A Dynamic Program for Valuing Corporate Securities

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Abstract

We design and implement a dynamic program (DP) for valuing corporate securities, seen as derivatives on a firm's assets, and computing the term structure of yield spreads and default probabilities. Our setting accommodates arbitrary corporate debts, multiple seniority classes, payouts, tax benefits, bankruptcy costs, and a reorganization process. This flexibility comes at the expense of a minor loss of efficiency; the analytical approach proposed in the literature is exchanged here for a quasi-analytical approach based on dynamic programming coupled with finite elements. We provide several theoretical properties of the debt- and equity-value functions. Finally, to assess our construction, we carry out a numerical investigation along with a complete sensitivity analysis. DP shows flexibility and efficiency.

Key words: Option theory, no-arbitrage pricing, structural models, corporate securities, corporate bonds, dynamic programming, finite elements.

1 Introduction

The aim of this paper is to design and implement a dynamic programming framework for valuing arbitrary corporate-bond portfolios and computing the term structure of their yield spreads and default probabilities. This program, based on the structural model, is flexible and efficient. Moving beyond an academic exercise, we endeavor offer a *realistic* setting for analyzing corporate credit risk. A corporate bankruptcy results in a loss of value for the firm's claimholders and a loss of positions for the firm's workers. Corporate credit-risk models are thus useful for market participants in that they help preclude financial distress and its adverse events.

The option-based approach for valuing corporate bonds goes back to Merton (1974). He considers a model for a firm with a simple capital structure with a pure bond and a common stock (equity). The author then interprets the stock as a European call option on the firm's assets, whose value follows a geometric Brownian motion (GBM), as set by Black and Scholes (1973). The option's expiry date and strike price are the bond's maturity date and principal amount, respectively. Holding the pure bond is equivalent to holding the entire firm and to selling a European call option to shareholders in order to buy the firm at the bond's maturity for the bond's principal amount.

Merton's (1974) pioneering paper has given rise to an extensive literature, known as the *structural model*, where the values of a firm's debt and equity are expressed as functions of time and the firm's asset value. The default event at a given payment date occurs when the state variable falls under a certain *default barrier*. The key attractive point of the structural model is that the (unobserved) asset value is inferred from the (observed) equity value and the nominal debt structure. The extensions to Merton's paper in the literature are twofold.

The first research stream solves the model in closed form. The barriers considered are exogenous (Black and Cox 1976, Longstaff and Schwartz 1995, Ericsson and Reneby 1998, Collin-Dufresne and Goldstein 2001, Abinzano et al. 2009, and Hsu, Saà-Requejo, and Santa-Clara 2010) or endogeneous (Black and Cox 1976, Geske 1977, Leland 1994, Anderson, Sundaresan, and Tychon 1996, Leland and Toft 1996, Mella-Barral and Perraudin 1997, François and Morellec 2004, Nivorozhkin 2005a and 2005b, Chen and Kou 2009, Fan and Sundaresan 2000, Bruche and Naqvi 2010, and Shibata and Tian 2012). The latter sub-family of papers assumes simple debt structures with a very limited number of endogenous barriers (one to two in all cases).

The second research stream refers to numerical methods: numerical integration (Moraux 2004), finite differences (Brennan and Schwartz 1978 and Anderson, Sundaresan, and Tychon 1996), binomial trees (Anderson and Sundaresan 1996, Broadie, Chernov, and Sundaresan 2007, and Broadie and Kaya 2007), dynamic programming (Annabi, Breton, and François 2012a and 2012b), and Monte Carlo simulation (Zhou 2001 and Galai, Raviv, and Wiener 2007). Some of these papers accommodate general debt structures (Anderson and Sundaresan 1996, Anderson, Sundaresan and Tychon 1996, Broadie, Chernov, and Sundaresan 2007, and Brodie and Kaya 2007). The others develop under the particular settings of Black and Cox (1976) or Leland (1994).

On the one hand, closed-form solutions, where available, are obviously preferred to approximations. They are extremely efficient; they assure the highest accuracy at a very low computing time. Closed-form solutions explicitly link the unknown parameters to their input parameters and, thus, allow for a direct sensitivity analysis. However, they rely on very simplified assumptions. On the other hand, more realistic models are solved by means of numerical procedures. Our dynamic program is an acceptable compromise in terms of flexibility and efficiency.

Among other objectives, the structural model attempts to explain the observed yield spreads and default frequencies. Despite its parsimony, the simplest structural model (Merton 1974) compares extremely well to the classic statistical approach for bankruptcy prediction (Hillegeist et al. 2004), and, to a lesser extent, to the neural-network approach (Aziz and Dar 2006). A hybrid approach can be developed, where some of the statistical risk factors are inferred from the structural model, e.g. the distance to default (Benos and Papanastasopoulos 2007). More complex structural models have further explained the observed yield spreads and default frequencies (Collin-Dufresne and Goldstein 2001, Delianedis and Geske 2001 and 2003, Huang and Huang 2012, Leland 2004, and Suo and Wang 2006). According to Delianedis and Geske (2001), the most important components of credit risk are default, recovery, tax benefits, jumps, liquidity, and market factors.

Black and Cox (1976) extend Merton's model to a corporate-bond portfolio made of a pure senior bond and a pure junior bond. Geske (1977) uses the theory of compound options, and further extends Merton's model to arbitrary corporate-bond portfolios. However, his analytical approach remains questionable when the number of coupon dates is high. Leland (1994) considers an annuity, which results in a constant default barrier over time. Then, by maximizing the present value of equity, he solves for the so-called *endogenous default barrier*. He considers tax benefits under the survival event and bankruptcy costs under the default event. These frictions allow Leland (1994) to discuss the notions of maximum debt capacity and optimal capital structure. The latter is a break-down of the Modigliani-Miller conjecture, which states that, in pure and perfect capital markets, the firm's asset value is independent of its capital structure.

We propose a dynamic-programming framework for valuing arbitrary corporate debts, seen as derivatives on a firm's assets, and computing the term structure of yield spreads and default probabilities. Our setting extends the models of Merton (1974), Black and Cox (1976), Geske (1977) and Leland (1994), for it accommodates arbitrary corporate debts, multiple seniority classes, payouts, tax benefits, bankruptcy costs, and a reorganization process. The default barriers inferred at payment dates are completely endogenous, and follow from an optimal decision process. These extensions come at the expense of a minor loss of efficiency. The analytical approach of these authors is exchanged here for a quasi-analytical approach based on dynamic programming coupled with finite elements.

This paper is organized as follows. Section 2 presents the model and provides several properties of the debt- and equity-value functions, and Section 3 solves the dynamic program. Section 4 proposes a reorganization process. Section 5 is a numerical investigation, which replicates reported results from the literature and carries out a complete sensitivity analysis. Section 6 concludes.

2 Model and notation

Consider a public company with the following capital structure: a portfolio of senior and junior bonds and a residual claim, that is, a common stock (equity). Let $\mathcal{P} = \{t_1, \ldots, t_n, \ldots, t_N = T\}$ be a set of payment dates, and $t_0 = 0$ be the origin. At time $t_n \in \mathcal{P}$, the firm is committed to paying $d_n^s + d_n^j = d_n > 0$ to its creditors (bondholders), where d_n^s and d_n^j are the outflows generated at t_n by the senior and junior bonds, respectively. The total outflow d_n includes interest as well as principal payments. The interest payments are indicated by d_n^{int}. The amounts d_n^s, d_n^j, and d_n^{int}, for $n = 1, \ldots, N$, are known to all investors from the very beginning. The last payment dates of the senior and junior debts, both in $\mathcal{P}$, are indicated by T^s and T^j, respectively.

Several authors consider a senior coupon bond and a junior coupon bond with a longer maturity, that is, $0 \leq T^s < T^j = T$. Senior bondholders are therefore assured payment before junior bondholders. This realistic case is embedded in our setting.

For $t \in [0, T]$, the (present) value of the firm's assets, tax benefits, bankruptcy costs, senior debt, junior debt, and equity are indicated by $A_t = a$, $\mathrm{TB}_t(a)$, $\mathrm{BC}_t(a)$, $D_t^s(a)$, $D_t^j(a)$, and $S_t(a)$, respectively. The total debt value is indicated by $D_t(a) = D_t^s(a) + D_t^j(a)$. These quantities are interpreted herein as financial derivatives on the firm's assets. Tax benefit, as a claim, is characterized by a cash-flow stream of $\mathrm{tb}_n = r_n^c d_n^{\mathrm{int}}$, under survival at t_n. The rate $r_n^c \in [0, 1]$ is the periodic corporate tax rate over $[t_n, t_{n+1}]$, for $n = 0, \ldots, N-1$; it is considered as a known constant. Bankruptcy cost, as a claim, is characterized by a cash flow of wA_τ under default, where τ is the hitting time at which the firm defaults. The proportion w can be interpreted as a *write-down* or a *loss-severity ratio*, and $1 - w$ as a *recovery rate*; it is considered a known constant.

The *state process* $\{A\}$, the firm's asset value, is an exogenous, strictly positive, and Markov process, for which the following transition parameters are supposed to be known in closed form:

$$\begin{aligned} T^0_{abc\Delta} &= E^* \left[\mathbb{I} \left(b < A_u \leq c \right) \mid A_t = a \right] \\ &= P^* \left(A_u \in (b, c] \mid A_t = a \right), \end{aligned} \tag{1}$$

and

$$T^1_{abc\Delta} = E^* \left[A_u \mathbb{I} \left(b < A_u \leq c \right) \mid A_t = a \right], \tag{2}$$

where $0 \leq t \leq u \leq T$, $\Delta = u - t$, and a, b, and $c \in \mathbb{R}_+^*$. Here, $E^* [. \mid A_t = a]$ represents the conditional expectation symbol under the risk-neutral probability measure $P^*(.)$, and $\mathbb{I}(.)$ the indicator function. For time-heterogenous Markov processes, the transition parameters also depend on t. These truncated moments can be seen as the minimum information required for the Markov process $\{A\}$ to play the role of a state process.

The conditions (1)–(2) accommodate a large family of pure-diffusion, jump-diffusion, and more general Markov processes. Examples include the GBM, the GBM coupled with Poisson jumps, and the GARCH processes. See Ben-Ameur, Breton, and L'Écuyer (2002) and Ben-Ameur, Breton, and François (2006) for valuing American-style Asian and installment options under the GBM assumption, respectively. Other Markov processes can be used along the same lines with some modifications. See Ben-Ameur, Chérif, and Remillard (2012) for valuing options under a jump-diffusion process, and Ben-Ameur, Breton, and Martinez (2009) under the family of Gaussian GARCH processes. Ben-Ameur et al. (2007) use a stochastic interest-rate model for valuing call and put options embedded in bonds. For simplicity, we focus herein on the GBM assumption, for which $\{A\}$ is characterized by

$$A_u = A_t e^{\left(r-\delta-\sigma^2/2\right)(u-t)+\sigma\sqrt{u-t}Z}, \quad \text{for } 0 \leq t \leq u \leq T,$$

where r is the risk-free rate, δ the firm's payout ratio, σ the volatility of the firm's asset value, and Z a standard normal random variable independent of the past of $\{A\}$ till time t. These parameters are considered as known positive constants.

The model builds on the following economic balance-sheet equality:

$$a + \mathrm{TB}_t(a) - \mathrm{BC}_t(a) = D_t(a) + S_t(a), \quad \text{for } t \in [0, T], \tag{3}$$

where $a = A_t$. Consistently with Leland (1994), the quantity at the left-hand side of eq. (3) is called the *total value of the firm*. Brennan and Schwartz (1978) consider a balance-sheet equality where the present values of dividends, tax benefits, and bankruptcy costs are exchanged for their associated current payoffs at payment dates. They solve the model numerically for a coupon bond and an exogenous default barrier. Nivorozhkin (2005a and 2005b) considers a senior and a junior discount bonds and bankruptcy costs in a one-period model à la Merton (1974) and à la Black and Cox (1976). Then, he solves for eq. (3) in closed form. Leland (1994) considers a perpetuity, and solves for eq. (3) and for its endogenous unique default barrier in closed form. Geske (1977) considers a coupon bond, and shows how compound options can be used to solve for eq. (3) in closed form, while this theory is useless for large N. These models are nested in our construction. We consider arbitrary senior and junior debts, and solve for eq. (3) and for its endogenous (many) default barriers in quasi-closed form.

We herein enforce the *strict priority rule* under default; however, other sharing rules between claimholders can be easily introduced.

Proposition 1 *All value functions and decisions at time $t \in [0, T]$ depend on (t, a), where $a = A_t$, and verify the balance-sheet equality in eq. (3). The default event at time $t_n \in \mathcal{P}$ is in the form $\{a \leq b_n^*\}$, where $a = A_{t_n}$. The default barriers b_n^*, for $n = 1, \ldots, N$, are inferred from an optimal decision process, and maximize the equity value. They are rightly named the endogenous default barriers.*

Proof. We propose a proof by induction. First, we show that the property holds at t_N. Then, we assume that the same property holds at a certain future date t_{n+1}, and show that it holds at $t \in (t_n, t_{n+1})$, then at t_n.

Consider the following cases at t_N that result from eq. (3):

Case 1: The firm survives at time $t_N = T$, that is,

$$S_{t_N} > 0 \quad \text{or} \quad a > d_N - \text{tb}_N = b_N^*, \tag{4}$$

where $a = A_{t_N}$. One has

$$\begin{aligned} \text{TB}_{t_N}(a) &= \text{tb}_N, \quad \text{BC}_{t_N}(a) = 0, \\ D_{t_N}^s(a) &= d_N^s, \quad D_{t_N}^j(a) = d_N^j, \\ S_{t_N}(a) &= a + \text{tb}_N - D_{t_N}(a) \\ &= a + \text{tb}_N - d_N > 0. \end{aligned}$$

Senior and junior bondholders are fully paid; whatever remains belongs to equityholders.

Case 2: The firm defaults at time $t_N = T$, that is,

$$a \leq b_N^*, \tag{5}$$

where $a = A_{t_N}$. One has

$$\begin{aligned} \text{TB}_{t_N}(a) &= 0, \quad \text{BC}_{t_N}(a) = wa, \\ D_{t_N}^s(a) &= \min\left(a(1-w), d_N^s\right), \\ D_{t_N}^j(a) &= \max\left(0, a(1-w) - D_{t_N}^s(a)\right) \\ S_{t_N}(a) &= 0. \end{aligned}$$

Senior bondholders are partially paid and junior bondholders are not when $D_{t_N}^s(a) = a(1-w)$, that is, $a(1-w) < d_N^s$. The former are fully paid and the latter are partially paid when $D_{t_N}^s(a) = d_N^s$, that is, $a(1-w) \geq d_N^s$. Clearly, all value functions at maturity are functions of $A_{t_N} = a$, and the balance-sheet equality in eq. (3) holds in all cases. Combining both cases, we can interpret the stock as a call option on the firm's assets as in Merton (1974), that is, $S_{t_N}(a) = \max\left(0, a - (d_N - \text{tb}_N)\right)$, for all $a > 0$.

