated with two lags. These choices for the forecasting VAR are similar to those made in Christiano et al. (2005), Del Negro et al. (2007) or Christiano et al. (2011). We experimented by increasing the lag number up to four, and by adding labor market variables as in Christiano et al. (2016) and all these choices do not materially change results.

C.4 Further time-series evidence on the belief wedges

Figure 1 in the main text and Panel A from Table 4 contain time-series characteristics of the belief wedges from the Michigan Survey constructed using survey data for the period 1982Q1–2015Q4, net of the corresponding VAR forecasts. This is our preferred time period because the Michigan Survey for this period contains better quality disaggregated survey data at monthly frequency that allow us to better align information sets (Appendix C.2), study the cross-sectional patterns between the belief wedges, as well as compare the Michigan Survey responses with available SPF forecasts.

We use the Michigan Survey responses aggregated at quarterly frequency for the period 1960Q1–2015Q4 as a robustness check. These results are reported in Panel B from Table 4. The patterns in the data are largely unchanged (information on the median inflation forecast is not available in the Michigan Survey for this time period). The belief wedges continue to be large, volatile, and countercyclical. The mean inflation wedge is somewhat smaller than in Panel A, and the lower correlation between output gap and GDP growth

Panel A: 1982Q1–2015Q4, VAR forecast			correlation matrix					
	mean	std	(1)	(2)	(3)	(4)	(5)	(6)
(1) Unemployment wedge $\Delta_t^{(4)}(u)$	0.58	0.54	1.00	0.87	0.23	0.21	−0.54	−0.32
(2) Unemployment wedge $\overline{\Delta}_t^{(4)}(u)$	0.54	0.45		1.00	0.20	0.22	−0.29	−0.43
(3) Mean inflation wedge $\Delta_t^{(4)}(\pi)$	1.25	1.03			1.00	0.94	−0.37	−0.53
(4) Median inflation wedge $\Delta_t^{(4)}(\pi)$	0.43	1.14				1.00	−0.32	−0.60
(5) Output gap $\log\left(\mathcal{Y}_t/\bar{\mathcal{Y}}_t\right)$	−1.75	1.93					1.00	0.61
(6) GDP growth $\log\left(\mathcal{Y}_t/\mathcal{Y}_{t-4}\right)$	2.67	2.03						1.00

Panel B: 1960Q2–2015Q4, VAR forecast			correlation matrix					
	mean	std	(1)	(2)	(3)	(4)	(5)	(6)
(1) Unemployment wedge $\Delta_t^{(4)}(u)$	0.43	0.63	1.00	0.89	0.17	—	−0.49	0.00
(2) Unemployment wedge $\overline{\Delta}_t^{(4)}(u)$	0.43	0.44		1.00	0.24	—	−0.40	−0.16
(3) Mean inflation wedge $\Delta_t^{(4)}(\pi)$	0.78	1.17			1.00	—	−0.49	−0.56
(4) Median inflation wedge $\Delta_t^{(4)}(\pi)$	—	—				—	—	—
(5) Output gap $\log\left(\mathcal{Y}_t/\bar{\mathcal{Y}}_t\right)$	−1.00	2.29					1.00	0.32
(6) GDP growth $\log\left(\mathcal{Y}_t/\mathcal{Y}_{t-4}\right)$	2.97	3.30						1.00

Panel C: 1982Q1–2015Q4, SPF forecast			correlation matrix					
	mean	std	(1)	(2)	(3)	(4)	(5)	(6)
(1) Unemployment wedge $\Delta_t^{(4)}(u)$	0.55	0.49	1.00	0.97	0.18	0.21	−0.38	−0.60
(2) Unemployment wedge $\overline{\Delta}_t^{(4)}(u)$	0.48	0.47		1.00	0.16	0.22	−0.18	−0.53
(3) Mean inflation wedge $\Delta_t^{(4)}(\pi)$	1.07	0.85			1.00	0.94	−0.14	−0.29
(4) Median inflation wedge $\Delta_t^{(4)}(\pi)$	0.43	1.14				1.00	−0.32	−0.60
(5) Output gap $\log\left(\mathcal{Y}_t/\bar{\mathcal{Y}}_t\right)$	−1.75	1.93					1.00	0.61
(6) GDP growth $\log\left(\mathcal{Y}_t/\mathcal{Y}_{t-4}\right)$	2.67	2.03						1.00