Suppose now that $\text{TB}_{t_{n+1}}(.)$, $\text{BC}_{t_{n+1}}(.)$, $D_{t_{n+1}}^s(.)$, $D_{t_{n+1}}^j(.)$, and $S_{t_{n+1}}(.)$ are functions of $A_{t_{n+1}}$ at a certain future date t_{n+1}, and that the balance-sheet equality in eq. (3) holds. For $t \in (t_n, t_{n+1})$, and, consequently, just after the payment date t_n, no-arbitrage pricing gives

$$\begin{aligned}
\mathrm{TB}_{t_n^+}\left(a'\right) &= E^*\left[\rho \mathrm{TB}_{t_{n+1}}\left(A_{t_{n+1}}\right) \mid A_{t_n^+} = a'\right] \\
\mathrm{BC}_{t_n^+}\left(a'\right) &= E^*\left[\rho \mathrm{BC}_{t_{n+1}}\left(A_{t_{n+1}}\right) \mid A_{t_n^+} = a'\right] \\
D^s_{t_n^+}\left(a'\right) &= E^*\left[\rho D^s_{t_{n+1}}\left(A_{t_{n+1}}\right) \mid A_{t_n^+} = a'\right] \\
D^j_{t_n^+}\left(a'\right) &= E^*\left[\rho D^j_{t_{n+1}}\left(A_{t_{n+1}}\right) \mid A_{t_n^+} = a'\right] \\
S_{t_n^+}\left(a'\right) &= E^*\left[\rho S_{t_{n+1}}\left(A_{t_{n+1}}\right) \mid A_{t_n^+} = a'\right],
\end{aligned} \tag{6}$$

where $\rho = e^{-r(t_{n+1}-t_n)}$ is the discount factor over (t_n, t_{n+1}). Eq. (6) implicitly assumes that the firm survives for all $t \in (t_n, t_{n+1})$ and $a = A_t > 0$, given survival until time t_n, which is obviously true since

$$P^*\left(S_{t_{n+1}}\left(A_{t_{n+1}}\right) > 0 \mid A_t = a\right) > 0.$$

Given the properties of the conditional expectation operator, all value functions at t_n^+ depend on $a' = A_{t_n^+}$. The martingale property of the discounted state process $\{e^{-rt}A_t\}$ reduces to the balance-sheet equality in eq. (3) at time $t \in (t_n, t_{n+1})$, and, consequently, at t_n^+ :

$$\begin{aligned}
& E^*\left[\rho\left(A_{t_{n+1}} + \mathrm{TB}_{t_{n+1}}\left(A_{t_{n+1}}\right) - \mathrm{BC}_{t_{n+1}}\left(A_{t_{n+1}}\right)\right) \mid A_{t_n^+} = a'\right] \\
= \; & E^*\left[\rho\left(D_{t_{n+1}}\left(A_{t_{n+1}}\right) + S_{t_{n+1}}\left(A_{t_{n+1}}\right)\right) \mid A_{t_n^+} = a'\right],
\end{aligned}$$

which is equivalent to

$$a' + \mathrm{TB}_{t_n^+}\left(a'\right) - \mathrm{BC}_{t_n^+}\left(a'\right) = D_{t_n^+}\left(a'\right) + S_{t_n^+}\left(a'\right). \tag{7}$$

Under survival at t_n, eq. (7) becomes

$$\begin{aligned}
& a + \mathrm{TB}_{t_n}(a) - \mathrm{BC}_{t_n}(a) \\
= \; & \left(d_n + D_{t_n^+}\left(a'\right)\right) + \left(S_{t_n^+}\left(a'\right) - d_n\right) \\
= \; & D_{t_n}(a) + S_{t_n}(a),
\end{aligned} \tag{8}$$

where $a = A_{t_n}$, $a' = A_{t_n^+} = a + \mathrm{tb}_n$, and $S_{t_n^+}(a') - d_n > 0$. Define equity as the residual claim $S_{t_n}(a) = S_{t_n^+}(a') - d_n > 0$ under survival, and $S_{t_n}(a) = 0$ under default at t_n. Eq. (8) shows that the balance-sheet equality in eq. (3) holds at t_n under survival, and that all value functions depend on $a = A_{t_n}$. As stated by Black and Cox (1976), debt cannot be financed by selling part of the firm's assets; rather, it is financed by issuing new shares of stock. This covenant applies here since $S_{t_n^+}(a') = S_{t_n}(a) + d_n$ under survival.

Consider now the following cases.

Case 1: The firm survives at time t_n, that is,

$$S_{t_n^+}(a') > d_n \quad \text{or} \quad a > b_n^*, \tag{9}$$

where b_n^* is the (endogenous) default barrier at time t_n. One has

$$\begin{aligned}
\mathrm{TB}_{t_n}(a) &= \mathrm{tb}_n + \mathrm{TB}_{t_n^+}(a'), \quad \mathrm{BC}_{t_n}(a) = \mathrm{BC}_{t_n^+}(a') \\
D_{t_n}^s(a) &= d_n^s + D_{t_n^+}^s(a'), \quad D_{t_n}^j(a) = d_n^j + D_{t_n^+}^j(a') \\
S_{t_n}(a) &= S_{t_n^+}(a') - d_n > 0.
\end{aligned}$$

Senior and junior bondholders are fully paid; whatever remains belongs to equityholders.

Case 2: The firm defaults at time t_n, that is,

$$a \leq b_n^*. \tag{10}$$

One has

$$\begin{aligned}
\mathrm{TB}_{t_n}(a) &= 0, \quad \mathrm{BC}_{t_n}(a) = wa \\
D_{t_n}^s(a) &= \min\Big(a(1-w), d_n^s + D_{t_n^+}^s(a)\Big) \\
D_{t_n}^j(a) &= \max\big(0, a(1-w) - D_{t_n}^s(a)\big) \\
S_{t_n}(a) &= 0.
\end{aligned}$$

Under default, tax benefits have no current nor future potentialities, and bankruptcy costs have current but no future potentialities. Senior bondholders are partially paid and junior bondholders are not when

$$a(1-w) < d_n^s + D_{t_n^+}^s(a),$$

that is, $a < c_n^*$ and $a < b_n^*$ or $a < b_n^{**} = \min(c_n^*, b_n^*)$. For $b_n^{**} \leq a \leq b_n^*$, senior bondholders are fully paid and junior bondholders are partially paid. Again, the balance-sheet equality in eq. (3) holds and all value functions depend on (t_n, a), where $a = A_{t_n}$, under default. ■

Table 1a: Value functions at $t_n \in \mathcal{P}$ without a reorganization process

Balance-sheet components	Default: $a \leq b_n^*$ $(0, b_n^{**}]$	$[b_n^{**}, b_n^*]$	Survival: $a > b_n^*$
$a = A_{t_n}$	a	a	a
$\mathrm{TB}_{t_n}(a)$	0	0	$\mathrm{TB}_{t_n^+}(a') + \mathrm{tb}_n$
$-\mathrm{BC}_{t_n}(a)$	$-wa$	$-wa$	$-\mathrm{BC}_{t_n^+}(a')$
$=$	$=$	$=$	$=$
$D^s_{t_n}(a)$	$a(1-w)$	$D^s_{t_n^+}(a) + d_n^s$	$D^s_{t_n^+}(a') + d_n^s$
$D^j_{t_n}(a)$	0	$a(1-w) - D^s_{t_n}(a)$	$D^j_{t_n^+}(a') + d_n^j$
$S_{t_n}(a)$	0	0	$S_{t_n^+}(a') - d_n$

In all cases, $D_{t_n}(a) = a(1-w)$ under default. The third (second) column of Table 1 reports the balance-sheet equality under default, when junior bondholders are partially paid (not paid at all). The third column collapses when $b_n^{**} = b_n^*$. The intuition is that, under high bankruptcy costs, it may happen that junior bondholders are never partially paid under default.

Table 1 presents a divergence from Leland (1994), who assumes that, under default, $D_{t_n}(a) = b^*(1-w)$ rather than $a(1-w)$, for $a < b^*$, where $D_{t_n}(a)$ is his present value of a perpetuity at (t_n, a), where $a = A_{t_n}$. Clearly, our setting is more realistic since the diffusion process $\{A\}$ can cross the fixed barrier b^* before t_n. This author implicitly assumes that $P^*(A_{t_n} < b^*) = 0$, which results in several incoherences. For example, a high enough coupon results in $b^* > A_0$ and a negative value for his $D_0(A_0)$. Our findings diverge from Leland's results as the perpetuity's regular coupon increases, and converge to them as it decreases. Furthermore, our setting is more flexible since it considers arbitrary corporate-bond portfolios and multiple seniority classes.

Proposition 2 *For $t \in \mathcal{P}$ (respectively $t \notin \mathcal{P}$), the debt and equity value functions $D_t(.)$ and $S_t(.)$ are non-negative (strictly positive), non-decreasing (strictly increasing), and continuous functions of $a = A_t > 0$.*

Proof. For simplicity, we provide a proof by induction that $D_t(a)$ is a continuous function of $a = A_t$, for a given $t \in [0, T]$. First, by eq. (4)–(5), the property holds at maturity. Now suppose that the property holds at a certain future date t_{n+1}. For $t \in (t_n, t_{n+1})$ and $a = A_t > 0$, the set of value functions

$$D_{t_{n+1}}\left(ae^{\left(r-\sigma^2/2\right)(t_{n+1}-t)+\sigma\sqrt{t_{n+1}-t}z}\right),$$

seen as functions of $z \in \mathbb{R}$, is bounded by

$$\sum_{m=n+1}^{N} d_m.$$

By eq. (6) and Lebesgue's dominated theorem (Cramér 1946), the value function $D_t(.)$ is continuous:

$$\begin{aligned}
& \lim_{a \to a_0} D_t(a) \\
= & \lim_{a \to a_0} \int_{\mathbb{R}} D_{t_{n+1}} \left(a e^{\left(r-\sigma^2/2\right)(t_{n+1}-t)+\sigma\sqrt{t_{n+1}-t}z} \right) e^{-r(t_{n+1}-t)} \varphi(z)\, dz \\
= & \int_{\mathbb{R}} \lim_{a \to a_0} D_{t_{n+1}} \left(a e^{\left(r-\sigma^2/2\right)(t_{n+1}-t)+\sigma\sqrt{t_{n+1}-t}z} \right) e^{-r(t_{n+1}-t)} \varphi(z)\, dz \\
= & \int_{\mathbb{R}} D_{t_{n+1}} \left(\lim_{a \to a_0} a e^{\left(r-\sigma^2/2\right)(t_{n+1}-t)+\sigma\sqrt{t_{n+1}-t}z} \right) e^{-r(t_{n+1}-t)} \varphi(z)\, dz \\
= & \int_{\mathbb{R}} D_{t_{n+1}} \left(a_0 e^{\left(r-\sigma^2/2\right)(t_{n+1}-t)+\sigma\sqrt{t_{n+1}-t}z} \right) e^{-r(t_{n+1}-t)} \varphi(z)\, dz \\
= & D_t(a_0),
\end{aligned}$$

where the first two steps are due to Lebesgue's dominated theorem, and the three last steps are due to the continuity of the value function $D_{t_{n+1}}(.)$. ■

Proposition 3 *For $t \in [0, T]$, the debt value function $D_t(.)$ verifies the additional properties*

$$\lim_{a \to 0} D_t(a) = 0 \quad \textit{and} \quad \lim_{a \to \infty} D_t(a) = M_t = \sum_{t_n \geq t} d_n e^{-(t_n - t)r}.$$

For a large enough $a = A_t$, the company is seen as risk free. Thus, the debt value is bounded, since $D_t(a) \in (0, M_t)$, for $t \in [0, T]$ and $a = A_t \in \mathbb{R}_+^*$. The required yield on the debt, $y \in (r, \infty)$, assumes survival till maturity, and sets at zero the net present value of the debt:

$$D_0 = \sum_{n=1}^{N} e^{-y t_n} d_n,$$

which, in turn, defines the yield spread as $y - r \geq 0$. The required yield can be seen as an internal rate of return. Along the same lines, the required yield and the yield spread can be defined either for an individual bond or a class of bonds.

Proof. Again, we propose a proof by induction. First, we show that the property holds at maturity $t_N = T$. Next, we assume that the property holds at a certain future date t_{n+1}, and we show that it holds at $t \in (t_n, t_{n+1})$, then at t_n. Obviously, the property holds at $t_N = T$; see eq. (4)–(5). Suppose now that the property holds at t_{n+1}. By eq. (6), one has

$$\begin{aligned} & D_t(a) \\ = \; & E^*\left[e^{-r(t_{n+1}-t)} D_{t_{n+1}}\left(A_{t_{n+1}}\right) \mid A_t = a\right] \\ = \; & E^*\left[e^{-r(t_{n+1}-t)} D_{t_{n+1}}\left(a e^{\left(r-\sigma^2/2\right)(t_{n+1}-t)+\sigma\sqrt{t_{n+1}-t}Z}\right)\right], \\ & \text{for } t \in (t_n, t_{n+1}), \end{aligned}$$

where Z follows the standard normal distribution. Again, by Lebesgue's dominated theorem and the continuity of $D_{t_{n+1}}(.)$, one has

$$\begin{aligned} \lim_{a\to 0} D_t(a) &= E^*\left[e^{-r(t_{n+1}-t)} \lim_{a\to 0} D_{t_{n+1}}\left(a e^{\left(r-\sigma^2/2\right)(t_{n+1}-t)+\sigma\sqrt{t_{n+1}-t}Z}\right)\right] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \lim_{a\to \infty} D_t(a) &= E^*\left[e^{-r(t_{n+1}-t)} \lim_{a\to \infty} D_{t_{n+1}}\left(a e^{\left(r-\sigma^2/2\right)(t_{n+1}-t)+\sigma\sqrt{t_{n+1}-t}Z}\right)\right] \\ &= E^*\left[e^{-r(t_{n+1}-t)} M_{t_{n+1}}\right] \\ &= e^{-r(t_{n+1}-t)} M_{t_{n+1}}, \end{aligned}$$

where the last two steps come from the induction hypothesis at time t_{n+1}. Clearly, the same result holds when $t \to t_n$ and $t > t_n$. Finally, by eq. (10), one has

$$\lim_{a\to 0} D_{t_n}(a) = 0,$$

and by eq. (9), one has

$$\begin{aligned} \lim_{a\to \infty} D_{t_n}(a) &= e^{-r(t_{n+1}-t_n)} M_{t_{n+1}} + d_n \\ &= M_{t_n}. \end{aligned}$$

■

The literature reports two definitions for the notion of default probability, one is unconditional and the other is conditional on late survival. The first, known as the *total default probability* up to time t_n, is

$$\begin{aligned}
\mathrm{TDP}_n &= P^*\left(\text{Default at } t_1 \text{ or } \dots \text{ or Default at } t_n\right) \\
&= P^*\left(\cup_{i=1}^n \left\{A_{t_i} \le b_i^*\right\}\right) \\
&= 1 - P^*\left(\overline{\cup_{i=1}^n \left\{A_{t_i} \le b_i^*\right\}}\right) \\
&= 1 - P^*\left(\cap_{i=1}^n \left\{A_{t_i} > b_i^*\right\}\right), \\
&= 1 - P^*\left(A_{t_1} > b_1^*, \dots, A_{t_n} > b_n^*\right),
\end{aligned}$$

and the second, known as the *conditional default probability* up to time t_n given late survival till t_{n-1}, is

$$\begin{aligned}
\mathrm{CDP}_n &= P^*\left(\text{Default at } t_n \mid \text{Survival till } t_{n-1}\right) \\
&= \frac{P^*\left(A_{t_1} > b_1^*, \dots, A_{t_{n-1}} > b_{n-1}^*, A_{t_n} \le b_n^*\right)}{P^*\left(A_{t_1} > b_1^*, \dots, A_{t_{n-1}} > b_{n-1}^*\right)} \\
&= 1 - \frac{P^*\left(A_{t_1} > b_1^*, \dots, A_{t_n} > b_n^*\right)}{P^*\left(A_{t_1} > b_1^*, \dots, A_{t_{n-1}} > b_{n-1}^*\right)}, \quad \text{for } n = 1, \dots, N.
\end{aligned}$$

TDP_n and CDP_n, for $n = 1, \dots, N$, define the *term structure of default probabilities*, total and conditional respectively.