Table 4: Time-series and business cycle statistics for the belief wedges. *Panel A*: Belief wedge relative to a VAR forecast, time period 1982Q1–2015Q4. *Panel B*: Belief wedge relative to a VAR forecast, time period 1960Q2–2015Q4 (median inflation forecast not available for this period). *Panel C*: Belief wedge relative to the SPF forecast, time period 1982Q1–2015Q4. For details see Appendix C.4.

implies that the wedges continue to be strongly countercyclical when using the output gap as the measure of economic activity but the relationship with GDP growth is weaker.

Finally, we also construct the belief wedges using the responses from the Survey of Professional Forecasters as a measure of forecasts under the data-generating measure. The inflation and unemployment wedges constructed this way are plotted in Figure 12. Panel C from Table 4 provides the time-series characteristic for these wedges. As in the previous cases, we obtain large and volatile belief wedges that are highly negatively correlated with the business cycle.

In all three panels, we report alternative specifications for the wedges. For the inflation wedge, we show the results for the mean and median inflation forecast for the Michigan Survey. For the unemployment wedge, we produced two wedges based on alternative interpretations of the relevant question in the Michigan Survey. The wedge $\Delta_t^{(4)}(u)$ is the wedge for the forecast of the unemployment rate four quarters ahead. The wedge $\overline{\Delta}_t^{(4)}(u)$ is the wedge for the forecast of the average unemployment rate during the next four quarters.

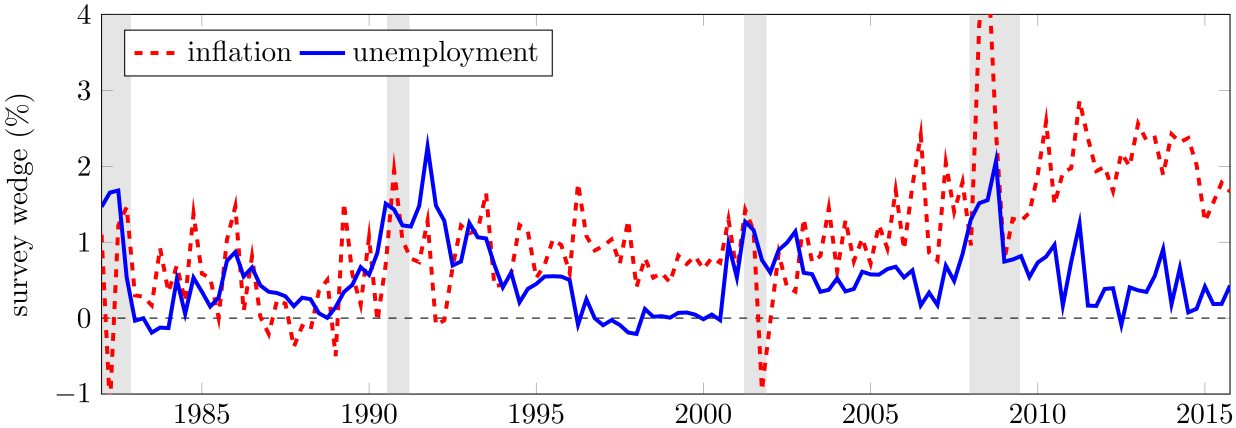

Figure 12: Difference between the mean one-year ahead forecasts from the Michigan Survey and corresponding SPF forecasts. Details on the construction of the data series are in Appendix C. NBER recessions shaded.

C.5 Further cross-sectional evidence on the belief wedges

In this section, we provide further evidence on the cross-sectional relationship between household-level survey answer biases for alternative questions, documented in the Michigan Survey and the FRBNY Survey.