These default proportions can be computed under the risk-neutral or the physical probability measure. Delianedis and Geske (2003) claim that the differences over time in the risk-neutral default probabilities are powerful predictors of corporate bankruptcy. They ignore the drift parameter of the GBM state process $\{A\}$ under the physical probability measure, known to carry a high estimation sampling error. Although less rigorous, the (total) default probabilities are generally preferred to the conditional default probabilities. The reason lies in the fact that conditional default probabilities, given late survival, are informative only for the very near future. Later on, given survival at time t_n, the firm will likely survive at time t_{n+1}.

As well, we can define the *senior term structure of loss probabilities* by

$$\mathrm{STDP}_n = 1 - P^*\left(A_{t_1} > b_1^{**}, \dots, A_{t_n} > b_n^{**}\right),$$

and

$$\mathrm{SCDP}_n = 1 - \frac{P^*\left(A_{t_1} > b_1^{**}, \dots, A_{t_n} > b_n^{**}\right)}{P^*\left(A_{t_1} > b_1^{**}, \dots, A_{t_{n-1}} > b_{n-1}^{**}\right)}, \quad \text{for } n = 1, \dots, N.$$

Proposition 4 *Let $\{X\}$ be a geometric Brownian motion with an initial position x_0, a drift μ, a volatility σ, and $c_1, \ldots, c_n \in \mathbb{R}$. One has*

$$P\left(X_{t_1} > c_1, \ldots, X_{t_n} > c_n\right) = \Phi\left(c_1', \ldots, c_n'\right),$$

where

$$c_m' = \frac{\log\left(x_0/c_m\right) + \left(\mu - \sigma^2/2\right) t_m}{\sigma}, \quad \textit{for } m = 1, \ldots, n,$$

and $\Phi(.)$ is the cumulative distribution function of the multivariate normal law $\mathcal{N}\left(0, \Sigma = CC^T\right)$ with

$$C = \begin{bmatrix} \sqrt{t_1 - t_0} & 0 & 0 & 0 & 0 \\ \sqrt{t_1 - t_0} & \sqrt{t_2 - t_1} & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & 0 & 0 \\ \sqrt{t_1 - t_0} & \sqrt{t_2 - t_1} & \cdots & \sqrt{t_{n-1} - t_{n-2}} & 0 \\ \sqrt{t_1 - t_0} & \sqrt{t_2 - t_1} & \cdots & \sqrt{t_{n-1} - t_{n-2}} & \sqrt{t_n - t_{n-1}} \end{bmatrix}.$$

Proof. The proof is based on the fact that

$$\begin{aligned} & X_{t_m} \\ = \; & x_0 e^{\left(\mu - \sigma^2/2\right) t_m - \sigma W_{t_m}} \\ = \; & x_0 e^{\left(\mu - \sigma^2/2\right) t_m - \sigma\left(\sqrt{t_1 - t_0} Z_1 + \cdots + \sqrt{t_m - t_{m-1}} Z_m\right)}, \end{aligned}$$

where $\{W\}$ is a standard Brownian motion, and $(Z_1, \ldots, Z_n)^{\mathrm{T}} \in \mathbb{R}^n$ follows the standard normal law $\mathcal{N}(0, \mathbb{I}_n)$, where $\mathbb{I}_n$ is the identity variance-covariance matrix of size n. For $m = 1, \ldots, n$, the event

$$\{X_{t_m} > c_m\}$$

is equivalent to

$$\left\{\sqrt{t_1 - t_0} Z_1 + \cdots + \sqrt{t_m - t_{m-1}} Z_m \leq c_m'\right\},$$

that is,

$$\left\{\text{The } m^{\text{th}} \text{ row of } CZ \leq c_m'\right\}.$$

■

3 A reorganization process

To date, we have assumed that a default is immediately followed by a liquidation. This contrasts with most of corporate bankruptcy laws; a reorganization process often takes place before liquidation. The rational for a reorganization process is to partially discharge a firm under default in an attempt to extend its business life, save its job positions, and reinforce its future reimbursements and potentialities. Discharging the firm is usually reported in terms of grace periods and grace payments, which increases the present value of equity.

In the absence of frictions, that is, $r^c = \omega = 0$, a reorganization process cannot result in a surplus. The balance-sheet equality is

$$a = D_0(a) + S_0(a),$$

so that, an increase in S_0 results in a decrease in D_0, while $a = A_0$ remains constant. In this context, Moraux (2004), Galai, Raviv, and Wiener (2007), and Abinzano et al. (2009) propose alternative reorganization processes, under Black and Cox' (1976) setting.

In the presence of frictions, the balance-sheet equality is

$$a + \mathrm{TB}_t(a) - \mathrm{RC}_t(a) - \mathrm{BC}_t(a) = D_t(a) + S_t(a),$$

where $\mathrm{RC}_t(a)$ is the present value of reorganization costs at time t when $A_t = a$. Similarly to bankruptcy costs, we assume a proportional reorganization cost, indicated by $w^r < w$. A discharge at default leads to a longer life for the firm; it usually results in a cumulative surplus from an increase in tax benefits and a decrease in bankruptcy costs. Thus, the total value of the firm

$$\overline{A}_0 = a + \mathrm{TB}_0(a) - \mathrm{RC}_0(a) - \mathrm{BC}_0(a)$$

usually increases, while its exogenous component $a = A_0$ remains constant. In this context, Anderson and Sundaresan (1996), Anderson, Sundaresan, and Tychon (1996), Ericsson and Reneby (1998), Mella-Barra and Perraudin (1997), Fan and Sundaresan (2000), François and Morellec (2004), Broadie, Chernov, and Sundaresan (2007), Broadie and Kaya (2007), Bruche and Naqvi (2010), Annabi, Breton, and François (2012a and 2012b), and Shibata et Tian (2012) propose alternative reorganization processes. Some of these papers accommodate general debt structures (Anderson and Sundaresan 1996, Anderson, Sundaresan, and Tychon 1996, Broadie, Chernov, and

Sundaresan 2007, and Brodie and Kaya 2007). The others develop under particular settings of Black and Cox (1976) or Leland (1994).

We propose a reorganization process, and assume debtors-in-possession under reorganization and the strict priority rule under liquidation. Our reorganization design depends on two parameters, that is, the *maximum number of grace periods a firm can call for*, indicated by $\overline{g} \in \{0, 1, \ldots, N\}$, and the *part of the due payment forgiven by obligors over a grace period*, $\eta \in [0, 1]$. The cases $\overline{g} = 0$ and/or $\eta = 0$ refer to the dynamic program without reorganization. All value functions have to be reworked to depend not only on time and the current level of the firm's asset value, but also on the number of grace periods called for by the firm before the current date. The generic value function $v_t(a)$ in eq. (3) is exchanged for $v_t^{\overline{g},\eta}(a, g)$, where $g = g_t \leq \overline{g}$ is the number of grace periods called for by the firm before time t. These value functions are expressed in Table 1b under survival, reorganization, and liquidation at time $t_n \in \mathcal{P}$.

Table 1b: Value functions at $t_n \in \mathcal{P}$ with a reorganization process

Balance-sheet components	Liquidation: $a \leq b_n^l$ $(0, b_n^{**}]$ and $g \leq \overline{g}$	$\left[b_n^{**}, b_n^l\right]$ and $g \leq \overline{g}$	Reorganization $b_n^l \leq a \leq b_n^r$ and $g < \overline{g}$	Survival $a > b_n^r$ and $g \leq \overline{g}$
$a = A_{t_n}$	a	a	a	a
$\mathrm{TB}_{t_n}(a, g)$	0	0	$\mathrm{TB}_{t_n^+}(a'', g+1) +$ $(1-\eta)\mathrm{tb}_n$	$\mathrm{TB}_{t_n^+}(a', g) +$ tb_n
$-\mathrm{BC}_{t_n}(a, g)$	$-wa$	$-wa$	$-\mathrm{BC}_{t_n^+}(a'', g+1)$	$-\mathrm{BC}_{t_n^+}(a', g)$
$-\mathrm{RC}_{t_n}(a, g)$	0	0	$-w_r a -$ $\mathrm{RC}_{t_n^+}(a'', g+1)$	$-\mathrm{RC}_{t_n^+}(a', g)$
$=$	$=$	$=$	$=$	$=$
$D_{t_n}^s(a)$	$a(1-w)$	$D_{t_n^+}^s(a, g) +$ d_n^s	$D_{t_n^+}^s(a'', g+1) +$ $(1-\eta)\, d_n^s$	$D_{t_n^+}^s(a', g) +$ d_n^s
$D_{t_n}^j(a)$	0	$a(1-w) -$ $D_{t_n}^s(a, g)$	$D_{t_n^+}^j(a'', g+1) +$ $(1-\eta)\, d_n^j$	$D_{t_n^+}^j(a', g) +$ d_n^j
$S_{t_n}(a)$	0	0	$S_{t_n^+}(a'', g+1) -$ $(1-\eta)\, d_n$	$S_{t_n^+}(a', g) -$ d_n

For ease of notation, the barriers $b_n^r(\overline{g})$, $b_n^l(\overline{g})$, and $b_n^{**}(\overline{g})$ are indicated by b_n^r, b_n^l, and b_n^{**}. The highest barrier b^r is called the *reorganization barrier* and the lowest barrier b^l the *liquidation barrier*. The event $\left\{b_n^{**} \leq a = A_{t_n} < b_n^l\right\}$ means that senior bondholders are fully paid and junior bondholders are partially paid under liquidation at time t_n. The barriers b_n^r, b_n^l, and b_n^{**} verify respectively

$$\begin{aligned} S_{t_n^+}(a', g) - d_n &= 0, \\ S_{t_n^+}(a'', g+1) - (1-\eta)\, d_n &= 0, \\ a\,(1-w) - \left(d_n^s + D_{t_n^+}^s(a, g) \right) &= 0, \end{aligned}$$

where the variables a' and a'' are obtained from the state variable $a = A_{t_n}$ as follows:

$$a' = a + \mathrm{tb}_n \quad \text{and} \quad a'' = (1 - w^r)\, a + (1-\eta)\, \mathrm{tb}_n.$$

In the absence of a reorganization process, that is, $\overline{g} = 0$, $\eta = 0$, and $w^r = 0$, set $b^r = b^l = b^*$, Table 1a and Table 1b coincide.

The optimal reorganization process can be defined as the solution of

$$\max_{(\overline{g}, \eta)} S_0(a, g), \tag{11}$$

or

$$\max_{(\overline{g}, \eta)} \overline{A}_0, \tag{12}$$

or

$$\begin{aligned} &\max_{(\overline{g}, \eta)} S_0(a, g) \\ &\text{u.c. } D_0(a, g) \geq D_0(a), \end{aligned} \tag{13}$$

where $v_0(a, g)$ stands for $v_0^{\overline{g}, \eta}(a, g)$ and $g = g_0$ for the number of grace periods called for by the firm before the origin.

4 Solving the dynamic program

Let $\mathcal{G} = \{a_1, \ldots, a_p\}$ be a mesh of grid points for the firm's asset value, $a_0 = 0$, and $a_{p+1} = \infty$. It is better for the grid points to be more concentrated where the firm's assets value is the most likely to happen and the functions to be approximated the most likely to vary. The optimal choice of $\mathcal{G}$ is not addressed here; however, the our dynamic program reaches any desired level of accuracy as long as a_1 and a_p are extreme enough and $a_{i+1} - a_i$, for

$i = 1, \ldots, p-1$, are small enough. We use the quantiles of the state process $\{A\}$ at time $T = t_N$ for grid construction.

To start with, we set $\overline{g} = 0$, then we discuss the case $\overline{g} \geq 1$. Suppose now that convergent approximations of all value functions are available on $\mathcal{G}$ at a certain future date t_{n+1}. They are indicated by $\widetilde{\mathrm{TB}}_{t_{n+1}}(.)$, $\widetilde{\mathrm{BC}}_{t_{n+1}}(.)$, $\widetilde{D}^s_{t_{n+1}}(.)$, $\widetilde{D}^j_{t_{n+1}}(.)$, and $\widetilde{S}_{t_{n+1}}(.)$. This is not really a strong assumption since these value functions are known in closed form at maturity $t_N = T$. The dynamic program works as follows:

1. Start the program at t_{n+1}. Use piecewise linear polynomials, and interpolate the value functions $\widetilde{\mathrm{TB}}_{t_{n+1}}(.)$, $\widetilde{\mathrm{BC}}_{t_{n+1}}(.)$, $\widetilde{D}^s_{t_{n+1}}(.)$, $\widetilde{D}^j_{t_{n+1}}(.)$, and $\widetilde{S}_{t_{n+1}}(.)$ from $\mathcal{G}$ to the overall state space $\mathbb{R}^*_+$. The interpolations are indicated by $\widehat{\mathrm{TB}}_{t_{n+1}}(.)$, $\widehat{\mathrm{BC}}_{t_{n+1}}(.)$, $\widehat{D}^s_{t_{n+1}}(.)$, $\widehat{D}^j_{t_{n+1}}(.)$, and $\widehat{S}_{t_{n+1}}(.)$.

2. Use eq. (1)–(2), and compute the transition parameters at time t_n in closed form, that is, $T^0_{kin} = T^0_{a_k a_i a_{i+1} \Delta_n}$ and $T^1_{kin} = T^1_{a_k a_i a_{i+1} \Delta_n}$, where $\Delta_n = t_{n+1} - t_n$.

3. Use eq. (6) and compute the value functions $\widetilde{\mathrm{TB}}_{t_n^+}(.)$, $\widetilde{\mathrm{BC}}_{t_n^+}(.)$, $\widetilde{D}^s_{t_n^+}(.)$, $\widetilde{D}^j_{t_n^+}(.)$, and $\widetilde{S}_{t_n^+}(.)$, defined on $\mathcal{G}$, from $\widehat{\mathrm{TB}}_{t_n^+}(.)$, $\widehat{\mathrm{BC}}_{t_n^+}(.)$, $\widehat{D}^s_{t_n^+}(.)$, $\widehat{D}^j_{t_n^+}(.)$, and $\widehat{S}_{t_n^+}(.)$, defined on the overall state space $\mathbb{R}^*_+$.

4. For $a_k \in \mathcal{G}$, search for k' such that $a'_k = a_k + \mathrm{tb}_n \in [a_{k'}, a_{k'+1})$.

5. Approximate the default barriers b^*_n and then b^{**}_n at t_n as follows:

$$
\begin{aligned}
\widetilde{b}^*_n &= \min\left\{a_k \in \mathcal{G} \text{ and } a_k \text{ such that } \widetilde{S}_{t_n^+}(a_{k'}) - d_n > 0\right\} \\
\widetilde{b}^{**}_n &= \min\left\{a_k \in \mathcal{G} \text{ such that } (1-w)\, a_k > \widetilde{D}^s_{t_n^+}(a_k) + d^s_n\right\} \wedge \widetilde{b}^*_n;
\end{aligned}
$$

6. Use eq. (6) and Table 1a, and compute $\widetilde{\mathrm{TB}}_{t_n}(.)$, $\widetilde{\mathrm{BC}}_{t_n}(.)$, $\widetilde{D}^s_{t_n}(.)$, $\widetilde{D}^j_{t_n}(.)$, and $\widetilde{S}_{t_n}(.)$, defined on $\mathcal{G}$;

7. If $t_n = 0$, then stop the program; else go to step 1 and repeat the procedure from time t_n to time t_{n-1}.

We make explicit Step 1 and Step 3 for a generic value function $v_t(.)$. Step 1 can be written as follows:

$$\widehat{v}_{t_{n+1}}(a) = \sum_{i=0}^{p} \left(\alpha_i^{n+1} + \beta_i^{n+1} a\right) \mathbb{I}\left(a_i \le a < a_{i+1}\right),$$

where the internal local coefficients α_i^{n+1} and β_i^{n+1}, for $i = 1, \ldots, p-1$, are

$$\begin{aligned}
\beta_i^{n+1} &= \frac{\widetilde{v}_{t_{n+1}}(a_{i+1}) - \widetilde{v}_{t_{n+1}}(a_i)}{a_{i+1} - a_i} \\
\alpha_i^{n+1} &= \frac{a_{i+1}\widetilde{v}_{t_{n+1}}(a_i) - a_i \widetilde{v}_{t_{n+1}}(a_{i+1})}{a_{i+1} - a_i},
\end{aligned}$$

and the external local coefficients are set to the values of their adjacent intervals, that is,

$$\left(\alpha_0^{n+1}, \beta_0^{n+1}\right) = \left(\alpha_1^{n+1}, \beta_1^{n+1}\right),$$

and

$$\left(\alpha_p^{n+1}, \beta_p^{n+1}\right) = \left(\alpha_{p-1}^{n+1}, \beta_{p-1}^{n+1}\right).$$

Step 3 can be written as

$$\begin{aligned}
\widetilde{v}_{t_n^+}(a_k) &= E^*\left[e^{-r(t_{n+1}-t_n)} \widehat{v}_{t_{n+1}}\left(A_{t_{n+1}}\right) \mid A_{t_n} = a_k\right] \\
&= e^{-r(t_{n+1}-t_n)} \sum_{i=0}^{p} \left(\alpha_i^{n+1} T_{kin}^0 + \beta_i^{n+1} T_{kin}^1\right), \\
&\quad \text{for } k = 1, \ldots, p,
\end{aligned}$$

whether the function $v_t(.)$ represents $\mathrm{TB}_t(.)$, $\mathrm{BC}_t(.)$, $D_t^s(.)$, $D_t^j(.)$, or $S_t(.)$. The transition tables T^0 and T^1 are known in closed form when the state process moves according to a GBM (Ben-Ameur, Breton, and L'Écuyer 2002). These tables represent a fixed cost for the dynamic program as long as the state process is time homogenous and $\Delta_n = t_{n+1} - t_n = \Delta$ is a fixed constant.

All value functions, just after a payment date, can be written as a sum of local future values multiplied by their associated transition parameters given the current position of the state process. This sum of small pieces is then discounted back at the risk-free rate. This dynamic program does respect the true dynamics of the state process $\{A\}$ through the transition tables in eq. (1)–(2).

This procedure can be improved further by using the transition tables T^2 or T^2 and T^3, that is, mixing between the dynamic program and piecewise quadratic or cubic polynomials, but the gain in accuracy will mostly be offset by the loss in computing time.

For $\overline{g} \geq 1$, solving the problems (11)–(13) can be done via a modification of our dynamic program, that is, an augmentation of the state variable from (t, a) to (t, a, g), where $a = A_t$ and $g = g_t \leq \overline{g}$ is the number of grace periods called for by the firm before time t. The generic value function $v_t(a)$ in eq. (3) is exchanged for $v_t^{\overline{g},\eta}(a, g)$. The modified dynamic program handles $\overline{g} + 1$ (times) as many value functions through the backward recursion.

The code is written in the C language, compiled with the GCC compiler, and run under Windows 7. The GSL library (Galassi et al. 2009) is used to achieve specific computing tasks, and the CUBATURE software package (Hahn 2005) to compute default probabilities.

5 A numerical investigation

We first compare DP values to selected closed-form solutions in the literature, and show that DP is a viable alternative to the analytical approach. DP is flexible and efficient. Next, we perform a sensitivity analysis of main value functions with respect to their input parameters. The results are interpreted according to the corporate finance theory.

5.1 DP versus selected closed-form solutions

Table 2 is based on information from Nivorozhkin (2005b), and compares DP values to Black and Cox (1976). Set $T = 1$ (year), $N = 1$ (period), $A_0 = \$100$, $d_1^s = \$70$, $d_1^j = \$30$, and $r = 10\%$ (per year). This is a portfolio made of a senior pure bond and a junior pure bond, both maturing in one year. B&C stands for Black and Cox and DP refers to dynamic programming.

Table 2 shows a clear convergence of DP values to their analytical (B&C) counterparts, as the DP grid size increases. The DP procedure shows accuracy at the sixth digit, while only four digits are reported. Default probabilities are mostly supported by junior bondholders. For example, for $\mu = 0.1$ and $\sigma = 0.1$, junior bondholders support a probability of 17.11% of loosing value, while senior bondholders are almost safe.

Table 3 is based on information from Anson et al. (2004), and compares DP values to Geske (1977). Set $T = 2$ (years), $N = 2$ (periods), $A_0 = \$200$, $d_1 = \$100$, $d_2 = \$100$, and $r = 5\%$ (per year). This is a portfolio of two (senior) pure bonds, one maturing in one year and the other in two years.

Again, Table 3 shows a clear convergence of DP values to their analytical (Geske) counterparts, as the DP grid size increases. The DP procedure is still accurate at the sixth digit. While default probabilities for the first year are significant, they collapse for the second year, given survival the first year. This fact characterizes the structural model.

Table 2: DP versus Black and Cox (1976)

		$\mu = .05$			$\mu = .10$		
		$\sigma = .1$	$\sigma = .2$	$\sigma = .3$	$\sigma = .1$	$\sigma = .2$	$\sigma = .3$
S_0	DP−500				10.3082	13.2698	16.7342
	DP−1000				10.3082	13.2697	16.7342
	DP−2000				10.3082	13.2697	16.7342
	DP−4000				10.3082	13.2697	16.7342
	B&C				10.3082	13.2697	16.7342
D_0^j	DP−500				26.3532	23.4525	20.5868
	DP−1000				26.3532	23.4526	20.5868
	DP−2000				26.3532	23.4526	20.5868
	DP−4000				26.3532	23.4526	20.5868
	B&C				26.3532	23.4526	20.5868
D_0^s	DP−0500				63.3386	63.2777	62.6790
	DP−1000				63.3386	63.2777	62.6790
	DP−2000				63.3386	63.2777	62.6790
	DP−4000				63.3386	63.2777	62.6790
	B&C				63.3386	63.2777	62.6790
CDP_1	DP−2000	0.3264	0.4404	0.4934	0.1711	0.3446	0.4273
	B&C	0.3264	0.4404	0.4934	0.1711	0.3446	0.4273
SCDP_1	DP−2000	0.0000	0.0266	0.1140	0.0000	0.0145	0.0850
	B&C	0.0000	0.0266	0.1140	0.0000	0.0145	0.0850

According to Delianedis and Geske (2003), what explains default frequencies is more the differences over time in the risk-neutral default probabilities than the default probabilities themselves. Thus, such an analysis should be done on a regular basis to keep track of the default probability movements over time.

Table 4 compares DP values to Geske (1977). Set $T = 2$ (years), $N = 2$ (two periods of one year each), $A_0 = \$100$, $d_1^s = \$70$, $d_2^j = \$30$, and $r = 10\%$ (per year). This is a portfolio made of a senior pure bond maturing in one year and a junior pure bond maturing in two years.

DP competes well against the analytical approach of Geske. The accuracy of the DP procedure is still at the sixth digit. Even though the nominal debt structure is mostly senior, default risk is almost entirely supported by junior bondholders. Given survival at year one, default probabilities drastically decrease.

Table 3: DP versus Geske (1977)

		$\mu = 0.05$			$\mu = 0.10$		
		$\sigma = .1$	$\sigma = .2$	$\sigma = .3$	$\sigma = .1$	$\sigma = .2$	$\sigma = .3$
S_0	DP−500	16.9325	23.6097	30.9027			
	DP−1000	16.9324	23.6092	30.9019			
	DP−2000	16.9323	23.6091	30.9017			
	DP−4000	16.9323	23.6091	30.9017			
	Geske	16.9323	23.6091	30.9017			
D_0	DP−500	183.0675	176.3903	169.0973			
	DP−1000	183.0676	176.3908	169.0981			
	DP−2000	183.0677	176.3909	169.0983			
	DP−4000	183.0677	176.3909	169.0983			
	Geske	183.0677	176.3909	169.0983			
CDP_1	DP−2000	0.2429	0.3922	0.4598	0.1157	0.3003	0.3945
	Geske	0.2429	0.3922	0.4598	0.1157	0.3003	0.3945
CDP_2	DP−2000	0.0000	0.0000	0.0030	0.0000	0.0000	0.0017
	Geske	0.0000	0.0000	0.0030	0.0000	0.0000	0.0017

Table 5 compares DP values to Leland (1994). Set $A_0 = \$100$, $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.5$. The debt is a perpetuity that promizes an annual coupon C (in dollars) forever. For the DP procedure to run, we fix the debt maturity at 150 years.

Here, DP values show a slower convergence to their theoretical counterparts, as the grid size increases. This is explained by the very long-term horizon of this program, that is, 150 years. In addition, as explained earlier, DP diverges from Leland's analytical approach, which assumes that $P(A_{t_n} < b^*) = 0$, where b^* is the (unique) endogenous default barrier of the perpetuity. Since Leland's endogenous default barrier b^* is an increasing function of the perpetuity regular coupon C, we expect that DP values diverge from Leland's values as the coupon increases. Table 5 highlights this fact.

Table 4: DP versus Geske (1977)

		$\mu = .05$			$\mu = .10$		
		$\sigma = .1$	$\sigma = .2$	$\sigma = .3$	$\sigma = .1$	$\sigma = .2$	$\sigma = .3$
S_0	DP−0500				12.5355	15.0301	18.2623
	DP−1000				12.5354	15.0299	18.2621
	DP−2000				12.5354	15.0299	18.2620
	DP−4000				12.5354	15.0299	18.2620
	Geske				12.5354	15.0299	18.2620
D_0^j	DP−0500				24.1259	21.6922	19.0588
	DP−1000				24.1260	21.6924	19.0589
	DP−2000				24.1260	21.6924	19.0590
	DP−4000				24.1260	21.6924	19.0590
	Geske				24.1260	21.6924	19.0590
D_0^s	DP−0500				63.3386	63.2777	62.6789
	DP−1000				63.3386	63.2777	62.6790
	DP−2000				63.3386	63.2777	62.6790
	DP−4000				63.3386	63.2777	62.6790
	Geske				63.3386	63.2777	62.6790
CDP_1	DP−2000	0.2298	0.3841	0.4549	0.1076	0.2929	0.3898
	Geske	0.2298	0.3841	0.4549	0.1076	0.2929	0.3898
CDP_2	DP−2000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Geske	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
SCDP_1	DP−2000	0.0000	0.0266	0.1140	0.0000	0.0145	0.0850
	Geske	0.0000	0.0266	0.1140	0.0000	0.0145	0.0850

Overall, DP is a viable alternative to the analytical approach not only because it shows convergence, robustness, and efficiency, but also for it shows great flexibility. DP allows one to perform realistic numerical and empirical investigations in a quasi-closed form.

Table 5: DP versus Leland (1994)

		$\sigma = .1$		$\sigma = .2$		$\sigma = .3$	
		$C = 3$	$C = 9$	$C = 3$	$C = 9$	$C = 3$	$C = 9$
S_0	DP−2000	68.4648	7.4663	68.5155	13.6453	69.5185	22.3734
	DP−4000	68.4648	7.4619	68.5153	13.6395	69.5176	22.3689
	DP−6000	68.4648	7.4611	68.5153	13.6387	69.5175	22.3683
	DP−8000	68.4648	7.4608	68.5153	13.6384	69.5174	22.3680
	Leland	67.5000	4.6182	67.6177	12.0311	68.9758	21.6565
D_0	DP−2000	48.5153	112.1499	48.2133	98.6652	44.9001	87.7548
	DP−4000	48.5153	112.8964	48.2171	98.2764	44.9108	88.1467
	DP−6000	48.5153	113.9280	48.2128	98.3798	44.9205	87.9861
	DP−8000	48.5153	113.5883	48.2139	98.6955	44.9227	87.9962
	Leland	50.0000	120.3449	49.4524	105.6438	45.6861	94.0040
TB_0	DP−2000	16.9801	34.1615	16.8367	28.7714	15.3788	25.3771
	DP−4000	16.9801	34.5474	16.8386	28.5683	15.3841	25.5791
	DP−6000	16.9801	35.0821	16.8364	28.6217	15.3891	25.4956
	DP−8000	16.9801	34.9059	16.8369	28.7853	15.3903	25.5008
	Leland	17.5000	37.6724	17.2466	31.9715	15.6458	28.4315
BC_0	DP−2000	0.0000	14.5453	0.1079	16.4609	0.9602	15.2489
	DP−4000	0.0000	14.1891	0.1062	16.6524	0.9557	15.0635
	DP−6000	0.0000	13.6930	0.1083	16.6032	0.9511	15.1412
	DP−8000	0.0000	13.8568	0.1077	16.4514	0.9502	15.1366
	Leland	0.0000	12.7093	0.1765	14.2966	0.9839	12.7710

5.2 Sensitivity analysis

To start with, we consider a coupon bond and, next, a corporate debt made up of a senior coupon bond and a junior coupon bond with a longer maturity. We conduct a sensitivity analysis with respect to the (junior) bond's coupon rate c (in % per year) and, then, to the firm's asset value $a = A_0$ (in dollars). In all cases, when c increases or, equivalently, a decreases, ceteris paribus, the firm moves from a healthy to a distressed situation.