For most of the cross-sectional analysis, we do not convert unemployment responses using the procedure described in Appendix C.1 but encode categorical household-level responses on the forecasted change in unemployment rate {down, same (or don't know), up} for household i in demographic group g and month t as $\widetilde{u}_{i,g,t} \in \{-1, 0, 1\}$, respectively. We drop respondents aged 65 and above and those with missing responses. Group-level averages $\widetilde{u}_{g,t}$ then represent the share of respondents who forecast an increase in unemployment minus the share that forecasts a decrease. For the inflation responses, we drop households who indicate 'don't know', have a missing response, or have extreme forecasts (above 20% or below −10%). The results are robust to keeping the extreme forecasts.

Table 5 reports the conditional time-series averages of the households' forecasts for different demographic groups in the Michigan Survey, displayed in Figure 5. While more educated respondents and respondents with higher incomes overpredict inflation and unemployment less on average, all demographic groups still overpredict both quantities. Moreover, demographic groups which on average overpredict inflation relatively more also overpredict unemployment relatively more.

Tables 6–8 provide further detail at the level of demographic groups and individual households that establishes the relationship between the survey responses about inflation and unemployment.

First, we ask whether in times when demographic group g on average overpredicts inflation more relative to population, the group also overpredict unemployment more relative to population. Table 6 summarizes the regression coefficients in regressions of the form

$$\widetilde{u}_{g,t} - \widetilde{u}_t = \alpha_g + \beta_g \left[\widetilde{\pi}_{g,t} - \widetilde{\pi}_t\right] + \varepsilon_{g,t}. \tag{56}$$

where $\widetilde{u}_{g,t}$, $\widetilde{\pi}_{g,t}$ are the average forecasts of demographic group g in month t, and $\widetilde{u}_t$, $\widetilde{\pi}_t$ are the average forecasts in month t for the whole population. The estimated regression coefficients $\widehat{\beta}_g$ are all positive and most of them are highly statistically significant.

Next, we investigate whether in times when *individual households* i overpredict inflation more relative to

	actual	SPF	all	18-34	35-44	45-54	55-64	W	NC	NE	S
π	2.69	2.91	3.96	4.03	3.94	3.91	3.81	3.87	3.90	3.91	4.09
u	6.22	6.29	6.83	6.69	6.85	6.92	6.93	6.81	6.83	6.88	6.81
u share	—	—	16.5	12.0	17.0	19.5	19.3	15.7	16.8	18.0	16.0
		male	female	bottom	2nd Q	3rd Q	top	HS	SC	COL	GS
π		3.38	4.46	4.90	4.20	3.67	3.17	4.50	3.89	3.46	3.34
u		6.74	6.90	6.98	6.85	6.80	6.68	6.92	6.83	6.72	6.73
u share		12.9	19.6	21.3	17.6	15.6	11.2	20.0	16.9	12.7	12.7

Table 5: Demographic characteristics of households' expectations on inflation (π) and unemployment rate. 'u share' is the percentage share of responses that unemployment rate will increase minus the percentage share stating that unemployment rate will decrease. u is the average fitted unemployment rate forecast computed as in Appendix C.1. Time-series averages, all values are annualized and in percent, time period 1982Q1–2015Q4. *Actual*: actual average inflation and unemployment rate; *SPF*: average SPF forecast; *all*: average household forecast; *18-34 etc*: age groups; *W*: West region; *NC*: North-Central; *NE*: North-East; *S*: South; *bottom, 2nd Q, 3rd Q, top*: income quartiles; *HS*: high-school education; *SC*: some college; *COL*: college degree; *GS*: graduate studies.