We report the value functions of corporate securities for several values of the firm's asset volatility $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), the risk-free rate $r \in \{4\%, 6\%, 8\%\}$ (per year), and the bankruptcy-cost parameter $\omega \in \{0, 0.25, 0.5\}$. The payout rate δ and the reorganization parameters $\overline{g}$, g_0, and η are set at zero, except for Figures 41–42.

Figures 1–8 represent the 1- equity value, 2- bond value, 3- yield spread, and 4- total value of the firm at the origin, as functions of the bond's coupon, when $r^c = 0$ and $w = 0$. The results show that, in the absence of frictions, the higher the coupon, the lower the equity and the higher the bond value, while the total value of the firm remains flat. The last result indicates that the total value of the firm, which reduces to its exogenous component, is independent of the firm's capital structure. Thus, in the absence of frictions, the Modigliani-Miller conjecture holds.

The bond value is subject to two conflicting mechanics when the coupon increases. First, a higher coupon increases the bond's future cash flows, and has a positive impact on the bond value. Second, a higher coupon increases the default probability, and has a negative impact on the bond value. In the absence of frictions, the overall impact of these two conflicting mechanics is positive; the first is dominant.

The bond's required rate y (in % per year) is unbounded, as a function of the bond's coupon rate c (in % per year), since it sets at zero the bond's net present value, assuming that its promised cash flows occur with certainty:

$$D_0 = \left[\sum_{n=1}^{N} c \times e^{-y \times n} + e^{-y \times N} \right] \times P \text{ (in dollars)},$$

where P and c are the bond's principal amount and coupon rate. This is explained by the fact that D_0 is bounded.

Next, when the firm's asset volatility σ rises, the equity value increases and the bond value decreases, while the total value of the firm remains constant. A *substitution effect* is possible in all cases, that is, shareholders benefit from any increase in the risk of the firm's operations at the expense of bondholders. Thus, in the absence of frictions, bondholders must be protected by special covenants against a potential substitution effect.

Furthermore, the results suggest the following:

$$\lim_{c\uparrow} S_0 = 0 \quad \text{and} \quad \lim_{c\uparrow} D_0 = A_0.$$

Finally, the firm's default probability is an increasing function of the firm's asset volatility σ for a healthy company, e.g., $c = 4\%$ per year, and a decreasing function for a distressed company, e.g., $c = 12\%$ per year. The last case is not reported in our analysis. This relationship is not monotone for intermediate coupon rates, e.g., $c = 10\%$ per year.

Figures 9–16 follow the same pattern as Figures 1–8, and represent a more realistic situation. We set $r^c = 35\%$ (per year) and $w = 0.25$. The equity value is still a decreasing function of the bond's coupon. Conversely, in the presence of frictions, the bond value first increases, and then decreases. The optimum can be seen as the *maximum debt capacity* of the firm. Clearly, a position right of the optimum is suboptimal; the company agrees to pay a high coupon for a given debt, while a much lower coupon would suffice. Similarly, the total value of the firm first increases, and then decreases. The maximum represents the *optimal capital structure*. This result indicates that the total value of the firm depends on the firm's capital structure. Thus, frictions break down the Modigliani-Miller conjecture.

This analysis triggers the following question: how far is a given public company from its maximum debt capacity and optimal capital structure? This question is not easy to answer in real life since the company is represented by a single point but not a complete curve. Combining the structural model and the statistical approach is probably the right way to address this important issue.

The bond value is still subject to two conflicting mechanics. An increase in the coupon rate results in an increase in the bond's future cash flows as well as the default probability. Their overall impact is positive for low coupons, and negative for high coupons. The substitution effect is still feasible only for investment-grade bonds. For high-yield bonds, though, an increase in the risk of the firm's operations, measured by σ, benefits both shareholders and bondholders. Protective covenants against the substitution effect are no longer valuable for bondholders under financial distress. The same two conflicting mechanics explain the shape of the tax-benefit and bankruptcy-cost curves.

The results suggest the following:

$$\begin{aligned} \lim_{c\uparrow} S_0 &= 0, \quad \lim_{c\uparrow} D_0 = \lim_{c\uparrow} A_0' = (1-w)\,A_0 \\ \lim_{c\uparrow} \mathrm{TB}_0 &= 0, \quad \lim_{c\uparrow} \mathrm{BC}_0 = wA_0. \end{aligned}$$

Main value functions, as functions of the firm's leverage $L = D_0/A_0'$, show a similar pattern, as if they were reported as functions of the bond's coupon. This investigation is motivated by the fact that the firm's leverage is an increasing function of the bond's coupon. The figures are not reported here.

The frictions introduced here, that is, $r^c = 35\%$ (per year) and $w = 0.25$, makes the firm healthier with respect to the first scenario, which assumes

$r^c = 0$ (per year) and $w = 0$. For a coupon rate $c = 10\%$ (per year), the default probability becomes an increasing function of the firm's asset volatility σ.

Figures 17–24 follow the same pattern as the previous scenario, except that we let the risk-free interest rate $r \in \{4\%, 6\%, 8\%\}$ vary instead of the firm's asset volatility σ. A relevant result is that, unlike riskless bonds, the value of a risky bond may be positively related to the risk-free interest rate. This is the case with high-yield bonds. Rather, investment-grade bonds behave as riskless bonds. This result is consistent with Longstaff and Schwartz (1995), but the explanation differs; the result arises more from frictions in a one-factor model than from market factor interactions in a two-factor model.

Figures 25–36 consider a bond portfolio made up of a senior coupon bond and a junior coupon bond with a longer maturity. We report the values of corporate securities as functions of the junior coupon rate c^j (in % per year), and, for each value function, we vary the bankruptcy-cost parameter $\omega \in \{0, 0.25, 0.5\}$.

Firstly, equity, tax benefits, junior default barriers, and default probabilities do not depend on the bankruptcy-cost parameter ω. This is a direct consequence of the strict priority rule, which sets the equity value and tax benefits at zero whenever the firm defaults. The parameter ω plays the role of a sharing parameter under default. This result is consistent with Leland (1994).

Secondly, senior bondholders are almost always protected by junior bondholders, except for high levels of w. Thus, for low and moderate levels of ω, costly junior debts do not alter senior bondholders' position. Alternatively, when ω rises, senior default barriers and default probabilities converge to their junior counterparts, since, for high levels of w, junior bondholders are never partially paid under default (see Table 1a).

The results suggest the following:

$$\begin{aligned} \lim_{c^j\uparrow} S_0 &= 0, \quad \lim_{c^j\uparrow} D_0 = \lim_{c^j\uparrow} A_0' = (1-w)\,A_0 \\ \lim_{c^j\uparrow} \mathrm{TB}_0 &= 0, \quad \lim_{c^j\uparrow} \mathrm{BC}_0 = wA_0. \end{aligned}$$

Figures 37–40 follow the same pattern as the previous scenario, but report the value functions of corporate securities as functions of the firm's asset value. The equity and bond value are both increasing. Leland (1994) claims that, in the presence of frictions, the equity value can switch to a concave function. We have searched for such a pattern without success. Unlike the equity value, the bonds' values are bounded and convergent, as shown in Proposition 2,

$$\begin{aligned}\lim_{a\to\infty} D_0^s &= \left[\sum_{n=1}^{5} 7\% \times e^{-6\%\times n} + e^{-6\%\times 5}\right] \times 70 \\ &= 72.3951 \text{ (in dollars),}\end{aligned}$$

and

$$\begin{aligned}\lim_{a\to\infty} D_0^j &= \left[\sum_{n=1}^{10} 10\% \times e^{-6\%\times n} + e^{-6\%\times 10}\right] \times 30 \\ &= 38.3538 \text{ (in dollars).}\end{aligned}$$

Bondholders take advantage of an increase in the firm's asset value up to a certain limit. Tax benefits and bankruptcy costs behave similarly as in Figures 9–16, where a low (high) coupon rate corresponds to a high (low) firm's asset value. This suggests the following properties:

$$\begin{aligned}\lim_{a\to\infty} \mathrm{TB}_0 &= \sum_{n=1}^{5} 7\% \times 70 \times 35\% \times e^{-6\%\times n} + \\ &\quad \sum_{n=1}^{10} 10\% \times 30 \times 35\% \times e^{-6\%\times n} \\ &= 14.8495 \text{ (in dollars),}\end{aligned}$$

and

$$\lim_{a\to\infty} \mathrm{BC}_0 = 0 \text{ (in dollars).}$$

Figures 41–42 show that equity value is an increasing function of $\overline{g}$ and η. Thus, the solution of pb. (11) is always $(\overline{g}, \eta) = (N, 100\%)$. All in all, if the main objective is to increase equity value, forgive all the time at the maximum rate. This solution is feasible at the theoretical level since bondholders can adjust their prices consequently, but it is not at the practical level since it obviously results in severe conflicting situations. Although the solution is case sensitive, a general property arises from our sensitivity analysis. For a fixed grace rate $\eta \in [0; 1]$, the first call period has the most important impact on equity value. Then, the latter quickly saturates. The solution of pb. (13) is usually obtained at $\overline{g} = 1$. This result supports the shortening of the average time under the U.S. bankruptcy law from three years to one year (Sheppard 1995). Alternatively, the solution of pb. (12) or pb. (13) is not always a corner solution. The optimal reorganization process of pb. (13) is attractive, but it comes up against the unobserved value function $D_0(A_0)$. For each case, we set in bold the solutions to pb. (11)–(13).

6 Conclusion

We propose a general and flexible dynamic program that extends the structural models of Merton (1974), Black and Cox (1976), Geske (1977), and Leland (1994). Our setting accommodates arbitrary corporate debts, multiple seniority classes, payouts, tax benefits, bankruptcy costs, and a reorganization process. These extensions come at the expense of a minor loss of efficiency. The analytical approach proposed in the literature is exchanged here for a quasi-analytical approach based on dynamic programming coupled with finite elements.

Our theoretical investigation provides several properties of the debt- and equity-value functions, and our numerical investigation shows complete consistency with the corporate finance literature. Examples discuss the substitution effect, maximum debt capacity, optimal capital structure, risk sharing between senior and junior bondholders, several limit properties, and optimal reorganization processes. DP shows flexibility and efficiency.

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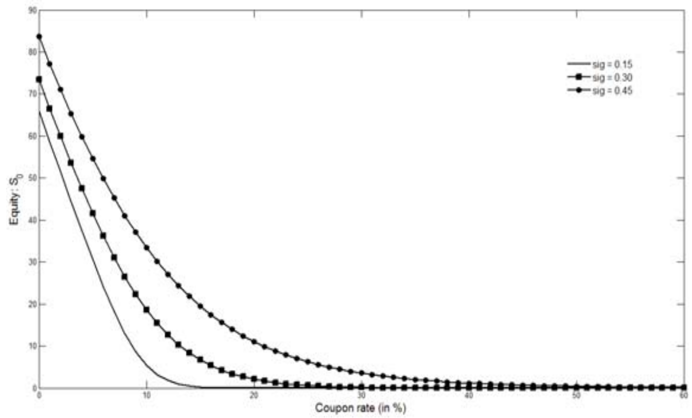

Figure 1: **Equity value as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 0\%$ (per year), and $w = 0$.

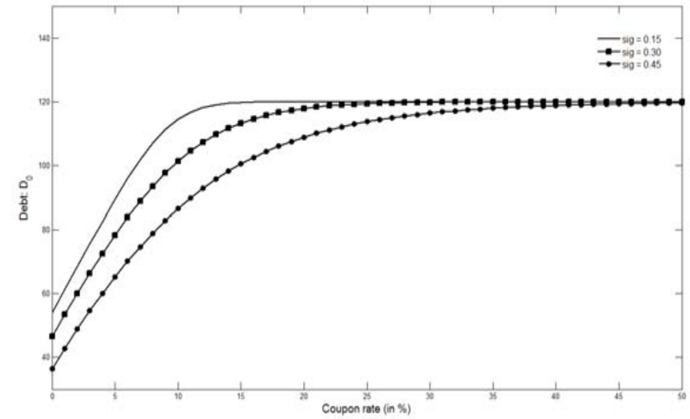

Figure 2: **Debt value as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 0\%$ (per year), and $w = 0$.

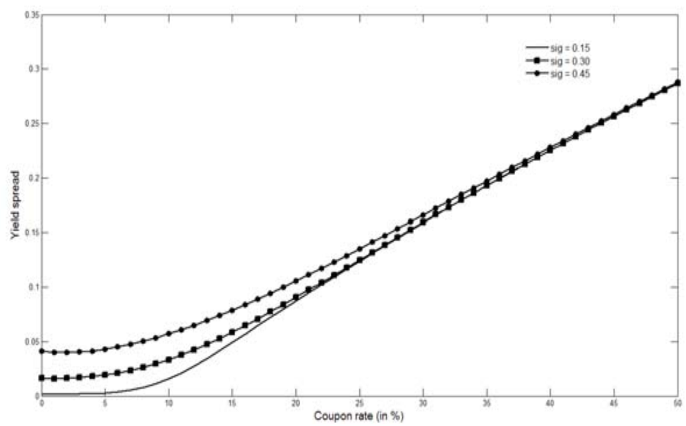

Figure 3: **Yield spread as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 0\%$ (per year), and $w = 0$.

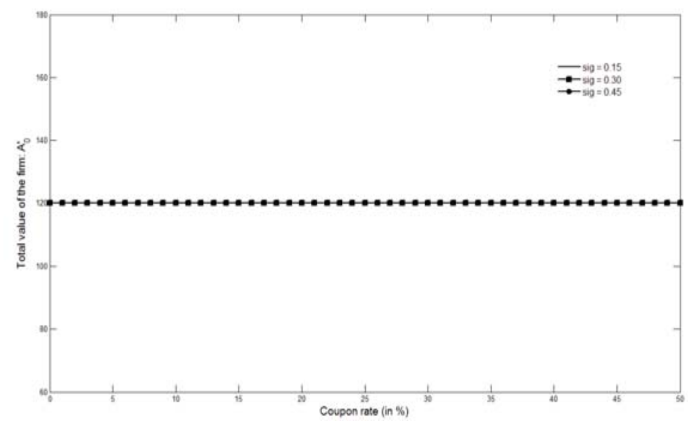

Figure 4: **Total value of the firm as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 0\%$ (per year), and $w = 0$.

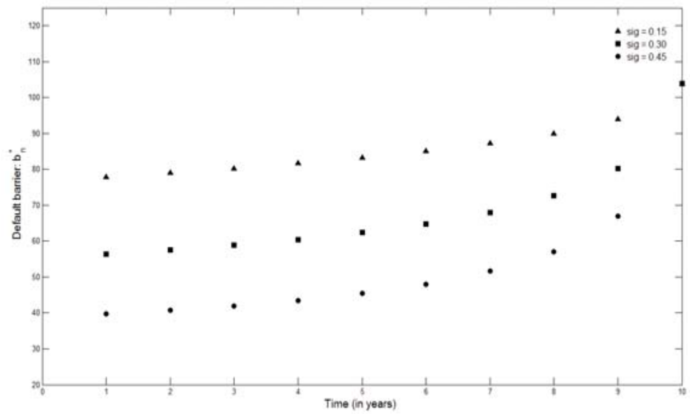

Figure 5: **Term structure of default barriers.** The debt is a coupon bond with a maturity of 10 years, a principal amount of \$100, and an annual coupon rate of 4%. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 0\%$ (per year), and $w = 0$.

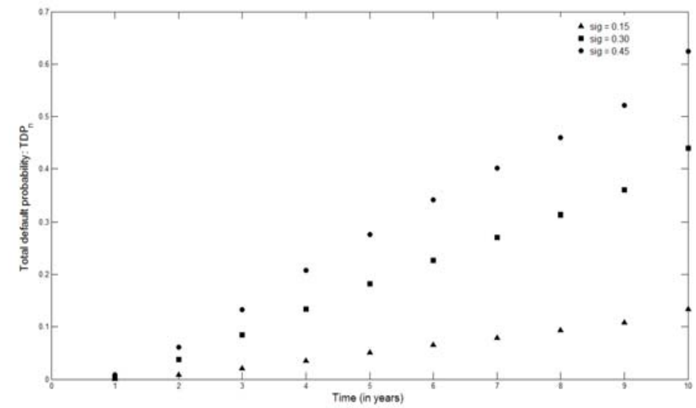

Figure 6: **Term structure of total default probabilities.** The debt is a coupon bond with a maturity of 10 years, a principal amount of \$100, and an annual coupon rate of 4%. Set Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 0\%$ (per year), and $w = 0$.

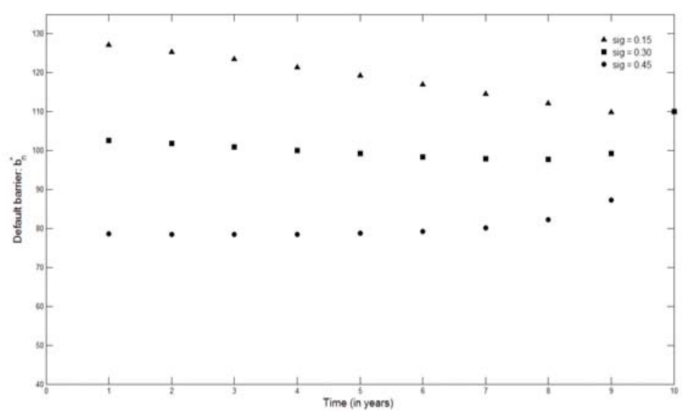

Figure 7: **Term structure of default barriers.** The debt is a coupon bond with a maturity of 10 years, a principal amount of \$100, and an annual coupon rate of 10%. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 0\%$ (per year), and $w = 0$.

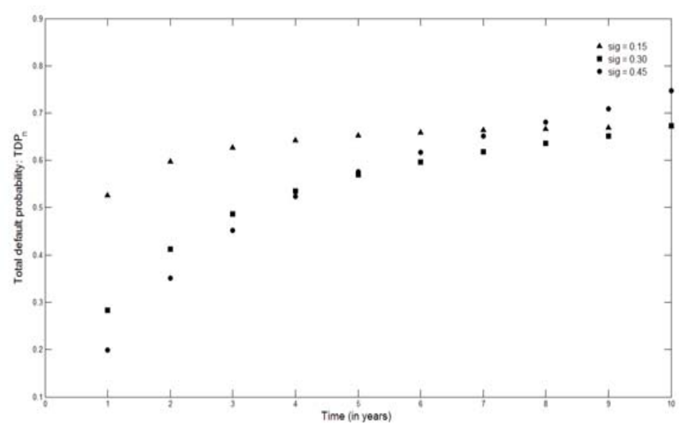

Figure 8: **Term structure of total default probabilities.** The debt is a coupon bond with a maturity of 10 years, a principal amount of \$100, and an annual coupon rate of 10%. Set Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 0\%$ (per year), and $w = 0$.

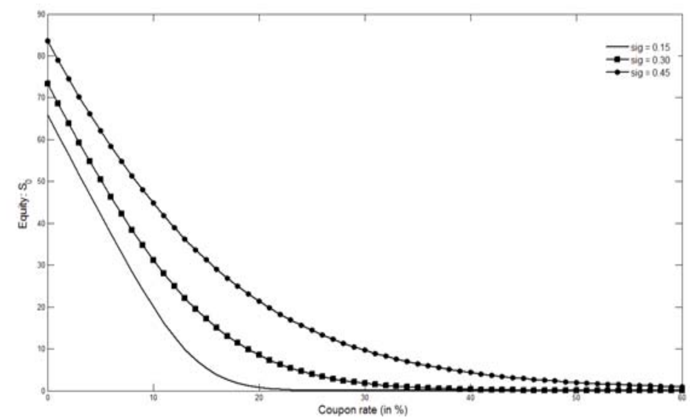

Figure 9: **Equity value as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

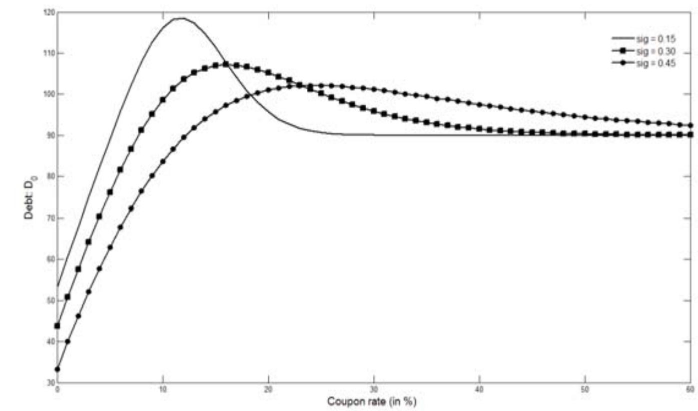

Figure 10: **Debt value as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

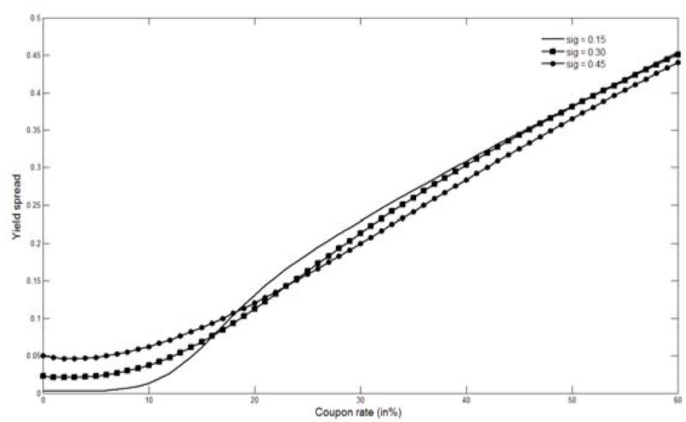

Figure 11: **Yield spread as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

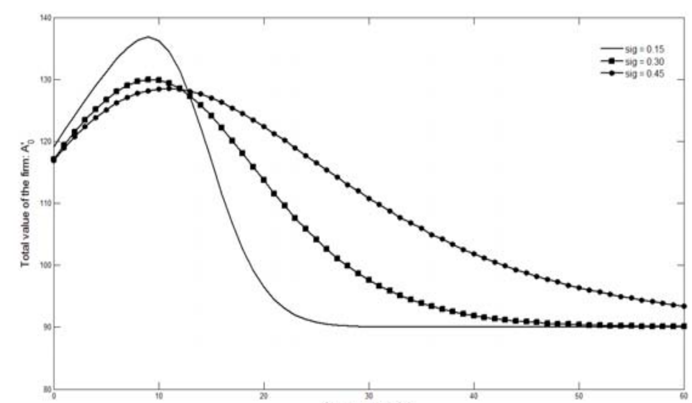

Figure 12: **Total value of the firm as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

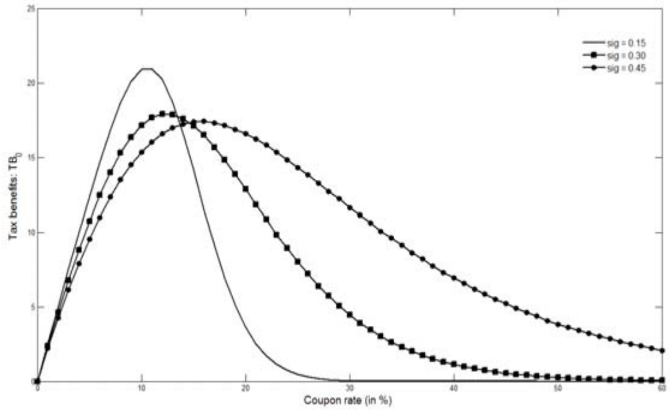

Figure 13: **Tax benefits as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

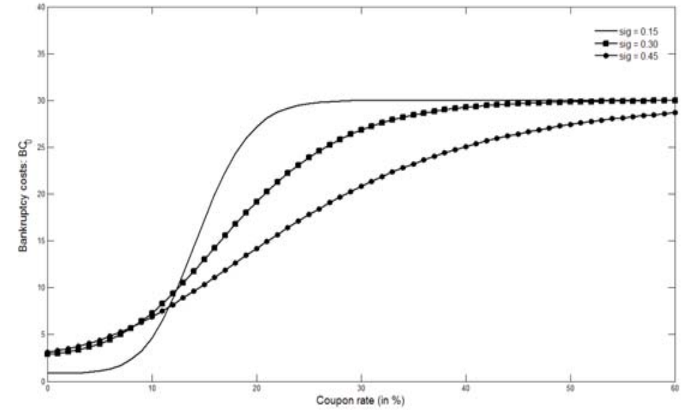

Figure 14: **Bankruptcy costs as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

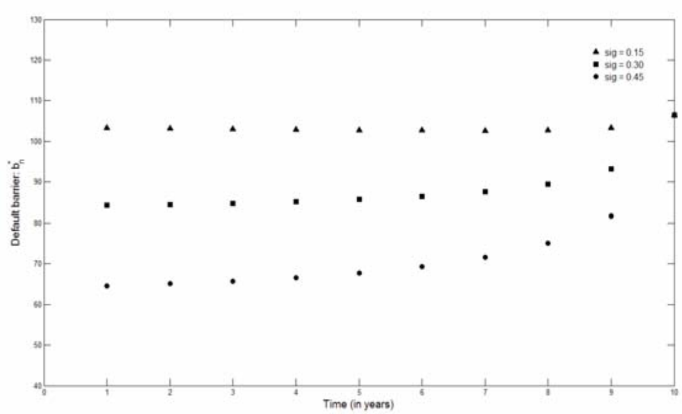

Figure 15: **Term structure of default barriers.** The debt is a coupon bond with a maturity of 10 years, a principal amount of \$100, and an annual coupon rate of 10%. Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

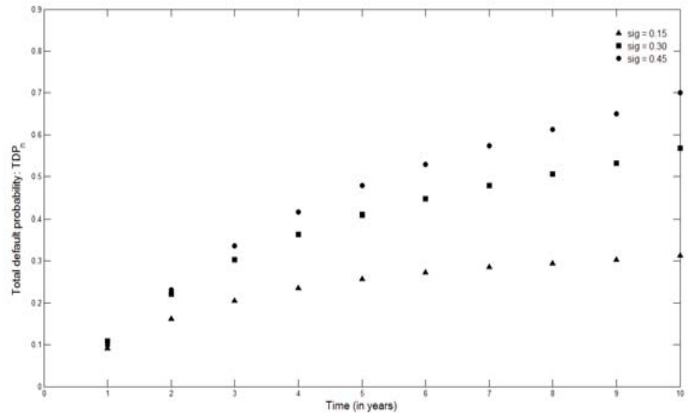

Figure 16: **Term structure of total default probabilities.** The debt is a coupon bond with a maturity of 10 years, a principal amount of \$100, and an annual coupon rate of 10%. Set Set $A_0 = \$120$, $\sigma \in \{15\%, 30\%, 45\%\}$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

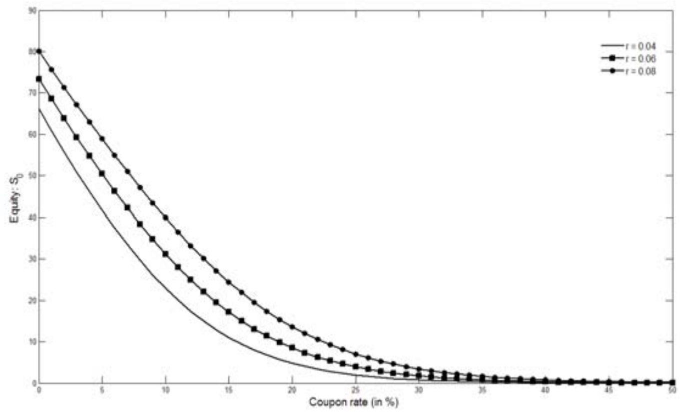

Figure 17: **Equity value as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r \in \{4\%, 6\%, 8\%\}$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

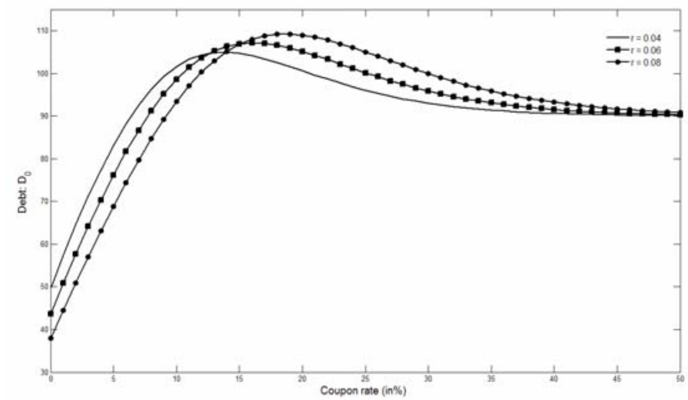

Figure 18: **Debt value as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r \in \{4\%, 6\%, 8\%\}$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

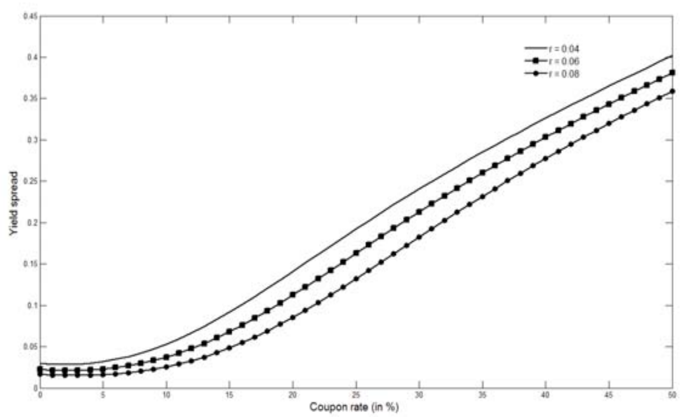

Figure 19: **Yield spread as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r \in \{4\%, 6\%, 8\%\}$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