the population, they also overpredict unemployment relatively more. The regression on the pooled sample with demographic controls yields

$$\widetilde{u}_{i,g,t} - \widetilde{u}_t = \alpha + \beta \left[\widetilde{\pi}_{i,g,t} - \widetilde{\pi}_t\right] + \varepsilon_{i,g,t}.$$

where $\widetilde{u}_{i,g,t}$, $\widetilde{\pi}_{i,g,t}$ are the forecasts of household i belonging to demographic group g in month t. The regression includes controls for demographic characteristics from Table 5. The estimated slope coefficient is $\widehat{\beta} = 2.08$ with a standard error of 0.04. We also run the same regression using differences between individual household forecasts and the group-specific average in the given month, instead of the population average:

$$\widetilde{u}_{i,g,t} - \widetilde{u}_{g,t} = \alpha_g + \beta_g \left[\widetilde{\pi}_{i,g,t} - \widetilde{\pi}_{g,t}\right] + \varepsilon_{i,g,t} \tag{57}$$

Table 7 reports the estimates of regression coefficients $\widehat{\beta}_g$.

Finally, we run the regressions month-by-month and for each demographic sorting (education, income, regions, age, sex):

$$\widetilde{u}_{i,g,t} - \widetilde{u}_{g,t} = \alpha_{g,t} + \beta_{g,t} \left[\widetilde{\pi}_{i,g,t} - \widetilde{\pi}_{g,t}\right] + \varepsilon_{i,g,t} \tag{58}$$

This is a variant of regression (1) where household-level forecasts are compared to the average within their demographic group rather than to the population average. Table 8 shows the mean and standard deviation of the *distribution* of estimated coefficients $\widehat{\beta}_{g,t}$. The column 'population' shows the same statistics for the population regressions (1), and Figure 13 plots the t-statistics for the $\widehat{\beta}_t$ coefficients.

Essentially all the monthly coefficients and their significance reported in Table 8 and Figure 13 confirm the positive relationship between the bias in the unemployment forecast and inflation forecast. Around 95% of all the estimated coefficients $\widehat{\beta}_{g,t}$ are positive and about two thirds of them have a t-statistic larger than 1.96. Figure 13 also shows that the significantly positive cross-sectional relationship between the belief wedges is not specific to a particular subperiod in the data.

We now turn to the evidence in the Survey of Consumer Expectations administered by the Federal Reserve Bank of New York (FRBNY Survey). Finally, we can also look at the NYFRB data. Table 9 reports the cross-sectional correlations for the following survey questions about forecasts of aggregate and

	18-34	35-44	45-54	55-65	W	NC	NE	S
$100 \times \widehat{\beta}_g$	3.37	2.32	2.16	2.92	2.89	1.57	1.98	4.67
std. err.	0.92	0.82	0.81	0.74	0.94	0.91	0.86	0.89

	male	female	bottom	2nd Q	3rd Q	top	HS	SC	COL	GS
$100 \times \widehat{\beta}_g$	3.95	4.52	0.56	0.72	2.97	0.85	4.41	5.50	2.60	5.50
std. err.	1.16	1.18	0.83	0.87	0.88	1.09	0.92	0.85	0.92	1.08

Table 6: Regression coefficients in regression (56) for alternative demographic groups g, listed in the caption of Table 5. $100 \times \widehat{\beta}_g$ scales the left-hand side in the regression to percentage shares.

	population	education	income	region	age	sex
$100 \times \widehat{\beta}_g$	2.19	2.15	2.14	2.19	2.20	2.12
std. err.	0.04	0.04	0.04	0.04	0.04	0.04

Table 7: Regression coefficients in regression (57) for alternative demographic groups, listed in the caption of Table 5. $100 \times \widehat{\beta}_g$ scales the left-hand side in the regression to percentage shares.

	population	education	income	region	age	sex
average $100 \times \widehat{\beta}_{g,t}$	2.32	2.27	2.27	2.32	2.33	2.26
std. dev. $100 \times \widehat{\beta}_{g,t}$	1.39	1.33	1.37	1.38	1.38	1.39
months	408	408	408	408	408	408
$\# \ t > 0$	392	393	396	394	392	389
$\# \ t > 1.96$	266	265	260	269	270	260

Table 8: Regression coefficients in regression (58) for alternative demographic groups, listed in the caption of Table 5. 'Months' indicates the number of monthly regressions we run in each case, and $\# \ t > 0$ and $\# \ t > 196$ indicate the number of regressions from that sample in which the estimate $\widehat{\beta}_{g,t}$ has a t-statistic large than zero or 1.96, respectively. $100 \times \widehat{\beta}_g$ scales the left-hand side in the regression to percentage shares.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1) inflation	1.00	0.27	0.78	0.87	0.85	0.56	0.70
(2) unemployment		1.00	0.21	0.29	0.11	0.49	0.22
(3) stock prices (−)			1.00	0.90	0.64	0.44	0.66
(4) earnings growth (−)				1.00	0.82	0.50	0.80
(5) income growth (−)					1.00	0.45	0.85
(6) job loss						1.00	0.58
(7) job finding (−)							1.00

Table 9: Cross-sectional correlations for responses in the pooled sample from the FRBNY Survey, 2013M06–2016M09. See text for details on individual questions. (−) denotes variable taken with a negative sign.

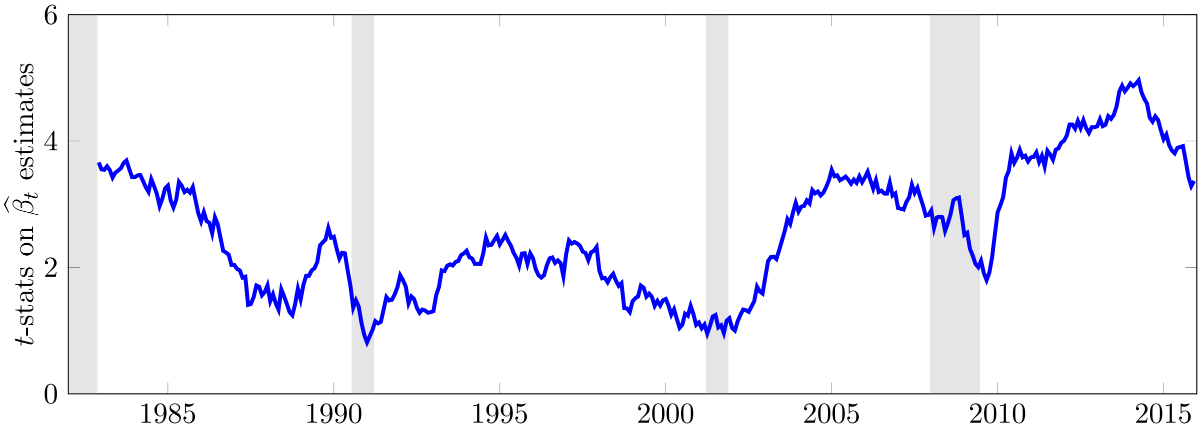

Figure 13: Smoothed (12-month moving average) t-statistics on the estimates $\widehat{\beta}_t$ in regression (1).

household-level variables (for more detail on the survey design and questions, see Armantier et al. (2016)):

(1) expected rate of inflation over the next 12 months;

(2) percent chance that 12 months from now the unemployment rate in the U.S. will be higher than it is now;

(3) percent chance that 12 months from now, on average, stock prices in the U.S. stock market will be higher than they are now;

(4) expected percent increase in individual earnings over the next 12 months conditional on staying in the same job;

(5) expected percent increase of total household income over the next 12 months;

(6) percent chance of losing job in the next 12 months;

(7) percent chance of finding a job in next three months conditional on losing job today

Variables that are positively correlated with the notion of 'good times' are taken with opposite signs. The first three variables represent forecasts of macroeconomic variables while the remaining four refer to households' individual outcomes. Table 9 shows that the correlations are all positive and mostly quite large, indicating that households who forecast higher inflation are also generally more pessimistic about aggregate and individual outcomes. Notice that stock prices, earnings and income growth are nominal variables, so the pessimism about real quantities for households who forecast higher inflation is even stronger. These results corroborate and extend our findings from the cross-sectional analysis of the Michigan Survey.

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