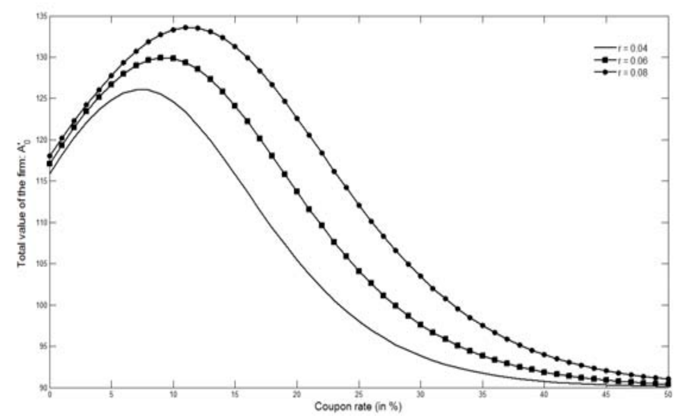

Figure 20: **Total value of the firm as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r \in \{4\%, 6\%, 8\%\}$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

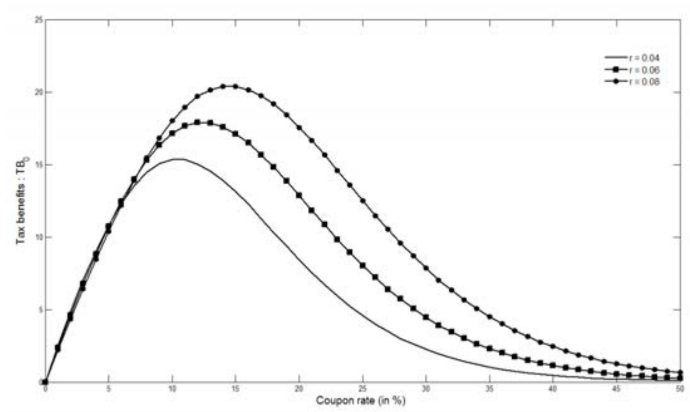

Figure 21: **Tax benefits as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r \in \{4\%, 6\%, 8\%\}$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

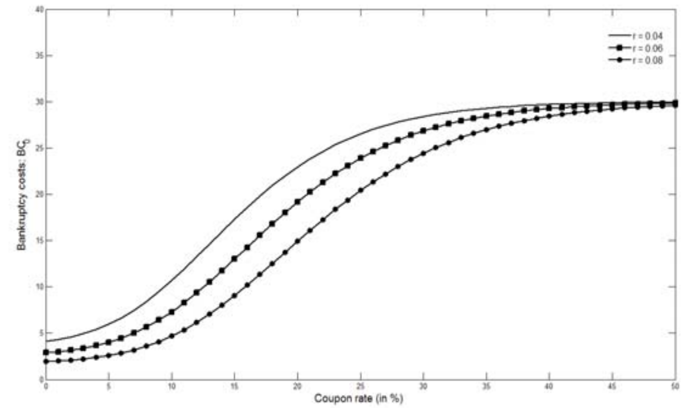

Figure 22: **Bankruptcy costs as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r \in \{4\%, 6\%, 8\%\}$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

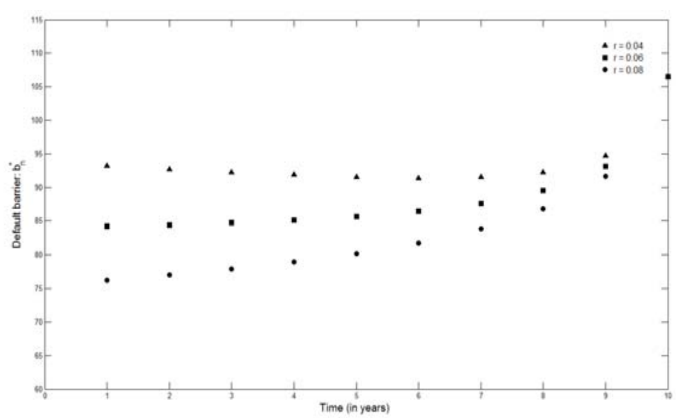

Figure 23: **Term structure of default barriers.** The debt is a coupon bond with a maturity of 10 years, a principal amount of \$100, and an annual coupon rate of 10%. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r \in \{4\%, 6\%, 8\%\}$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

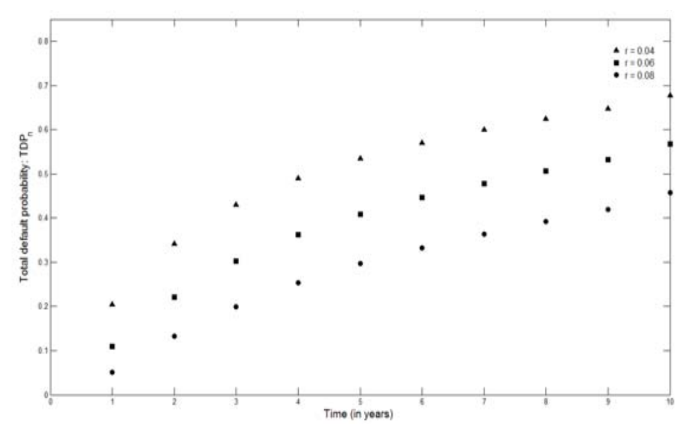

Figure 24: **Term structure of total default probabilities.** The debt is a coupon bond with a maturity of 10 years, a principal amount of \$100, and an annual coupon rate of 10%. Set Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r \in \{4\%, 6\%, 8\%\}$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

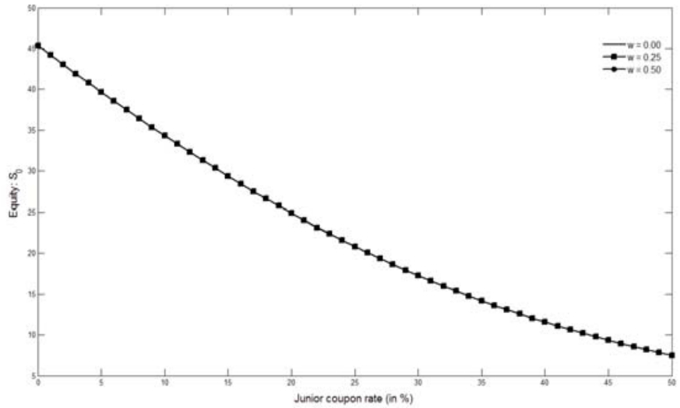

Figure 25: **Equity value as a function of the junior bond's coupon rate.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years and a principal amount of \$30. Set $A_0 =$ \$120, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w \in \{0, 0.25, 0.50\}$.

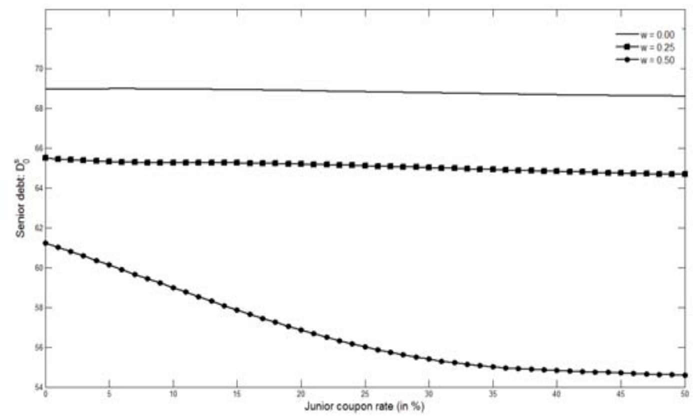

Figure 26: **Senior debt value as a function of the junior bond's coupon rate.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years and a principal amount of \$30. Set $A_0 =$ \$120, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w \in \{0, 0.25, 0.50\}$.

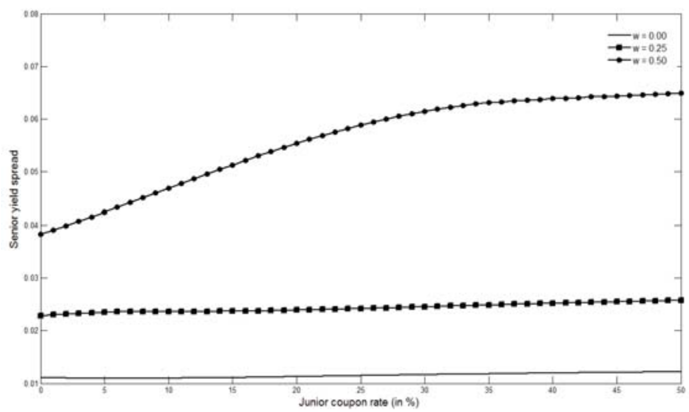

Figure 27: **Senior yield spread as a function of the junior bond's coupon rate.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years and a principal amount of \$30. Set $A_0 =$ \$120, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w \in \{0, 0.25, 0.50\}$.

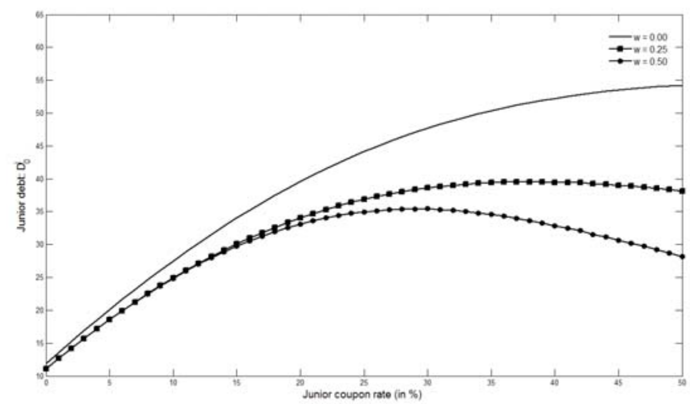

Figure 28: **Junior debt value as a function of the junior bond's coupon rate.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years and a principal amount of \$30. Set $A_0 =$ \$120, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w \in \{0, 0.25, 0.50\}$.

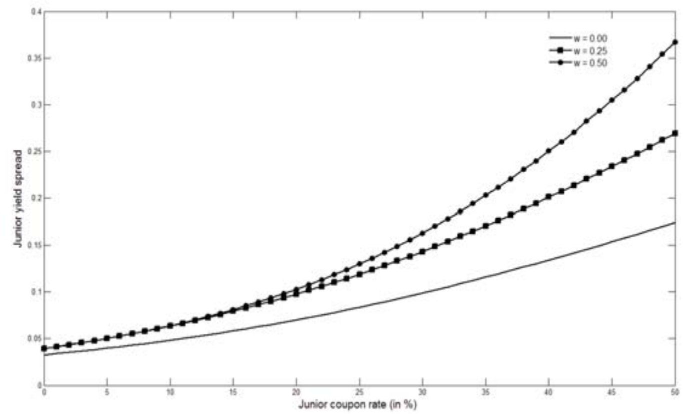

Figure 29: **Junior yield spread as a function of the junior bond's coupon rate.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years and a principal amount of \$30. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w \in \{0, 0.25, 0.50\}$.

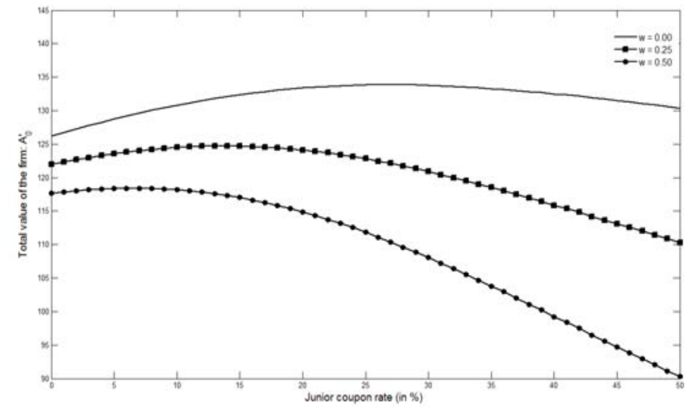

Figure 30: **Total value of the firm as a function of the junior bond's coupon rate.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years and a principal amount of \$30. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w \in \{0, 0.25, 0.50\}$.

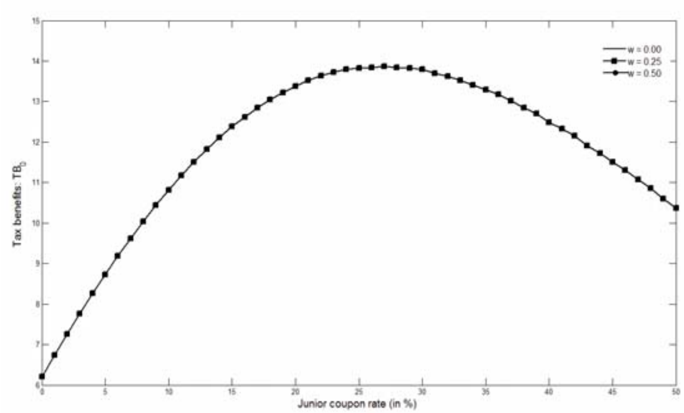

Figure 31: **Tax benefits as a function of the junior bond's coupon rate.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years and a principal amount of \$30. Set $A_0 =$ \$120, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w \in \{0, 0.25, 0.50\}$.

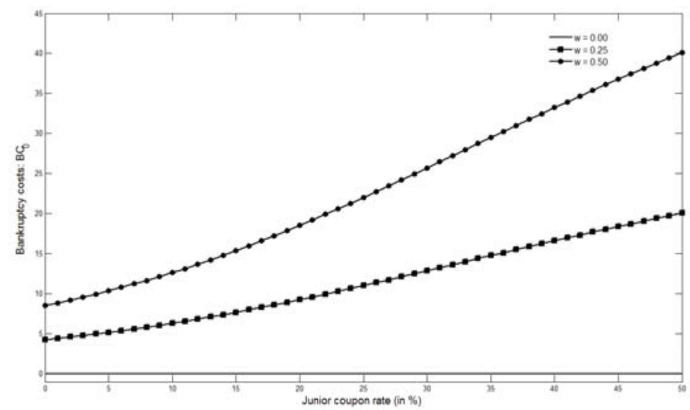

Figure 32: **Bankruptcy costs as a function of the junior bond's coupon rate.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years and a principal amount of \$30. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w \in \{0, 0.25, 0.50\}$.

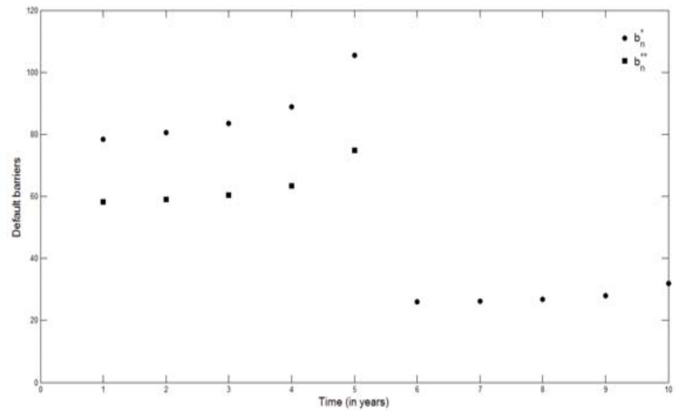

Figure 33: **Term structure of default barriers.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years, a principal amount of \$30, and an annual coupon rate of 10%. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0$.

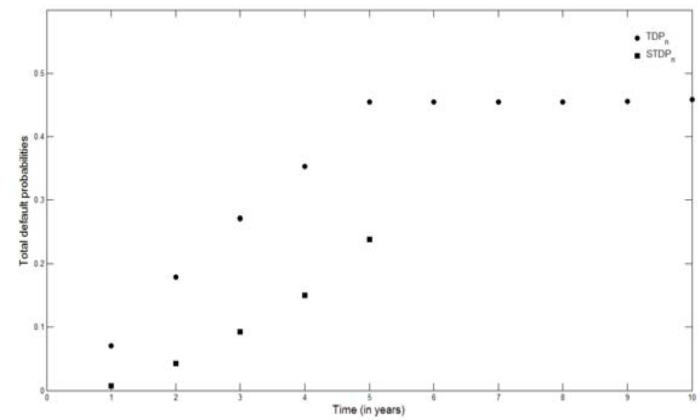

Figure 34: **Term structure of total default probabilities.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years, a principal amount of \$30, and an annual coupon rate of 10%. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0$.

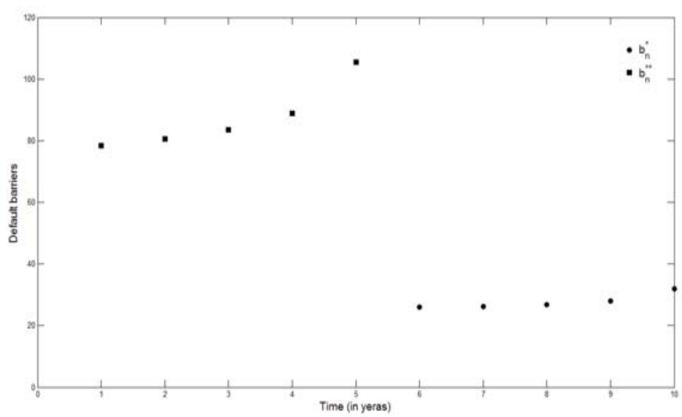

Figure 35: **Term structure of default barriers.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years, a principal amount of \$30, and an annual coupon rate of 10%. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.5$.

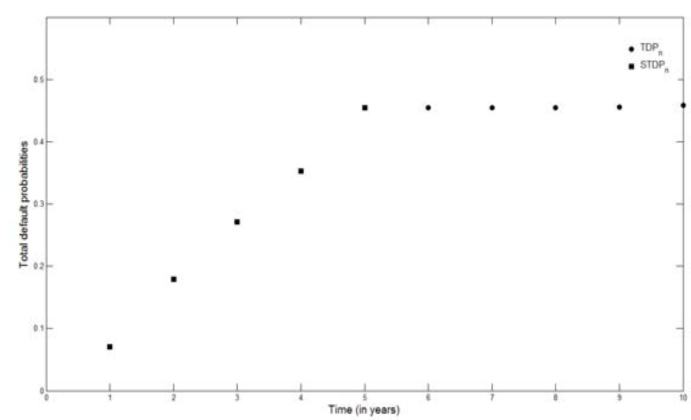

Figure 36: **Term structure of total default probabilities.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years, a principal amount of \$30, and an annual coupon rate of 10%. Set $A_0 = \$120$, $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.5$.

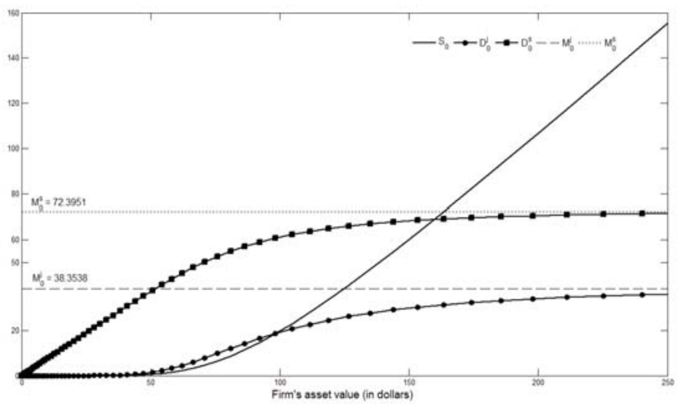

Figure 37: **Debt and equity as functions of the firm's value.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years, a principal amount of \$30, and an annual coupon rate of 10%. Set $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

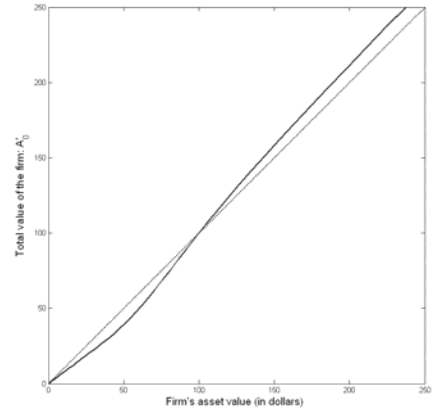

Figure 38: **Total value of the firm as a function of the firm's value.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years, a principal amount of \$30, and an annual coupon rate of 10%. Set $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

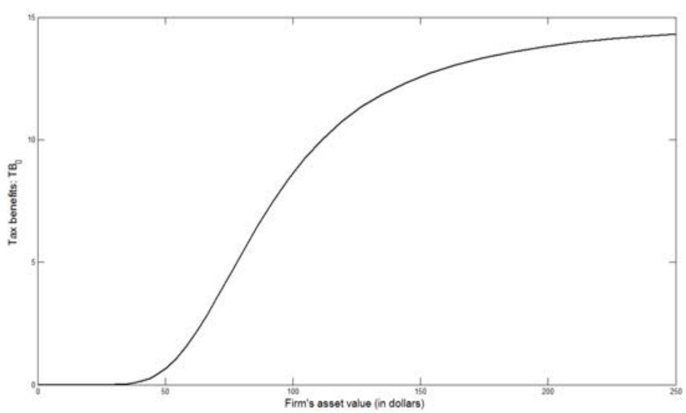

Figure 39: **Tax benefits as a function of the firm's value.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years, a principal amount of \$30, and an annual coupon rate of 10%. Set $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

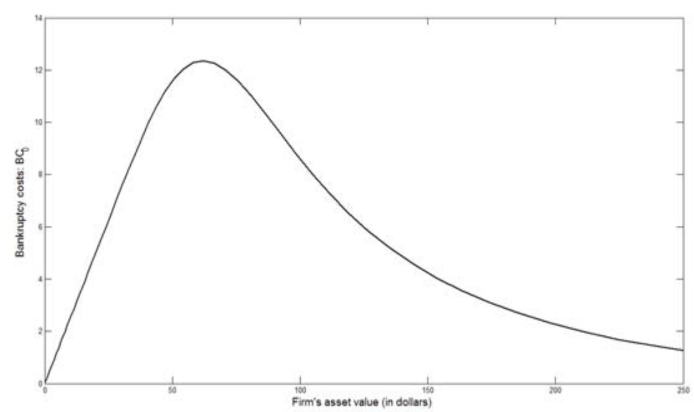

Figure 40: **Bankruptcy costs as a function of the firm's value.** The senior debt is a coupon bond with a maturity of 5 years, a principal amount of \$70, and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years, a principal amount of \$30, and an annual coupon rate of 10%. Set $\sigma = 30\%$ (per year), $r = 6\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

$\bar{g}$ \ η	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
10	(54.8,70.3,125.1)	(54.9,70.6,125.5)	(55.1,70.9,126.0)	(55.5,71.3,126.8)	(56.0,71.5,127.5)	(56.8,71.4,128.2)	(57.9,71.0,128.9)	(59.2,70.2,129.4)	(60.7,69.0,129.7)	(62.4,67.3,129.7)	(64.3,65.5,129.8)
9	(54.8,70.3,125.1)	(54.9,70.6,125.5)	(55.1,70.9,126.0)	(55.5,71.3,126.8)	(56.0,71.5,127.5)	(56.8,71.4,128.2)	(57.9,71.0,128.9)	(59.2,70.2,129.4)	(60.7,69.0,129.7)	(62.4,67.3,129.7)	(64.3,65.5,129.8)
8	(54.8,70.3,125.1)	(54.9,70.6,125.5)	(55.1,70.9,126.0)	(55.5,71.3,126.8)	(56.0,71.5,127.5)	(56.8,71.4,128.2)	(57.9,71.0,128.9)	(59.2,70.2,129.4)	(60.7,69.0,129.7)	(62.4,67.3,129.7)	(64.3,65.5,129.8)
7	(54.8,70.3,125.1)	(54.9,70.6,125.5)	(55.1,70.9,126.0)	(55.5,71.3,126.8)	(56.0,71.5,127.5)	(56.8,71.4,128.2)	(57.9,71.0,128.9)	(59.2,70.2,129.4)	(60.7,69.0,129.7)	(62.4,67.3,129.7)	(64.3,65.5,129.8)
6	(54.8,70.3,125.1)	(54.9,70.6,125.5)	(55.1,70.9,126.0)	(55.5,71.3,126.8)	(56.0,71.5,127.5)	(56.8,71.4,128.2)	(57.9,71.0,128.9)	(59.2,70.2,129.4)	(60.7,69.0,129.7)	(62.4,67.3,129.7)	(64.3,65.5,129.8)
5	(54.8,70.3,125.1)	(54.9,70.6,125.5)	(55.1,70.9,126.0)	(55.5,71.3,126.8)	(56.0,71.5,127.5)	(56.8,71.4,128.2)	(57.9,71.0,128.9)	(59.2,70.2,129.4)	(60.7,69.0,129.7)	(62.4,67.3,129.7)	(64.3,65.5,129.8)
4	(54.8,70.3,125.1)	(54.9,70.6,125.5)	(55.1,70.9,126.0)	(55.5,71.3,126.8)	(56.0,71.5,127.5)	(56.8,71.4,128.2)	(57.9,71.0,128.9)	(59.2,70.2,129.4)	(60.7,69.0,129.7)	(62.4,67.3,129.7)	(64.3,65.5,129.8)
3	(54.8,70.3,125.1)	(54.9,70.6,125.5)	(55.1,70.9,126.0)	(55.5,71.3,126.8)	(56.0,71.5,127.5)	(56.8,71.4,128.2)	(57.9,71.0,128.9)	(59.2,70.2,129.4)	(60.7,68.9,129.6)	(62.4,67.3,129.7)	(64.3,65.5,129.8)
2	(54.8,70.3,125.1)	(54.9,70.6,125.5)	(55.1,70.9,126.0)	(55.5,71.3,126.8)	(56.0,71.4,127.4)	(56.8,71.3,128.1)	(57.9,70.9,128.8)	(59.2,70.1,129.3)	(60.7,68.9,129.6)	(62.4,67.3,129.7)	(64.3,65.5,129.8)
1	(54.8,70.3,125.1)	(54.9,70.6,125.5)	(55.1,70.9,126.0)	(55.5,71.0,126.5)	(56.0,71.1,127.1)	(56.8,71.0,127.8)	**(57.8,70.6,128.4)**	(59.1,69.9,129.0)	(60.7,68.8,129.5)	(62.4,67.3,129.7)	**(64.3,65.5,129.8)**
0	(54.8,70.3,125.1)	(54.8,70.3,125.1)	(54.8,70.3,125.1)	(54.8,70.3,125.1)	(54.8,70.3,125.1)	(54.8,70.3,125.1)	(54.8,70.3,125.1)	(54.8,70.3,125.1)	(54.8,70.3,125.1)	(54.8,70.3,125.1)	(54.8,70.3,125.1)

Figure 41: **Equity, debt, and firm's asset value.** The debt is a 4% coupon bond with a maturity of 10 years and a principal amount of \$100. Set A_0 = \$120, σ = 30% (per year), r = 6% (per year), r^c = 35% (per year), w^r = 0.02, and w = 0.25.

$\overline{g}$ \ η	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
10	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.3,105.9)	(4.6,101.3,105.9)	(4.7,101.3,106.0)	(4.7,101.3,106.0)	(4.7,101.5,106.2)	(4.8,101.4,106.2)	(5.2,99.9,105.0)	**(9.7,68.5,78.2)**
9	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.3,105.9)	(4.6,101.3,105.9)	(4.7,101.3,106.0)	(4.7,101.3,106.0)	(4.7,101.5,106.2)	(4.8,101.4,106.2)	(5.2,99.9,105.0)	(8.0,70.1,78.1)
8	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.3,105.9)	(4.6,101.3,105.9)	(4.7,101.3,106.0)	(4.7,101.3,106.0)	(4.7,101.5,106.2)	(4.8,101.4,106.2)	(5.2,99.9,105.1)	(7.1,70.4,77.5)
7	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.3,105.9)	(4.6,101.3,105.9)	(4.7,101.3,106.0)	(4.7,101.3,106.0)	(4.7,101.5,106.2)	(4.8,101.4,106.2)	(5.2,99.9,105.1)	(6.9,70.4,77.3)
6	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.3,105.9)	(4.6,101.3,105.9)	(4.7,101.3,106.0)	(4.7,101.3,106.0)	(4.7,101.5,106.2)	(4.8,101.4,106.2)	(5.2,99.9,105.1)	(6.8,72.3,79.1)
5	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.3,105.9)	(4.6,101.3,105.9)	(4.7,101.3,106.0)	(4.7,101.3,106.0)	(4.7,101.5,106.2)	(4.8,101.4,106.2)	(5.2,99.9,105.1)	(6.7,75.4,81.1)
4	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.3,105.9)	(4.6,101.3,105.9)	(4.7,101.3,106.0)	(4.7,101.3,106.0)	(4.7,101.5,106.2)	(4.8,101.4,106.2)	(5.2,100.0,105.2)	(6.6,79.6,86.2)
3	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.3,105.9)	(4.6,101.3,105.9)	(4.7,101.3,106.0)	(4.7,101.3,106.0)	(4.7,101.5,106.2)	(4.8,101.4,106.2)	(5.2,100.1,105.3)	(6.4,85.3,91.7)
2	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.3,105.9)	(4.6,101.3,105.9)	(4.7,101.3,106.0)	(4.7,101.3,106.0)	(4.7,101.5,106.2)	**(4.8,101.4,106.2)**	(5.1,100.4,105.5)	(5.9,92.6,98.5)
1	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.3,105.9)	(4.6,101.3,105.9)	(4.7,101.3,106.0)	(4.7,101.3,106.0)	(4.7,101.5,106.2)	(4.7,101.5,106.2)	(4.7,101.7,106.4)	**(4.8,101.8,106.6)**
0	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)	(4.6,101.2,105.8)

Figure 42: **Equity, debt, and firm's asset value.** The debt is a 24% coupon bond with a maturity of 10 years and a principal amount of \$100. Set A_0 = \$120, σ = 30% (per year), r = 6% (per year), r^c = 35% (per year), w^r = 0.20, and w = 0.25.